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*Research article*

# Refined estimates and generalization of some recent results with applications

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**Abstract:** In this paper, we firstly give improvement of Hermite-Hadamard type and Fejér type inequalities. Next, we extend Hermite-Hadamard type and Fejér types inequalities to a new class of functions. Further, we give bounds for newly defined class of functions and finally presents refined estimates of some already proved results. Furthermore, we obtain some new discrete inequalities for univariate harmonic convex functions on linear spaces related to a variant most recently presented by Baloch *et al.* of Jensen-type result that was established by S. S. Dragomir.

**Keywords:** convex functions; harmonic convex functions; Hermite-Hadamard inequality; Fejér type inequality; Hermite-Hadamard type inequality; Jensen-type inequality; a variant of Jensen type inequality; Hölder's inequality; weighted HGA inequality
**Mathematics Subject Classification:** 26A15, 26A51, 26D10, 26D15

## 1. Introduction

Functions with convexity property have been frequently deployed in numerous fields of pure and applied mathematics, by way of illustration in function theory, mathematical analysis, functional analysis, probability theory, optimization theory, operational research and information theory. Briefly, convex functions entail a strong and elegant interaction between analysis and geometry, and convex functions got prominence in the mathematical inequalities and applications, and most of inequalities are regularly used in solving several problems of the applied sciences like as, Jensen's inequality [1] which is lord among inequalities because it make out at once the main part of the other classical inequalities (e.g., those by Hölder, Minkowski, Beckenbach-Dresher and Young, the A-G inequality etc.) that holds for the class of convex functions under certain conditions. Some recent work on the applications of mathematical inequalities can be found in [2–9]. The following inequality holds for any convex function $\Phi$ defined on $\mathbb{R}$

$$\Phi\left(\frac{w_1+w_2}{2}\right) \leq \frac{1}{w_2-w_1}\int_{w_1}^{w_2}\Phi(w)dw \leq \frac{\Phi(w_1)+\Phi(w_2)}{2}, \quad w_1,\ w_2\ \in\mathbb{R},\ w_1\neq w_2. \tag{1.1}$$

Further, the simple generalization to a convex function extensively widens our scope for analysis, and during the investigation of convexity, many researchers founded new classes of functions which are not convex in general. Some of them are the so called harmonic convex functions [10], harmonic $(w_1, m)$-convex functions [11], harmonic $(s, m)$-convex functions [12, 13] and harmonic $(p,(s,m))$-convex functions [14]. For a quick glance on importance of these classes and applications see [15–29] and references therein.

It is import to bring into your kind knowledge that the harmonic property has taken a significant apart in different fields of pure and applied sciences. In [30], the authors have emphasized on the important character of the harmonic mean in Asian stock options. More interestingly, harmonic means are applied in electric circuit theory. Specifically, the total resistance of a set of parallel resistor is just half of the harmonic mean of the total resistors. For instance, suppose $R_1$ and $R_2$ are the resistances of two parallel resistors, then the total resistance $R_T$ is computed by the following formula:

$$R_T = \frac{R_1R_2}{R_1+R_2}.$$

**Definition 1.1.** A function $\Phi : \mathbf{I}\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is said to be harmonic convex function on $\mathbf{I}$ if

$$\Phi\left(\frac{w_1w_2}{tw_1+(1-t)w_2}\right)\leq t\Phi(w_2)+(1-t)\Phi(w_1) \tag{1.2}$$

holds for all $w_1, w_2 \in \mathbf{I}$ and $t\in[0,1]$. If the inequality is reversed, then $\Phi$ is said to be harmonic concave.

**Example 1.2.** Here, we present some non-trivial examples

- The functions $\Phi_1(w) = \ln w$, $\Phi_2(w) = \sqrt{w}$ for all $w \in (0,\infty)$, are examples of harmonic convex functions as well as concave functions.
- The functions $\Phi_3(w) = \frac{(w-1)^2+1}{w}$ and $\Phi_4(w) = \frac{1}{w^2}$ for all $w\in(0,\infty)$, are examples of harmonic convex function which are also convex functions.
- The function $\Psi(w) = \begin{cases}\frac{1-w}{w}, & if\ 0<w\leq 1\\ 0, & if\ 1<w\leq 2\\ \frac{w-2}{w}, & if\ w>2\end{cases}$ is an example of harmonic convex function that is neither convex nor concave function.

Clearly, the function $\pi(w) = -\ln w$, is example of convex functions which is not harmonic convex function. In the presence of these example, Baloch *et al.* [31] claimed that class of harmonic convex functions is neither exactly that of convex functions nor it is entirely different from class of convex functions. More precisely, they investigated remarkable relations between these two classes under certain conditions as follows:

**Lemma 1.3.** *Let $\boldsymbol{I} \subseteq \mathbb{R}/\{0\}$ be a real interval. Define $\boldsymbol{I}^{-1} = \{w_2 \in \overline{\mathbb{R}}, w_2 = \frac{1}{w_1}, w_1 \in \boldsymbol{I}\}$. A function $\Phi : \boldsymbol{I} \to \mathbb{R}$ is harmonic convex if and only if $\Psi : \boldsymbol{I}^{-1} \to \mathbb{R}$ is convex, where $\Psi$ is defined as $\Psi(w_2) = \Phi(w_1)$.*

**Lemma 1.4.** *Let $\boldsymbol{I} \subseteq (0,\infty)$ and $\boldsymbol{I}^{-1}$ has similar definition as given in Lemma 1.3. A function $\Phi : \boldsymbol{I} \to \mathbb{R}$ is harmonic convex if and only if $\Psi : \boldsymbol{I} \to \mathbb{R}$ is convex, where $\Psi$ is defined as $\Psi(w) = w\Phi(w)$ .*

**Proposition 1.5.** *Let $\boldsymbol{I} \subseteq \mathbb{R}/\{0\}$ be a real interval and $\Phi : \boldsymbol{I} \to \mathbb{R}$ is a function, then;*

*1). If $\boldsymbol{I} \subset (0,\infty)$ and $\Phi$ is convex and nondecreasing function, then $\Phi$ is harmonic convex function.*
*2). If $\boldsymbol{I} \subset (0,\infty)$ and $\Phi$ is harmonic convex and nonincreasing function, then $\Phi$ is convex function.*
*3). If $\boldsymbol{I} \subset (-\infty,0)$ and $\Phi$ is harmonic convex and nondecreasing function, then $\Phi$ is convex function.*
*4). If $\boldsymbol{I} \subset (-\infty,0)$ and $\Phi$ is convex and nonincreasing function, then $\Phi$ is harmonic convex function.*

In [10], İ. İşcan proved the following result, which is known as Hermite-Hadamard type inequality for harmonic convex functions.

**Theorem 1.6.** *Let $\Phi : \boldsymbol{I} \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be a harmonic convex function and $w_1, w_2 \in \boldsymbol{I}$ with $w_1 < w_2$. If $\Phi \in L[w_1, w_2]$, then the following inequalities hold*

$$\Phi\left(\frac{2w_1w_2}{w_1+w_2}\right) \le \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2}\frac{\Phi(w)}{w^2}dw \le \frac{\Phi(w_1)+f(w_2)}{2}. \tag{1.3}$$

In [32], Chen *et al.* proved Fejér type inequalities for harmonic convex functions as follows:

**Theorem 1.7.** *Let $\Phi : \boldsymbol{I} \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be a harmonic convex function and $w_1, w_2 \in \boldsymbol{I}$ with $w_1 < w_2$. If $\Phi \in L[w_1, w_2]$, then the following inequalities hold*

$$\Phi\left(\frac{2w_1w_2}{w_1+w_2}\right)\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw \le \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2}\frac{\Phi(w)\Psi(w)}{w^2}dw \le \frac{\Phi(w_1)+\Phi(w_2)}{2}\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw, \tag{1.4}$$

*where $\Psi : [w_1, w_2] \to \mathbb{R}$ is non-negative, integral and satisfies*

$$\Psi\left(\frac{w_1w_2}{w}\right) = \Psi\left(\frac{w_1w_2}{w_1+w_2-w}\right) \tag{1.5}$$

**Note**: Here, we mention the worth of the class of harmonic convex functions in view of following applications in the field of mathematics.

- Baloch *et al.* [31] proved many outstanding facts and disclosed that harmonic convexity gives an analytic tool to estimate several known definite integrals like $\int_{w_1}^{w_2}\frac{e^w}{w^n}dw$, $\int_{w_1}^{w_2}e^{w^2}dw$, $\int_{w_1}^{w_2}\frac{\sin w}{w^n}dw$ and $\int_{w_1}^{w_2}\frac{\cos w}{w^n}dw$, $\forall n \in \mathbb{N}$, where $w_1, w_2 \in (0,\infty)$.
- Secondly, Baloch *et al.* [33] explored that the inequality (1.6) provides a very nice short proof of Discrete form of Hölder's inequality.
- Thirdly, Baloch *et al.* [34] used inequalities (1.6) and (1.7) to establish a very simple short proof of weighted HGA inequality.

In [35], S. S. Dragomir proved the following result, which is known as Jensen-type inequality for harmonic convex functions.

**Theorem 1.8.** *Let $\boldsymbol{I} \subseteq (0,\infty)$ be an interval. If $\Phi : \boldsymbol{I} \to \mathbb{R}$ is harmonic convex function, then*

$$\Phi\left(\frac{1}{\sum_{k=1}^{n}\frac{a_k}{w_k}}\right) \leq \sum_{k=1}^{n} a_k \Phi(w_k), \tag{1.6}$$

*holds for all $w_1, ..., w_n \in \boldsymbol{I}$ and $a_k \in [0,1]$ with $\sum_{k=1}^{n} a_k = 1$.*

In [33], Baloch *et al.* proved the subsequent result which is a variant of Jensen-type inequality for harmonic convex functions.

**Theorem 1.9.** *Let $[m, M] \subseteq (0,\infty)$ be an interval. If $\Phi : [m, M] \to \mathbb{R}$ is harmonic convex function, then for any finite sequence $(w_k)_{k=1}^{n} \in [m, M]$ and $a_k \in [0,1]$ with $\sum_{k=1}^{n} a_k = 1$, we have*

$$\Phi\left(\frac{1}{\frac{1}{m} + \frac{1}{M} - \sum_{k=1}^{n}\frac{a_k}{w_k}}\right) \leq \Phi(m) + \Phi(M) - \sum_{k=1}^{n} a_k \Phi(w_k) \tag{1.7}$$

In [36], Baloch *et al.* also proved the following nice result:

**Theorem 1.10.** *If $\Phi : I \to \mathbb{R}$ is harmonic convex function on harmonic convex subset $\boldsymbol{I} \subseteq \mathbb{R}\backslash\{0\}$ , then for any finite positive sequence $\{w_k\}_{k=1}^{n} \in \boldsymbol{I}$ and $a_k$ with $A_n := \sum_{k=1}^{n} a_k > 0$, we have*

$$\begin{aligned} n \min_{1\leq k\leq n}\{a_k\} & \left[\frac{1}{n}\sum_{k=1}^{n}\Phi(w_k) - \Phi\left(\frac{1}{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{w_k}}\right)\right] \\ & \leq \frac{1}{A_n}\sum_{k=1}^{n} a_k\Phi(w_k) - \Phi\left(\frac{1}{\frac{1}{A_n}\sum_{k=1}^{n}\frac{a_k}{w_k}}\right) \\ & \leq n \max_{1\leq k\leq n}\{a_k\} \left[\frac{1}{n}\sum_{k=1}^{n}\Phi(w_k) - \Phi\left(\frac{1}{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{w_k}}\right)\right] \end{aligned} \tag{1.8}$$

## 2. Results

In this section, we firstly give refinement of Hermite-Hadamard type inequalities (1.2) for univariate harmonic convex functions.

**Theorem 2.1.** *Let $\Phi : \boldsymbol{I} \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be a harmonic convex function and $w_1, w_2 \in \boldsymbol{I}$ with $w_1 < w_2$. If $\Phi \in L[w_1, w_2]$, then for all $\lambda \in [0,1]$, the following inequalities hold*

$$\Phi\left(\frac{2w_1w_2}{w_1+w_2}\right) \leq l(\lambda) \leq \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2}\frac{\Phi(w)}{w^2}dw \leq L(\lambda) \leq \frac{\Phi(w_1)+\Phi(w_2)}{2} \tag{2.1}$$

*where*

$$l(\lambda) := \lambda\Phi\left(\frac{2w_1w_2}{\lambda w_1 + (2-\lambda)w_2}\right) + (1-\lambda)\Phi\left(\frac{2w_1w_2}{(1-\lambda)w_2 + (1+\lambda)w_1}\right)$$

*and*

$$L(\lambda) := \frac{\lambda\Phi(w_1) + \Phi\left(\frac{w_1w_2}{\lambda w_1 + (1-\lambda)w_2}\right) + (1-\lambda)\Phi(w_2)}{2}$$

*Proof.* Let $\Phi$ be harmonic convex function on $\mathbf{I}$, applying (1.6) on the sub-interval $[w_1, \frac{w_1w_2}{\lambda w_1+(1-\lambda)w_2}]$, with $\lambda \neq 0$, we get

$$\Phi\left(\frac{2w_1w_2}{\lambda w_1+(2-\lambda)w_2}\right) \leq \frac{w_1w_2}{\lambda(w_2-w_1)}\int_{w_1}^{\frac{w_1w_2}{\lambda w_1+(1-\lambda)w_2}}\frac{\Phi(w)}{w^2}dw \leq \frac{\Phi(w_1)+\Phi\left(\frac{w_1w_2}{\lambda w_1+(1-\lambda)w_2}\right)}{2}. \tag{2.2}$$

Applying (1.6) again on the sub-interval $[\frac{w_1w_2}{\lambda w_1+(1-\lambda)w_2}, w_2]$, with $\lambda \neq 0$, we get

$$\Phi\left(\frac{2w_1w_2}{(1-\lambda)w_2+(1+\lambda)w_1}\right) \leq \frac{w_1w_2}{(1-\lambda)(w_2-w_1)}\int_{\frac{w_1w_2}{\lambda w_1+(1-\lambda)w_2}}^{w_2}\frac{\Phi(w)}{w^2}dw \leq \frac{\Phi\left(\frac{w_1w_2}{\lambda w_1+(1-\lambda)w_2}\right)+\Phi(w_2)}{2}. \tag{2.3}$$

Multiply (2.2) by $\lambda$, (2.3) by $1-\lambda$ and adding the resulting inequalities, we get

$$l(\lambda) \leq \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2}\frac{\Phi(w)}{w^2}dw \leq L(\lambda) \tag{2.4}$$

where $l(\lambda)$ and $L(\lambda)$ are same as defined in Theorem 2.1.
Using the fact that $\Phi$ is harmonic convex, we obtain

$$\begin{aligned}
\Phi\left(\frac{2w_1w_2}{w_1+w_2}\right) &= \Phi\left(\frac{1}{\lambda\frac{\lambda w_1+(2-\lambda)w_2}{2w_1w_2}+(1-\lambda)\frac{(1-\lambda)w_2+(1+\lambda)w_1}{2w_1w_2}}\right) \qquad (2.5)\\
&\leq \lambda\Phi\left(\frac{2w_1w_2}{\lambda w_1+(1-\lambda)w_2+w_2}\right)+(1-\lambda)\Phi\left(\frac{2w_1w_2}{(1-\lambda)w_2+\lambda w_1+w_1}\right)\\
&\leq \frac{1}{2}\left[\lambda\Phi\left(\frac{2w_1w_2}{\lambda w_1+(1-\lambda)w_2}\right)+\lambda\Phi(w_1)+(1-\lambda)\Phi\left(\frac{2w_1w_2}{\lambda w_1+(1-\lambda)w_2}\right)+(1-\lambda)\Phi(w_2)\right]\\
&\leq \frac{\Phi(w_1)+\Phi(w_2)}{2}.
\end{aligned}$$

Then by (2.4) and (2.5), we get (2.1). □

**Corollary 2.2.** *Under the assumptions of Theorem 2.1, we have*

$$\Phi\left(\frac{2w_1w_2}{w_1+w_2}\right) \leq l \leq \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2}\frac{\Phi(w)}{w^2}dw \leq L \leq \frac{\Phi(w_1)+\Phi(w_2)}{2} \tag{2.6}$$

*where*

$$l := \frac{1}{2}\left[\Phi\left(\frac{4w_1w_2}{w_1+3w_2}\right)+\Phi\left(\frac{4w_1w_2}{3w_1+w_2}\right)\right]$$

*and*

$$L := \frac{\Phi(w_1)+2\Phi(\frac{2w_1w_2}{w_1+w_2})+f\Phi(w_2)}{4}$$

**Corollary 2.3.** *Under the assumptions of Theorem 2.1, we have*

$$\Phi\Big(\frac{2w_1w_2}{w_1+w_2}\Big) \leq \sup_{\lambda\in[0,1]} l(\lambda) \leq \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2}\frac{\Phi(w)}{w^2}dw \leq \inf_{\lambda\in[0,1]} L(\lambda) \leq \frac{\Phi(w_1)+\Phi(w_2)}{2}. \tag{2.7}$$

**Theorem 2.4.** *Let $\Phi : \boldsymbol{I} \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be a harmonic convex function and $w_1, w_2 \in \boldsymbol{I}$ with $w_1 < w_2$. If $\Phi, \Psi \in L[w_1, w_2]$, $\Psi$ is non-negative and satisfies condition (1.8) , then for all $\lambda \in [0, 1]$, the following inequalities hold*

$$\begin{aligned} \Phi\Big(\frac{2w_1w_2}{w_1+w_2}\Big)\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw &\leq l(\lambda)\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw \\ &\leq \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2}\frac{\Phi(w)\Psi(w)}{w^2}dw \\ &\leq L(\lambda)\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw \\ &\leq \frac{\Phi(w_1)+\Phi(w_2)}{2}\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw, \end{aligned} \tag{2.8}$$

*where $l(\lambda)$ and $L(\lambda)$ are as defined in Theorem 2.1.*

*Proof.* The proof follows on the same lines as that of Theorem 2.1. □

**Corollary 2.5.** *Under the assumptions of Theorem 2.4, we have*

$$\begin{aligned} \Phi\Big(\frac{2w_1w_2}{w_1+w_2}\Big)\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw &\leq l\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw \\ &\leq \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2}\frac{\Phi(w)\Psi(w)}{w^2}dw \\ &\leq L\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw \\ &\leq \frac{\Phi(w_1)+\Phi(w_2)}{2}\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw, \end{aligned} \tag{2.9}$$

*where $l$ and $L$ are same as defined in Corollary 2.13.*

**Corollary 2.6.** *Under the assumptions of Theorem 2.4, we have*

$$\begin{aligned} \Phi\Big(\frac{2w_1w_2}{w_1+w_2}\Big)\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw &\leq \sup_{\lambda\in[0,1]} l(\lambda)\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw \\ &\leq \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2}\frac{\Phi(w)\Psi(w)}{w^2}dw \\ &\leq \inf_{\lambda\in[0,1]} L(\lambda)\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw \\ &\leq \frac{\Phi(w_1)+\Phi(w_2)}{2}\int_{w_1}^{w_2}\frac{\Psi(w)}{w^2}dw \end{aligned} \tag{2.10}$$

For a function $\Phi : \mathbf{I} \subset \mathbb{R}/\{0\} \to \mathbb{C}$, we consider the symmetrical transform of $\Phi$ on interval $\mathbf{I}$ denoted by $\breve{\Phi}_{\mathbf{I}}$ or simply $\breve{\Phi}$ as defined by

$$\breve{\Phi}(t) := \frac{1}{2}\Big[\Phi(t) + \Phi\Big(\frac{w_1 w_2 t}{(w_1 + w_2)t - w_1 w_2}\Big)\Big],\ t \in \mathbf{I}.$$

The anti-symmetrical transform of $\Phi$ on interval $I$ denoted by $\tilde{\Phi}_{\mathbf{I}}$ or simply $\breve{\Phi}$ as defined by

$$\tilde{\Phi}(t) := \frac{1}{2}\Big[\Phi(t) - \Phi\Big(\frac{w_1 w_2 t}{(w_1 + w_2)t - w_1 w_2}\Big)\Big],\ t \in \mathbf{I}.$$

It is obvious for any function $\Phi$, we have $\Phi = \breve{\Phi} + \tilde{\Phi}$ and further, if $\Phi$ is harmonic convex on $\mathbf{I}$ then $\breve{\Phi}$ is also harmonic convex on $\mathbf{I}$ but reverse is not true in general.

**Definition 2.7.** A function $\Phi : \mathbf{I} \subset \mathbb{R}/\{0\} \to \mathbb{C}$ is said to be symmetrized harmonic convex (concave) on $\mathbf{I}$ if $\breve{\Phi}$ is harmonic convex (concave) on $\Phi$.

**Theorem 2.8.** *Assume that function $\Phi : \boldsymbol{I} \subset \mathbb{R}/\{0\} \to \mathbb{C}$ is symmetrized harmonic convex, integrable on $\boldsymbol{I}$ and $\Psi$ is non-negative integrable function that satisfies condition*

$$\int_{w_1}^{w_2} \frac{\Phi(w)\Psi\Big(\frac{w_1 w_2 t}{(w_1+w_2)t - w_1 w_2}\Big)}{w^2} dw = \int_{w_1}^{w_2} \frac{\Phi(x)\Psi(w)}{w^2} dw, \tag{2.11}$$

*then we have inequalities (1.6) and (1.7).*

*Proof.* Since, function $\Phi : \mathbf{I} \subset \mathbb{R}/\{0\} \to \mathbb{C}$ is symmetrized harmonic convex, integrable on $\mathbf{I}$, then by Hermite-Hadamard type inequality (1.6) for $\breve{\Phi}$, we have

$$\breve{\Phi}\left(\frac{2w_1 w_2}{w_1 + w_2}\right) \le \frac{w_1 w_2}{w_2 - w_1} \int_{w_1}^{w_2} \frac{\breve{\Phi}(w)}{w^2} dw \le \frac{\breve{\Phi}(w_1) + \breve{\Phi}(w_2)}{2}. \tag{2.12}$$

After some simple calculations, we see that $\breve{\Phi}\left(\frac{2w_1 w_2}{w_1+w_2}\right) = \Phi\left(\frac{2w_1 w_2}{w_1+w_2}\right),\ \breve{\Phi}(w_1) + \breve{\Phi}(w_2) = \Phi(w_1) + \Phi(w_2)$ and

$$\int_{w_1}^{w_2} \frac{\breve{\Phi}(w)}{w^2} dw = \int_{w_1}^{w_2} \frac{\Phi(w)}{w^2} dw$$

Therefore, by substituting these values in (2.12), we get (1.6).
Similarly, we can prove (1.7) for symmetrized harmonic convex function $\Phi$. □

**Corollary 2.9.** *Assume that function $\Phi : \boldsymbol{I} \subset \mathbb{R}/\{0\} \to \mathbb{C}$ is symmetrized harmonic convex, integrable on $\boldsymbol{I}$ and $\Psi$ is non-negative integrable function that satisfies condition (1.8), then we have (1.7).*

**Theorem 2.10.** *Assume that function $\Phi : [w_1, w_2] \subset \mathbb{R}/\{0\} \to \mathbb{C}$ is symmetrized harmonic convex on $[w_1, w_2]$, then for any $w \in [w_1, w_2]$ we have bounds*

$$\Phi\left(\frac{2w_1 w_2}{w_1 + w_2}\right) \le \breve{\Phi}(w) \le \frac{\Phi(w_1) + \Phi(w_2)}{2}. \tag{2.13}$$

*Proof.* Since, $\breve{\Phi}$ is harmonic convex on $[w_1, w_2]$, then for any $w \in [w_1, w_2]$ we have

$$\breve{\Phi}\left(\frac{2w_1w_2}{w_1+w_2}\right) \leq \frac{\breve{\Phi}(w) + \breve{\phi}\left(\frac{w_1w_2t}{(w_1+w_2)t-w_1w_2}\right)}{2}$$

and simple calculations shows that $\breve{\Phi}\left(\frac{2w_1w_2}{w_1+w_2}\right) = \Phi\left(\frac{2w_1w_2}{w_1+w_2}\right)$, $\breve{\Phi}(w) = \Phi(w)$ and

$$\breve{\Phi}\left(\frac{w_1w_2t}{(w_1+w_2)t-w_1w_2}\right) = \Phi\left(\frac{w_1w_2t}{(w_1+w_2)t-w_1w_2}\right),$$

we get first inequality in (2.13).
Also, by the harmonic convexity of $\breve{f}$ on $[w_1, w_2]$, we have

$$\begin{aligned}\breve{\Phi}(w) &\leq \frac{w_2(w_1-w)}{w(w_1-w_2)}\breve{\Phi}(w_1) + \frac{w_1(w-w_2)}{w(w_1-w_2)}\breve{\Phi}(w_2)\\ &= \frac{w_2(w_1-w)}{w(w_1-w_2)}\frac{\Phi(w_1)+\Phi(w_2)}{2} + \frac{w_1(w-w_2)}{w(w_1-w_2)}\frac{\Phi(w_1)+\Phi(w_2)}{2}\\ &= \frac{\Phi(w_1)+\Phi(w_2)}{2},\end{aligned}$$

which gives second inequality in (2.13). □

More precisely,

**Corollary 2.11.** *Assume that function $\Phi : [w_1, w_2] \subset \mathbb{R}/\{0\} \to \mathbb{C}$ is symmetrized harmonic convex on $[w_1, w_2]$, then for any $w \in [w_1, w_2]$ we have*

$$\inf_{w\in[w_1,w_2]} \breve{\Phi}(w) = \breve{\Phi}\left(\frac{2w_1w_2}{w_1+w_2}\right) = \Phi\left(\frac{2w_1w_2}{w_1+w_2}\right),$$

*and*

$$\sup_{w\in[w_1,w_2]} \breve{\Phi}(w) = \breve{\Phi}(w_1) = \breve{\Phi}(w_2) = \frac{\Phi(w_1)+\Phi(w_2)}{2}.$$

Next, we obtain some new discrete inequalities for univariate harmonic convex functions that can be seen as counterparts of Baloch's *et al.* result in Theorem 1.10.

**Theorem 2.12.** *Let $[m, M] \subseteq (0, \infty)$ be an interval. If $\Phi : [m, M] \to \mathbb{R}$ is harmonic convex function, then for any finite sequence $(w_k)_{k=1}^n \in [m, M]$ and $a_k \in [0, 1]$ with $\sum_{k=1}^n a_k = 1$, we have*

$$\begin{aligned}&\Phi\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^n\frac{a_k}{w_k}}\right) \qquad (2.14)\\ \geq\ &\Phi\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^n\frac{a_k}{w_k}}\right) - \min_{1\leq k\leq n}\{a_k\}\ \left[\sum_{k=1}^n\Phi(w_k) - n\Phi\left(\frac{1}{\frac{1}{n}\sum_{k=1}^n\frac{1}{w_k}}\right)\right]\\ \geq\ &\Phi\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^n\frac{a_k}{w_k}}\right) - \left[\sum_{k=1}^n a_k\Phi(w_k) - \Phi\left(\frac{1}{\sum_{k=1}^n\frac{a_k}{w_k}}\right)\right]\\ \geq\ &2\Phi\left(\frac{2mM}{m+M}\right) - \sum_{k=1}^n a_k\Phi(w_k).\end{aligned}$$

*Proof.* Since, $\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^{n}\frac{a_k}{w_k}}, \frac{2mM}{m+M} \in [m, M]$, then by the harmonic convexity of $\Phi$ on $[m, M]$, we have

$$\begin{aligned} & \frac{1}{2}\left[\Phi\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^{n}\frac{a_k}{\Phi_k}}\right)+\Phi\left(\frac{1}{\sum_{k=1}^{n}\frac{a_k}{w_k}}\right)\right] \qquad (2.15)\\ \geq\ & \Phi\left(\frac{1}{\frac{1}{2}\left[\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^{n}\frac{a_k}{w_k}+\sum_{k=1}^{n}\frac{a_k}{w_k}\right]}\right)\\ =\ & \Phi\Big(\frac{2mM}{m+M}\Big). \end{aligned}$$

Equivalently

$$\Phi\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^{n}\frac{a_k}{w_k}}\right)+\Phi\left(\frac{1}{\sum_{k=1}^{n}\frac{a_k}{w_k}}\right) \geq 2\Phi\Big(\frac{2mM}{m+M}\Big). \qquad (2.16)$$

By subtracting in both sides of (2.16)the same quantity $\sum_{k=1}^{n} a_k\Phi(w_k)$, we get

$$\begin{aligned} & \Phi\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^{n}\frac{a_k}{w_k}}\right)-\left[\sum_{k=1}^{n}a_k\Phi(w_k)-\Phi\left(\frac{1}{\sum_{k=1}^{n}\frac{a_k}{w_k}}\right)\right] \qquad (2.17)\\ \geq\ & 2\Phi\Big(\frac{2mM}{m+M}\Big)-\sum_{k=1}^{n}a_k\Phi(w_k). \end{aligned}$$

By using first inequality in (1.8), we get

$$\begin{aligned} & -\Big[\sum_{k=1}^{n}a_k\Phi(w_k)-\Phi\Big(\frac{1}{\sum_{k=1}^{n}\frac{a_k}{w_k}}\Big)\Big]\\ \leq & -\min_{1\leq k\leq n}\{a_k\}\ \Big[\sum_{k=1}^{n}\Phi(w_k)-n\Phi\Big(\frac{1}{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{w_k}}\Big)\Big], \end{aligned}$$

which implies

$$\begin{aligned} & \left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^{n}\frac{a_k}{w_k}}\right)-\Big[\sum_{k=1}^{n}a_k\Phi(w_k)-\Phi\Big(\frac{1}{\sum_{k=1}^{n}\frac{a_k}{w_k}}\Big)\Big] \qquad (2.18)\\ \leq\ & \left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^{n}\frac{a_k}{w_k}}\right)-\min_{1\leq k\leq n}\{a_k\}\ \Big[\sum_{k=1}^{n}\Phi(w_k)-n\Phi\Big(\frac{1}{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{w_k}}\Big)\Big]. \end{aligned}$$

By making the use of inequalities (2.17) and (2.18), we get second and third inequalities in (2.14). □

**Corollary 2.13.** *With the assumptions of Theorem 2.1, we have*

$$\begin{aligned} & \frac{1}{2}\left[\Phi\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^{n}\frac{a_k}{w_k}}\right)+\sum_{k=1}^{n}a_k\Phi(w_k)\right]-\Phi\Big(\frac{2mM}{m+M}\Big) \qquad (2.19)\\ \geq\ & \frac{1}{2}\Big[\sum_{k=1}^{n}a_k\Phi(w_k)-\Phi\Big(\frac{1}{\sum_{k=1}^{n}\frac{a_k}{w_k}}\Big)\Big]-\frac{1}{2}\min_{1\leq k\leq n}\{a_k\}\ \Big[\sum_{k=1}^{n}\Phi(w_k)-n\Phi\Big(\frac{1}{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{w_k}}\Big)\Big]\\ \geq\ & 0, \end{aligned}$$

*for all* $(w_k)_{k=1}^{n} \in [m, M]$ *and* $a_k \in [0, 1]$ *with* $\sum_{k=1}^{n} a_k = 1$.

## 3. Applications

Throughout this section, $l(\lambda)$ and $L(\lambda)$ are same as in Theorem 2.1.

Consider the harmonic convex function $\Phi : [w_1, w_2] \subset (0, \infty) \to \mathbb{R}$, $\Phi(w) = w$,, then by using (2.1) we get

$$\frac{2w_1w_2}{w_1+w_2} \leq l(\lambda) \leq \frac{w_1w_2}{w_2-w_1}(\ln w_2 - \ln w_1) \leq L(\lambda) \leq \frac{w_1+w_2}{2}, \tag{3.1}$$

the inequality (3.1) is a refinement of inequality presented in [31].
For harmonic convex function $\Phi : [w_1, w_2] \subset (0, \infty) \to \mathbb{R}$, $\Phi(w) = \ln w$, using (2.1) we get

$$\frac{2w_1w_2}{w_1+w_2} \leq \exp(l(\lambda)) \leq e\left(\frac{w_1^{w_2}}{w_2^{w_1}}\right)^{\frac{1}{w_2-w_1}} \leq \exp(L(\lambda)) \leq \sqrt{w_1w_2}, \tag{3.2}$$

the inequality (3.2) is a refinement of inequality presented in [31].
For harmonic convex function $\Phi : [w_1, w_2] \subset (0, \infty) \to \mathbb{R}$, $\Phi(w) = e^w$, using (2.1) we get

$$e^{\frac{2w_1w_2}{w_1+w_2}} \leq l(\lambda) \leq \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2} \frac{e^w}{w^2}dw \leq L(\lambda) \leq \frac{e^{w_1}+e^{w_2}}{2}, \tag{3.3}$$

the inequality (3.3) is a refinement of inequality presented in [31].
For harmonic convex function $\Phi : [w_1, w_2] \subset (0, \infty) \to \mathbb{R}$, $\Phi(w) = w^2e^{w^2}$, using (2.1) we get

$$\left(\frac{2w_1w_2}{w_1+w_2}\right)^2 e^{(\frac{2w_1w_2}{w_1+w_2})^2} \leq l(\lambda) \leq \frac{w_1w_2}{w_2-w_1}\int_{w_1}^{w_2} e^{w^2}dw \leq L(\lambda) \leq \frac{w_1^2e^{w_1^2}+w_2^2e^{w_2^2}}{2}, \tag{3.4}$$

the inequality (3.4) is a refinement of inequality presented in [31].
Now by considering the harmonic convex function $\Phi : [m, M] \subset (0, \infty) \to \mathbb{R}$, $\Phi(w) = \ln w$, $w_k \in [m, M]$, $a_k \geq 0$, for $k \in \{1, ..., n\}$ and such that $\sum_{k=1}^n a_k = 1$. Then by using (2.19) we get

$$\begin{aligned} &\frac{1}{2}\Big[\ln\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^n \frac{a_k}{w_k}}\right) + \sum_{k=1}^n a_k \ln(w_k)\Big] - \ln\left(\frac{2mM}{m+M}\right) \qquad (3.5)\\ \geq\ &\frac{1}{2}\Big[\sum_{k=1}^n a_k \ln(w_k) - \ln\left(\frac{1}{\sum_{k=1}^n \frac{a_k}{w_k}}\right)\Big] - \frac{1}{2}\min_{1\leq k\leq n}\{a_k\}\ \Big[\sum_{k=1}^n \ln(w_k) - n\ln\left(\frac{1}{\frac{1}{n}\sum_{k=1}^n \frac{1}{w_k}}\right)\Big]\\ \geq\ &0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\left(\frac{\prod_{k=1}^n w_k^{a_k}}{\frac{1}{m}+\frac{1}{M}-\sum_{k=1}^n \frac{a_k}{w_k}}\right)^{\frac{1}{2}}\left(\frac{m+M}{2mM}\right) \qquad (3.6)\\ \geq &\left(\frac{\left(\prod_{k=1}^n w_k^{a_k}\right)\left(\sum_{k=1}^n \frac{a_k}{w_k}\right)}{\left[\left(\prod_{k=1}^n w_k\right)\left(\frac{1}{n}\sum_{k+1}^n \frac{1}{w_k}\right)^n\right]^{\min_{1\leq k\leq n}\{a_k\}}}\right)^{\frac{1}{2}} \geq 1 \end{aligned}$$

Now, we consider the harmonic convex function $\Phi : [m, M] \subset (0, \infty) \to \mathbb{R}$, $\Phi(w) = w$, $w_k \in [m, M]$, $a_k = \frac{1}{n}$, for $k \in \{1, ..., n\}$. Then by using (2.19) and (1.7), we get

$$\frac{1}{2}\left[\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{w_k}}\right)+\frac{1}{n}\sum_{k=1}^{n}w_k\right] \geq \frac{2mM}{m+M}, \tag{3.7}$$

and

$$\frac{1}{2}\left[\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{w_k}}\right)+\frac{1}{n}\sum_{k=1}^{n}w_k\right] \leq \frac{m+M}{2}. \tag{3.8}$$

From (3.7) and (3.8), we get

$$\frac{2mM}{m+M} \leq \frac{1}{2}\left[\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{w_k}}\right)+\frac{1}{n}\sum_{k=1}^{n}w_k\right] \leq \frac{m+M}{2}, \tag{3.9}$$

and hence, using (3.9), we get another improvement of inequality (2.2) presented by Baloch *et al.* in [33] as follow:

$$\begin{aligned}
\left(\frac{2mM}{m+M}\right)^2 &\leq \frac{1}{4}\left[\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{w_k}}\right)+\frac{1}{n}\sum_{k=1}^{n}w_k\right]^2 \\
&\leq \left(\frac{m+M}{2}\right)^2 \\
&\leq \frac{1}{3}(m^2+mM+M^2) \\
&\leq \frac{m^2+M^2}{2}.
\end{aligned} \tag{3.10}$$

## 4. Conclusions

We presented refinements of Hermite-Hadamard type and Fejér types inequalities. Further, we generalized Hermite-Hadamard type and Fejér types inequalities for a class which is not harmonic convex and next we gave bounds for functions of this new class. Moreover, we discussed the importance of this class by giving lot of applications in the theory of inequalities. Our techniques and results are new in the field of mathematics for the class of harmonic convex functions and believe that it will be source of motivation for further research.

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## Conflict of interests

The authors declare no conflict of interest.

## Authors' contributions

Imran Abbas Baloch provided the main idea, carried out the proof of Theorems 2.1, 2.4, 2.8, 2.10, 2.12 and presented related Corollaries with applications. Thabet Abdeljawad, Aiman Mukheimer, Deeba Afzal and Aqeel Ahmad Mughal reviewed whole mathematics of article, completed the final revision of the article. All authors read and approved the final manuscript.

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