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Research article

Accelerated non-monotonic explicit proximal-type method for solving equilibrium programming with convex constraints and its applications

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Abstract: The main objective of this study is to introduce a new two-step proximal-type method to solve equilibrium problems in a real Hilbert space. This problem is a general mathematical model and includes a number of mathematical problems as a special case, such as optimization problems, variational inequalities, fixed point problems, saddle time problems and Nash equilibrium point problems. A new method is analogous to the famous two-step extragradient method that was used to solve variational inequality problems in a real Hilbert space established previously. The proposed iterative method uses an inertial scheme and a new non-monotone stepsize rule based on local bifunctional values rather than any line search method. A strong convergence theorem for the constructed method is proven by letting mild conditions on a bifunction. These results are being used to solve fixed point problems as well as variational inequalities. Finally, we considered two test problems, and the computational performance was presented to show the performance and efficiency of the proposed method.

Keywords: strong convergence; Lipschitz-type condition; equilibrium problem; variational inequalities; fixed point problems
Mathematics Subject Classification: 47H05, 47H10, 65K15, 65Y05, 68W10

1. Introduction

Assume that $\mathcal{D}$ is a nonempty, closed and convex subset of a real Hilbert space $\mathcal{E}$. Let $f : \mathcal{E}\times\mathcal{E} \to \mathbb{R}$ be a bifunction with $f(v, v) = 0$ for all $v \in \mathcal{D}$. An *equilibrium problem* (shortly, EP) for f on $\mathcal{D}$ is defined in the following manner: Find $\zeta^* \in \mathcal{D}$ such that

$$f(\zeta^*, v) \geq 0, \quad \forall\, v \in \mathcal{D}. \tag{EP}$$

Moreover, the solution set of an equilibrium is denoted by S_{EP}. In this study, the problem (EP) is studied based on the following conditions. A bifunction $f : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ is said to be (see for more details [3, 4]):
(**C1**) *pseudomonotone* on $\mathcal{D}$ if

$$f(v_1, v_2) \geq 0 \Longrightarrow f(v_2, v_1) \leq 0, \quad \forall\, v_1, v_2 \in \mathcal{D}; \tag{1.1}$$

(**C2**) *Lipschitz-type continuous* [15] on $\mathcal{D}$ if there exists two constants $k_1, k_2 > 0$ such that

$$f(v_1, v_3) \leq f(v_1, v_2) + f(v_2, v_3) + k_1\|v_1 - v_2\|^2 + k_2\|v_2 - v_3\|^2, \quad \forall\, v_1, v_2, v_3 \in \mathcal{D}; \tag{1.2}$$

(**C3**) For any weakly convergent sequence $\{v_n\} \subset \mathcal{D}$ $(v_n \rightharpoonup v^*)$ the following inequality holds

$$\limsup_{n\to\infty} f(v_n, v) \leq f(v^*, v), \quad \forall\, v \in \mathcal{D}; \tag{1.3}$$

(**C4**) $f(v, \cdot)$ is convex and subdifferentiable on $\mathcal{E}$ for each fixed $v \in \mathcal{E}$.

The general format of the problem (EP) has become attractive and has received a lot of attention from several authors in recent years. Mathematically, the problem (EP) can be considered as a generalization of many mathematical models, such as the fixed-point problems, scalar and vector minimization problems, the complementarity problems, the variational inequalities problems, the Nash equilibrium problems in non-cooperative games, the saddle point problems and the inverse minimization problems [4, 12, 17]. The equilibrium problem (EP) has applications in economics [8] or the dynamics of offer and demand [1], continuing to exploit the theoretical structure of non-cooperative games and the Nash equilibrium idea [18, 19]. To the best of our knowledge, the term “equilibrium problem” was first used in the literature in 1992 by Muu and Oettli [17] and was later studied further by Blum and Oettli [4].

By using the idea of the Korpelevich extragradient method [13], Flam et al. [10] and Quoc et al. [21] introduced the following method for solving equilibrium problems involving pseudomonotone and Lipschitz-type bifunction. Choose a random starting point of $u_0 \in \mathcal{D}$; looking the given iteration u_n and choose the next iteration using the iterative scheme:

$$\begin{cases} v_n = \arg\min\limits_{v\in\mathcal{D}}\{\rho f(u_n, v) + \frac{1}{2}\|u_n - v\|^2\}, \\ u_{n+1} = \arg\min\limits_{v\in\mathcal{D}}\{\rho f(v_n, v) + \frac{1}{2}\|u_n - v\|^2\}, \end{cases} \tag{1.4}$$

where $0 < \rho < \min\{\frac{1}{2k_1}, \frac{1}{2k_2}\}$ and k_1, k_2 are two Lipschitz-type constants of a bifunction (1.2). The method (1.4) has been extended and modified in various ways [16, 24–27, 29] and others in [2, 6, 22, 30, 32, 34–36].

It deserves mention that the above well-established method carries two significant drawbacks. The first is the constant stepsize that requires the knowledge or approximation of the Lipschitz constant of the relevant bifunction and it only converges weakly in Hilbert spaces. From the computational point of view, it might be hard to use a fixed stepsize, and hence, the convergence rate and usefulness of the method could be influenced. The inertial-type algorithms are of particular interest here. These algorithms are derived from an oscillator equation with damping and a conservative restoring force. This second-order dynamical system is known as Heavy Ball with Friction, and it was first studied by Polyak [20]. In general, the main feature of the inertial-type algorithms is that we can use the two previous iterations to construct the next one. Recently, inertial-type algorithms have been widely studied for the special cases of the problem (EP).

A natural question therefore arises:

Is it possible to introduce a new inertial strongly convergent extragradient method with a non-monotone stepsize rule to determine the numerical solution of the problem (EP) *involves a pseudomonotone bifunction?*

In this study, we provide a positive answer to this question, i.e., the gradient method still operate in the case of a non-monotonic stepsize rule for solving equilibrium problems accompanied by a pseudomonotone bifunction and obtain a strong convergence of the iterative sequence. We introduce a new extragradient-type method to solve the problem (EP) in the context of an infinite-dimensional real Hilbert space, inspired by the works of [7, 20, 21]. The key contributions to this research are given below:

(•) We introduce a new self-adaptive subgradient extragradient method by using an inertial scheme and a non-monotone stepsize rule to solve equilibrium problems. Also, we confirm that the generated sequence is strongly convergent. This result can be regarded as a modification of the method (1.4).

(•) The applications of our main results are considered in order to solve particular classes of equilibrium problems in a real Hilbert space.

(•) The numerical experiments regarding Algorithm 1 with Algorithm 3.1 in [11], Algorithm 1 in [28] and Algorithm 3 in [31]. The numerical results have indicated that the suggested method is appropriate and performed better compared to the existing ones.

The rest of the study has been arranged as follows: Section 2 includes basic definitions and key lemmas that are used throughout this manuscript. Section 3 consists of the proposed iterative scheme with a variable stepsize rule and a theorem of convergence analysis. Section 4 sets out the application of the proposed results to solve the problems of variational inequalities and fixed point problems. Section 5 gives numerical results to illustrate the performance of the new algorithms and equate them with the two existing algorithms.

2. Preliminaries

Let $\mathcal{D}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{E}$. The *metric projection* $P_{\mathcal{D}}(u)$ of $u \in \mathcal{E}$ onto a closed and convex subset $\mathcal{D}$ of $\mathcal{E}$ is defined by

$$P_{\mathcal{D}}(u) = \arg\min_{v \in \mathcal{D}} \|v - u\|.$$

Definition 2.1. Let $\mathcal{D}$ be a subset of a real Hilbert space $\mathcal{E}$ and $\varkappa : \mathcal{D} \to \mathbb{R}$ a given convex function.

(1). The *subdifferential of set* $\varkappa$ at $u \in \mathcal{D}$ is defined by

$$\partial\varkappa(u) = \{z \in \mathcal{E} : \varkappa(v) - \varkappa(u) \geq \langle z, v - u\rangle, \ \forall\, v \in \mathcal{D}\}.$$

(2). The *normal cone* at $u \in \mathcal{D}$ is defined by

$$N_{\mathcal{D}}(u) = \{z \in \mathcal{E} : \langle z, v - u\rangle \leq 0, \ \forall\, v \in \mathcal{D}\}.$$

Lemma 2.2. [23] *Assume that* $\varkappa : \mathcal{D} \to \mathbb{R}$ *is a convex, subdifferentiable and lower semicontinuous function on* $\mathcal{D}$*. An element* $u \in \mathcal{D}$ *is a minimizer of a function* $\varkappa$ *if and only if*

$$0 \in \partial\varkappa(u) + N_{\mathcal{D}}(u),$$

where $\partial\varkappa(u)$ *stands for the subdifferential of* $\varkappa$ *at* $u \in \mathcal{D}$ *and* $N_{\mathcal{D}}(u)$ *the normal cone of* $\mathcal{D}$ *at* u*.*

Lemma 2.3. [33] *Assume that* $\{a_n\} \subset (0, +\infty)$ *is a sequence satisfying the following inequality*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n, \ \ \forall\, n \in \mathbb{N}.$$

Moreover, $\{b_n\} \subset (0, 1)$ *and* $\{c_n\} \subset \mathbb{R}$ *are sequences such that*

$$\lim_{n\to+\infty} b_n = 0, \ \sum_{n=1}^{+\infty} b_n = +\infty \ \text{and} \ \limsup_{n\to+\infty} c_n \leq 0.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. [14] *Assume that* $\{a_n\} \subset \mathbb{R}$ *is a sequence and there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_{i+1}}$ *for all* $i \in \mathbb{N}$*. Then, there exists a nondecreasing sequence* $m_k \subset \mathbb{N}$ *such that* $m_k \to \infty$ *as* $k \to \infty$*, and the subsequent conditions are fulfilled by all (sufficiently large) numbers* $k \in \mathbb{N}$*:*

$$a_{m_k} \leq a_{m_{k+1}} \ \text{and} \ a_k \leq a_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.

3. Main algorithm and its convergence analysis

Now, we introduce a new variant of Algorithm (1.4) in which the constant stepsize ρ is chosen adaptively.

Algorithm 1 (Inertial Non-monotone Strongly Convergent Iterative Scheme)

Step 0: Choose $u_0, u_1 \in \mathcal{D}$, $\phi > 0$, $0 < \sigma < \min\left\{1, \frac{1}{2k_1}, \frac{1}{2k_2}\right\}$, $\mu \in (0,1)$, $\rho_1 > 0$. Moreover, select $\{\psi_n\} \subset (0,1)$ satisfies the conditions, i.e.,

$$\lim_{n\to+\infty} \psi_n = 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \psi_n = +\infty.$$

Step 1: Compute $\chi_n = u_n + \phi_n(u_n - u_{n-1}) - \psi_n[u_n + \phi_n(u_n - u_{n-1})]$ where

$$0 \le \phi_n \le \hat{\phi}_n \quad \text{and} \quad \hat{\phi}_n = \begin{cases} \min\left\{\frac{\phi}{2}, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \frac{\phi}{2} & \text{otherwise,} \end{cases} \tag{3.1}$$

where $\epsilon_n = \circ(\psi_n)$ is a positive sequence such that $\lim_{n\to+\infty} \frac{\epsilon_n}{\psi_n} = 0$.
Step 2: Compute $v_n = \arg\min_{v\in\mathcal{D}}\{\rho_n f(\chi_n, v) + \frac{1}{2}\|\chi_n - v\|^2\}$.
If $\chi_n = v_n$, then STOP. Otherwise, go to Step 3.
Step 3: Firstly choose $\omega_n \in \partial_2 f(\chi_n, v_n)$ satisfying $\chi_n - \rho_n\omega_n - v_n \in N_{\mathcal{D}}(v_n)$ and generate a half-space

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle \chi_n - \rho_n\omega_n - v_n, z - v_n\rangle \le 0\}$$

and compute

$$u_{n+1} = \arg\min_{v\in\mathcal{E}_n}\{\rho_n f(v_n, v) + \frac{1}{2}\|\chi_n - v\|^2\}.$$

Step 4: Next, the stepsize rule ρ_{n+1} is updated as follows:

$$\rho_{n+1} = \begin{cases} \min\left\{\sigma, \frac{\mu f(v_n, u_{n+1})}{f(\chi_n, u_{n+1}) - f(\chi_n, v_n) - k_1\|\chi_n - v_n\|^2 - k_2\|u_{n+1} - v_n\|^2 + 1}\right\}, \\ \quad \text{if} \quad \frac{\mu f(v_n, u_{n+1})}{f(\chi_n, u_{n+1}) - f(\chi_n, v_n) - k_1\|\chi_n - v_n\|^2 - k_2\|u_{n+1} - v_n\|^2 + 1} > 0, \\ \sigma \qquad\qquad \text{otherwise.} \end{cases} \tag{3.2}$$

Set $n = n + 1$ and go back to Step 1.

Remark 3.1. By the use of ρ_{n+1} in expression (3.2), we obtain

$$\rho_{n+1}[f(\chi_n, u_{n+1}) - f(\chi_n, v_n) - k_1\|\chi_n - v_n\|^2 - k_2\|v_n - u_{n+1}\|^2] \le \mu f(v_n, u_{n+1}). \tag{3.3}$$

Lemma 3.1. *Suppose that the conditions* (C1)–(C4) are satisfied and $\{u_n\}$ be a sequence generated by Algorithm 1. Then, we have

$$\begin{aligned} \|u_{n+1} - \zeta^*\|^2 \le \|\chi_n - \zeta^*\|^2 - (1 - \rho_{n+1})\|u_{n+1} - \chi_n\|^2 \\ - \rho_{n+1}(1 - 2k_1\rho_n)\|\chi_n - v_n\|^2 - \rho_{n+1}(1 - 2k_2\rho_n)\|u_{n+1} - v_n\|^2. \end{aligned} \tag{3.4}$$

Proof. By the use of definition of u_{n+1}, we obtain

$$0 \in \partial_2\Big\{\rho_n f(v_n, \cdot) + \frac{1}{2}\|\chi_n - \cdot\|^2\Big\}(u_{n+1}) + N_{\mathcal{D}}(u_{n+1}).$$

Thus, there exists $\omega_n \in \partial_2 f(v_n, u_{n+1})$ and $\overline{\omega}_n \in N_{\mathcal{D}}(u_{n+1})$ such that

$$\rho_n\omega_n + u_{n+1} - \chi_n + \overline{\omega}_n = 0.$$

The above expression implies that

$$\langle \chi_n - u_{n+1}, v - u_{n+1}\rangle = \rho_n\langle \omega_n, v - u_{n+1}\rangle + \langle \overline{\omega}_n, v - u_{n+1}\rangle, \ \forall\, v \in \mathcal{D}.$$

Due to $\overline{\omega}_n \in N_{\mathcal{D}}(u_{n+1})$ imply that $\langle \overline{\omega}_n, v - u_{n+1}\rangle \leq 0$, for every $v \in \mathcal{D}$. Thus, we obtain

$$\rho_n\langle \omega_n, v - u_{n+1}\rangle \geq \langle \chi_n - u_{n+1}, v - u_{n+1}\rangle, \ \forall\, v \in \mathcal{D}. \tag{3.5}$$

By given $\omega_n \in \partial_2 f(v_n, u_{n+1})$, we have

$$f(v_n, v) - f(v_n, u_{n+1}) \geq \langle \omega_n, v - u_{n+1}\rangle, \ \forall\, v \in \mathcal{D}. \tag{3.6}$$

From expressions (3.5) and (3.6), we obtain

$$\rho_n f(v_n, v) - \rho_n f(v_n, u_{n+1}) \geq \langle \chi_n - u_{n+1}, v - u_{n+1}\rangle, \ \forall\, v \in \mathcal{D}. \tag{3.7}$$

In the similar way, v_n gives that

$$\rho_n\{f(\chi_n, v) - f(\chi_n, v_n)\} \geq \langle \chi_n - v_n, v - v_n\rangle, \ \forall\, v \in \mathcal{D}. \tag{3.8}$$

By the use of $v = u_{n+1}$ into expression (3.8), we get

$$\rho_n\{f(\chi_n, u_{n+1}) - f(\chi_n, v_n)\} \geq \langle \chi_n - v_n, u_{n+1} - v_n\rangle. \tag{3.9}$$

By the use of $v = \zeta^*$ into expression (3.7), we obtain

$$\rho_n f(v_n, \zeta^*) - \rho_n f(v_n, u_{n+1}) \geq \langle \chi_n - u_{n+1}, \zeta^* - u_{n+1}\rangle. \tag{3.10}$$

Since $\zeta^* \in S_{EP}$ implies that $f(\zeta^*, v_n) \geq 0$ and pseudomonotonicity of a bifunction f gives that $f(v_n, \zeta^*) \leq 0$. Thus, expression (3.10) implies that

$$\langle \chi_n - u_{n+1}, u_{n+1} - \zeta^*\rangle \geq \rho_n f(v_n, u_{n+1}). \tag{3.11}$$

From expression (3.2), we have

$$f(v_n, u_{n+1}) \geq \rho_{n+1}\Big[f(\chi_n, u_{n+1}) - f(\chi_n, v_n) - k_1\|\chi_n - v_n\|^2 - k_2\|v_n - u_{n+1}\|^2\Big]. \tag{3.12}$$

Combining expressions (3.11) and (3.12) gives that

$$\begin{aligned}\langle \chi_n - u_{n+1}, u_{n+1} - \zeta^*\rangle \geq \rho_{n+1}\Big[&\rho_n\{f(\chi_n, u_{n+1}) - f(\chi_n, v_n)\}\\ &- k_1\rho_n\|\chi_n - v_n\|^2 - k_2\rho_n\|u_{n+1} - v_n\|^2\Big].\end{aligned} \tag{3.13}$$

From expressions (3.9) and (3.13), we obtain

$$\begin{aligned}2\langle \chi_n - u_{n+1}, u_{n+1} - \zeta^*\rangle \geq \rho_{n+1}\Big[&2\langle \chi_n - v_n, u_{n+1} - v_n\rangle\\ &- 2k_1\rho_n\|\chi_n - v_n\|^2 - 2k_2\rho_n\|u_{n+1} - v_n\|^2\Big].\end{aligned} \tag{3.14}$$

By the use of following formulas:

$$2\langle \chi_n - u_{n+1}, u_{n+1} - \zeta^* \rangle = \|\chi_n - \zeta^*\|^2 - \|u_{n+1} - \chi_n\|^2 - \|u_{n+1} - \zeta^*\|^2.$$

$$2\langle \chi_n - v_n, u_{n+1} - v_n \rangle = \|\chi_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|\chi_n - u_{n+1}\|^2.$$

Finally, we have

$$\begin{aligned} \|u_{n+1} - \zeta^*\|^2 \leq \|\chi_n - \zeta^*\|^2 &- (1-\rho_{n+1})\|u_{n+1} - \chi_n\|^2 \\ &- \rho_{n+1}(1 - 2k_1\rho_n)\|\chi_n - v_n\|^2 - \rho_{n+1}(1 - 2k_2\rho_n)\|u_{n+1} - v_n\|^2. \end{aligned} \tag{3.15}$$

□

Theorem 3.2. *Assume that conditions* (C1)–(C4) are satisfied. Then, the sequence $\{u_n\}$ generated by Algorithm 1 converges strongly to an element $\zeta^* = P_{S_{EP}(0)}$.

Proof. Thus, expression (3.1) implies that

$$\lim_{n\to+\infty} \frac{\phi_n}{\psi_n}\left\|u_n - u_{n-1}\right\| \leq \lim_{n\to+\infty} \frac{\epsilon_n}{\psi_n} = 0. \tag{3.16}$$

By the use of definition of $\{\chi_n\}$ and inequality (3.16), we obtain

$$\begin{aligned} \left\|\chi_n - \zeta^*\right\| &= \left\|u_n + \phi_n(u_n - u_{n-1}) - \psi_n u_n - \phi_n\psi_n(u_n - u_{n-1}) - \zeta^*\right\| \\ &= \left\|(1-\psi_n)(u_n - \zeta^*) + (1-\psi_n)\phi_n(u_n - u_{n-1}) - \psi_n\zeta^*\right\| \qquad (3.17) \\ &\leq (1-\psi_n)\left\|u_n - \zeta^*\right\| + (1-\psi_n)\phi_n\left\|u_n - u_{n-1}\right\| + \psi_n\left\|\zeta^*\right\| \\ &\leq (1-\psi_n)\|u_n - \zeta^*\| + \psi_n K_1, \qquad (3.18) \end{aligned}$$

where

$$(1-\psi_n)\frac{\phi_n}{\psi_n}\left\|u_n - u_{n-1}\right\| + \left\|\zeta^*\right\| \leq K_1.$$

By the use of Lemma 3.1, we obtain

$$\|u_{n+1} - \zeta^*\|^2 \leq \|\chi_n - \zeta^*\|^2, \ \forall\, n > 1. \tag{3.19}$$

Combining (3.18) with (3.19), we obtain

$$\begin{aligned} \|u_{n+1} - \zeta^*\| &\leq (1-\psi_n)\|u_n - \zeta^*\| + \psi_n K_1 \\ &\leq \max\{\|u_n - \zeta^*\|, K_1\} \\ &\vdots \\ &\leq \max\{\|u_2 - \zeta^*\|, K_1\}. \end{aligned} \tag{3.20}$$

Thus, we infer that the sequence $\{u_n\}$ is bounded. Indeed, by (3.18) we have

$$\begin{aligned} \left\|\chi_n - \zeta^*\right\|^2 &\leq (1-\psi_n)^2\|u_n - \zeta^*\|^2 + \psi_n^2 K_1^2 + 2K_1\psi_n(1-\psi_n)\|u_n - \zeta^*\| \\ &\leq \|u_n - \zeta^*\|^2 + \psi_n[\psi_n K_1^2 + 2K_1(1-\psi_n)\|u_n - \zeta^*\|] \\ &\leq \|u_n - \zeta^*\|^2 + \psi_n K_2, \end{aligned} \tag{3.21}$$

for some $K_2 > 0$. Combining the expressions (3.4) with (3.21), we have

$$\|u_{n+1} - \zeta^*\|^2 \le \|u_n - \zeta^*\|^2 + \psi_n K_2 - (1 - \rho_{n+1})\|u_{n+1} - \chi_n\|^2 - \rho_{n+1}(1 - 2k_1\rho_n)\|\chi_n - v_n\|^2 - \rho_{n+1}(1 - 2k_2\rho_n)\|u_{n+1} - v_n\|^2. \tag{3.22}$$

Due to the Lipschitz-continuity and pseudomonotonicity of f implies that the solution set S_{EP} is a closed and convex set (for further details see [21]). It is given that $\zeta^* = P_{S_{EP}(0)}$ such that

$$\langle 0 - \zeta^*, v - \zeta^* \rangle \le 0, \ \forall\, v \in S_{EP}. \tag{3.23}$$

The remainder of the proof is divided into the following two cases:
Case 1: Assume that there exists a fixed number $N_1 \in \mathbb{N}$ such that

$$\|u_{n+1} - \zeta^*\| \le \|u_n - \zeta^*\|, \ \forall\, n \ge N_1. \tag{3.24}$$

Thus, above expression implies that $\lim_{n\to+\infty} \|u_n - \zeta^*\|$ exists and let $\lim_{n\to+\infty} \|u_n - \zeta^*\| = l$, for some $l \ge 0$. From the expression (3.22), we have

$$(1 - \rho_{n+1})\|u_{n+1} - \chi_n\|^2 + \rho_{n+1}(1 - 2k_1\rho_n)\|\chi_n - v_n\|^2 + \rho_{n+1}(1 - 2k_2\rho_n)\|u_{n+1} - v_n\|^2 \le \|u_n - \zeta^*\|^2 + \psi_n K_2 - \|u_{n+1} - \zeta^*\|^2. \tag{3.25}$$

Due to existence of limit of the sequence $\|u_n - \zeta^*\|$ and $\psi_n \to 0$, we conclude that

$$\|\chi_n - v_n\| \to 0 \quad \text{and} \quad \|u_{n+1} - v_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.26}$$

It continues from (3.25) that

$$\lim_{n\to+\infty} \|u_{n+1} - \chi_n\| = 0. \tag{3.27}$$

Next, we have to compute

$$\begin{aligned}\|\chi_n - u_n\| &= \|u_n + \phi_n(u_n - u_{n-1}) - \psi_n[u_n + \phi_n(u_n - u_{n-1})] - u_n\| \\ &\le \phi_n\|u_n - u_{n-1}\| + \psi_n\|u_n\| + \phi_n\psi_n\|u_n - u_{n-1}\| \\ &= \psi_n\frac{\phi_n}{\psi_n}\|u_n - u_{n-1}\| + \psi_n\|u_n\| + \psi_n^2\frac{\phi_n}{\psi_n}\|u_n - u_{n-1}\| \longrightarrow 0.\end{aligned} \tag{3.28}$$

The above expression implies that

$$\lim_{n\to+\infty} \|u_n - u_{n+1}\| \le \lim_{n\to+\infty} \|u_n - \chi_n\| + \lim_{n\to+\infty} \|\chi_n - u_{n+1}\| = 0. \tag{3.29}$$

he above explanation guarantees that the sequences $\{\chi_n\}$ and $\{v_n\}$ are also bounded. Due to the reflexivity of $\mathcal{E}$ and the boundedness of $\{u_n\}$ guarantees that there exists a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\} \rightharpoonup \hat{u} \in \mathcal{E}$ as $k \to +\infty$. Next, we have to prove that $\hat{u} \in S_{EP}$. Due to the inequality (3.7) we have

$$\begin{aligned}\rho_{n_k} f(v_{n_k}, v) &\ge \rho_{n_k} f(v_{n_k}, u_{n_k+1}) + \langle \chi_{n_k} - u_{n_k+1}, v - u_{n_k+1} \rangle \\ &\ge \rho_{n_k}\rho_{n_k+1} f(\chi_{n_k}, u_{n_{k+1}}) - \rho_{n_k}\rho_{n_k+1} f(\chi_{n_k}, v_{n_k}) - k_1\rho_{n_k}\rho_{n_k+1}\|\chi_{n_k} - v_{n_k}\|^2 \\ &\quad - k_2\rho_{n_k}\rho_{n_k+1}\|v_{n_k} - u_{n_k+1}\|^2 + \langle \chi_{n_k} - u_{n_k+1}, v - u_{n_k+1} \rangle\end{aligned}$$

$$\geq \rho_{n_k+1}\langle \chi_{n_k} - v_{n_k}, u_{n_k+1} - v_{n_k}\rangle - k_1\rho_{n_k}\rho_{n_k+1}\|\chi_{n_k} - v_{n_k}\|^2 - k_2\rho_{n_k}\rho_{n_k+1}\|v_{n_k} - u_{n_k+1}\|^2 + \langle \chi_{n_k} - u_{n_k+1}, v - u_{n_k+1}\rangle, \tag{3.30}$$

where v is an arbitrary element in $\mathcal{E}_n$. It continues from that (3.26)–(3.29) and the boundedness of $\{u_n\}$ that the right-hand side goes to zero. From $\rho_n > 0$, the condition (1.3) and $v_{n_k} \rightharpoonup \hat{u}$, we have

$$0 \leq \limsup_{k\to+\infty} f(v_{n_k}, v) \leq f(\hat{u}, v), \ \forall\, v \in \mathcal{E}_n. \tag{3.31}$$

It implies that $f(\hat{u}, v) \geq 0$, $\forall\, v \in \mathcal{D}$, and hence $\hat{u} \in S_{EP}$. Next, we have

$$\begin{aligned}&\limsup_{n\to+\infty}\langle \zeta^*, \zeta^* - u_n\rangle \\ &= \lim_{k\to+\infty}\langle \zeta^*, \zeta^* - u_{n_k}\rangle = \langle \zeta^*, \zeta^* - \hat{u}\rangle \leq 0.\end{aligned} \tag{3.32}$$

By the use of $\lim_{n\to+\infty}\|u_{n+1} - u_n\| = 0$. Thus, expression (3.32) implies that

$$\begin{aligned}&\limsup_{n\to+\infty}\langle \zeta^*, \zeta^* - u_{n+1}\rangle \\ &\leq \limsup_{n\to+\infty}\langle \zeta^*, \zeta^* - u_n\rangle + \limsup_{n\to+\infty}\langle \zeta^*, u_n - u_{n+1}\rangle \leq 0.\end{aligned} \tag{3.33}$$

By the use of expression (3.17), we have

$$\begin{aligned}&\|\chi_n - \zeta^*\|^2 \\ &= \|u_n + \phi_n(u_n - u_{n-1}) - \psi_n u_n - \phi_n\psi_n(u_n - u_{n-1}) - \zeta^*\|^2 \\ &= \|(1-\psi_n)(u_n - \zeta^*) + (1-\psi_n)\phi_n(u_n - u_{n-1}) - \psi_n\zeta^*\|^2 \\ &\leq \|(1-\psi_n)(u_n - \zeta^*) + (1-\psi_n)\phi_n(u_n - u_{n-1})\|^2 + 2\psi_n\langle -\zeta^*, \chi_n - \zeta^*\rangle \\ &= (1-\psi_n)^2\|u_n - \zeta^*\|^2 + (1-\psi_n)^2\phi_n^2\|u_n - u_{n-1}\|^2 \\ &\quad + 2\phi_n(1-\psi_n)^2\|u_n - \zeta^*\|\|u_n - u_{n-1}\| + 2\psi_n\langle -\zeta^*, \chi_n - u_{n+1}\rangle + 2\psi_n\langle -\zeta^*, u_{n+1} - \zeta^*\rangle \\ &\leq (1-\psi_n)\|u_n - \zeta^*\|^2 + \phi_n^2\|u_n - u_{n-1}\|^2 + 2\phi_n(1-\psi_n)\|u_n - \zeta^*\|\|u_n - u_{n-1}\| \\ &\quad + 2\psi_n\|\zeta^*\|\|\chi_n - u_{n+1}\| + 2\psi_n\langle -\zeta^*, u_{n+1} - \zeta^*\rangle \\ &= (1-\psi_n)\|u_n - \zeta^*\|^2 + \psi_n\Big[\phi_n\|u_n - u_{n-1}\|\frac{\phi_n}{\psi_n}\|u_n - u_{n-1}\| \\ &\quad + 2(1-\psi_n)\|u_n - \zeta^*\|\frac{\phi_n}{\psi_n}\|u_n - u_{n-1}\| + 2\|\zeta^*\|\|\chi_n - u_{n+1}\| + 2\langle \zeta^*, \zeta^* - u_{n+1}\rangle\Big].\end{aligned} \tag{3.34}$$

From expressions (3.19) and (3.34) we obtain

$$\begin{aligned}&\|u_{n+1} - \zeta^*\|^2 \\ &\leq (1-\psi_n)\|u_n - \zeta^*\|^2 + \psi_n\Big[\phi_n\|u_n - u_{n-1}\|\frac{\phi_n}{\psi_n}\|u_n - u_{n-1}\| \\ &\quad + 2(1-\psi_n)\|u_n - \zeta^*\|\frac{\phi_n}{\psi_n}\|u_n - u_{n-1}\| + 2\|\zeta^*\|\|\chi_n - u_{n+1}\| + 2\langle \zeta^*, \zeta^* - u_{n+1}\rangle\Big].\end{aligned} \tag{3.35}$$

By the use of (3.27), (3.33), (3.35) and applying Lemma 2.3, conclude that $\lim_{n\to+\infty}\left\|u_n-\zeta^*\right\|=0$.
Case 2: Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i}-\zeta^*\|\leq\|u_{n_{i+1}}-\zeta^*\|,\ \forall\, i\in\mathbb{N}.$$

By using Lemma 2.4 there exists a sequence $\{m_k\}\subset\mathbb{N}$ as $\{m_k\}\to+\infty$ such that

$$\|u_{m_k}-\zeta^*\|\leq\|u_{m_{k+1}}-\zeta^*\| \quad\text{and}\quad \|u_k-\zeta^*\|\leq\|u_{m_{k+1}}-\zeta^*\|,\ \text{for all } k\in\mathbb{N}. \tag{3.36}$$

As similar to Case 1, the expression (3.25) implies that

$$\begin{aligned}&(1-\rho_{m_k+1})\|u_{m_k+1}-\chi_{m_k}\|^2+\rho_{m_k+1}(1-2k_1\rho_{m_k})\|\chi_{m_k}-v_{m_k}\|^2\\&+\rho_{m_k+1}(1-2k_2\rho_{m_k})\|u_{m_k+1}-v_{m_k}\|^2\\&\leq\|u_{m_k}-\zeta^*\|^2+\psi_{m_k}K_2-\|u_{m_k+1}-\zeta^*\|^2.\end{aligned} \tag{3.37}$$

Due to $\psi_{m_k}\to 0$, we deduce the following:

$$\lim_{k\to+\infty}\|\chi_{m_k}-v_{m_k}\|=\lim_{k\to+\infty}\|u_{m_k+1}-v_{m_k}\|=0. \tag{3.38}$$

It follows that

$$\lim_{k\to+\infty}\|u_{m_{k+1}}-\chi_{m_k}\|=0. \tag{3.39}$$

Next, we have to evaluate

$$\begin{aligned}\|\chi_{m_k}-u_{m_k}\|&=\|u_{m_k}+\phi_{m_k}(u_{m_k}-u_{m_k-1})-\psi_{m_k}[u_{m_k}+\phi_{m_k}(u_{m_k}-u_{m_k-1})]-u_{m_k}\|\\&\leq\phi_{m_k}\|u_{m_k}-u_{m_k-1}\|+\psi_{m_k}\|u_{m_k}\|+\phi_{m_k}\psi_{m_k}\|u_{m_k}-u_{m_k-1}\|\\&=\psi_{m_k}\frac{\phi_{m_k}}{\psi_{m_k}}\|u_{m_k}-u_{m_k-1}\|+\psi_{m_k}\|u_{m_k}\|+\psi_{m_k}^2\frac{\phi_{m_k}}{\psi_{m_k}}\|u_{m_k}-u_{m_k-1}\|\longrightarrow 0.\end{aligned} \tag{3.40}$$

It follows that

$$\lim_{k\to+\infty}\|u_{m_k}-u_{m_k+1}\|\leq\lim_{k\to+\infty}\|u_{m_k}-\chi_{m_k}\|+\lim_{k\to+\infty}\|\chi_{m_k}-u_{m_k+1}\|=0. \tag{3.41}$$

By using the same explanation as in the Case 1, such that

$$\limsup_{k\to+\infty}\langle\zeta^*,\zeta^*-u_{m_k+1}\rangle\leq 0. \tag{3.42}$$

By using the expressions (3.35) and (3.36), we obtain

$$\begin{aligned}&\left\|u_{m_k+1}-\zeta^*\right\|^2\\&\leq(1-\psi_{m_k})\left\|u_{m_k}-\zeta^*\right\|^2+\psi_{m_k}\Big[\phi_{m_k}\left\|u_{m_k}-u_{m_k-1}\right\|\frac{\phi_{m_k}}{\psi_{m_k}}\left\|u_{m_k}-u_{m_k-1}\right\|\\&\quad+2(1-\psi_{m_k})\left\|u_{m_k}-\zeta^*\right\|\frac{\phi_{m_k}}{\psi_{m_k}}\left\|u_{m_k}-u_{m_k-1}\right\|+2\left\|\zeta^*\right\|\left\|\chi_{m_k}-u_{m_k+1}\right\|+2\langle\zeta^*,\zeta^*-u_{m_k+1}\rangle\Big]\\&\leq(1-\psi_{m_k})\left\|u_{m_{k+1}}-\zeta^*\right\|^2+\psi_{m_k}\Big[\phi_{m_k}\left\|u_{m_k}-u_{m_k-1}\right\|\frac{\phi_{m_k}}{\psi_{m_k}}\left\|u_{m_k}-u_{m_k-1}\right\|\end{aligned}$$

$$+ 2(1-\psi_{m_k})\left\|u_{m_k}-\zeta^*\right\|\frac{\phi_{m_k}}{\psi_{m_k}}\left\|u_{m_k}-u_{m_k-1}\right\| + 2\left\|\zeta^*\right\|\left\|\chi_{m_k}-u_{m_k+1}\right\| + 2\langle\zeta^*, \zeta^*-u_{m_k+1}\rangle\Big]. \tag{3.43}$$

Thus, above expression implies that

$$\begin{aligned}&\left\|u_{m_k+1}-\zeta^*\right\|^2\\ &\leq \Big[\phi_{m_k}\left\|u_{m_k}-u_{m_k-1}\right\|\frac{\phi_{m_k}}{\psi_{m_k}}\left\|u_{m_k}-u_{m_k-1}\right\|\\ &\quad + 2(1-\psi_{m_k})\left\|u_{m_k}-\zeta^*\right\|\frac{\phi_{m_k}}{\psi_{m_k}}\left\|u_{m_k}-u_{m_k-1}\right\| + 2\left\|\zeta^*\right\|\left\|\chi_{m_k}-u_{m_k+1}\right\| + 2\langle\zeta^*, \zeta^*-u_{m_k+1}\rangle\Big]. \end{aligned} \tag{3.44}$$

Since $\psi_{m_k} \to 0$, and $\left\|u_{m_k}-\zeta^*\right\|$ is a bounded sequence. Therefore, expressions (3.42) and (3.44) implies that

$$\|u_{m_k+1}-\zeta^*\|^2 \to 0, \text{ as } k \to +\infty. \tag{3.45}$$

It implies that

$$\lim_{n\to+\infty} \|u_k-\zeta^*\|^2 \leq \lim_{n\to+\infty} \|u_{m_k+1}-\zeta^*\|^2 \leq 0. \tag{3.46}$$

As a consequence $u_n \to \zeta^*$. This completes the proof of the theorem. □

4. Applications

In this section, we have written about the new results from our main proposed methods to solve variational inequalities. In the last few years, variational inequalities have drawn a considerable amount of attention from both researchers and readers. It is well established that variational inequalities deal with a large variety of topics in partial differential equations, optimal control, optimization techniques, applied mathematics, engineering, finance, and operational science. The variational inequality problem for an operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is defined as follows:

$$\text{Find } \zeta^* \in \mathcal{D} \text{ such that } \langle\mathcal{A}(\zeta^*), v-\zeta^*\rangle \geq 0, \ \forall\, v \in \mathcal{D}. \tag{VIP}$$

We consider the following conditions to study the variational inequalities.

($\mathcal{A}$1) The solution set of the problem (VIP) is denoted by $VI(\mathcal{A}, \mathcal{D})$ and it is nonempty;

($\mathcal{A}$2) An operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is said to be a pseudomonotone if

$$\langle\mathcal{A}(u), v-u\rangle \geq 0 \Longrightarrow \langle\mathcal{A}(v), u-v\rangle \leq 0, \ \forall\, u, v \in \mathcal{D};$$

($\mathcal{A}$3) An operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is said to be a *Lipschitz continuous* if there exists a constants $L > 0$ such that

$$\|\mathcal{A}(u) - \mathcal{A}(v)\| \leq L\|u-v\|, \ \forall\, u, v \in \mathcal{D};$$

($\mathcal{A}$4) An operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is said to be *sequentially weakly continuous*, i.e., $\{\mathcal{A}(u_n)\}$ weakly converges to $\mathcal{A}(u)$ for every sequence $\{u_n\}$ converges weakly to u.

On the other hand, we have also developed some results to solve fixed point problems. The existence of a solution to a theoretical or real-world problem should be analogous to the existence of a fixed point for an appropriate map or operator. Fixed-point theorems are thus extremely important in many fields of mathematics, engineering, and science. In many cases, it is not difficult to find an exact solution. Therefore, it is crucial to create effective techniques to approximate the desired result. The fixed point problem for an operator $\mathcal{B} : \mathcal{E} \to \mathcal{E}$ is defined as follows:

$$\text{Find } \zeta^* \in \mathcal{D} \text{ such that } \mathcal{B}(\zeta^*) = \zeta^*. \tag{FPP}$$

The following conditions are required to study fixed point theorems.

($\mathcal{B}$1) The solution set of the problem (FPP) is denoted by $Fix(\mathcal{B}, \mathcal{D})$ is nonempty;
($\mathcal{B}$2) $\mathcal{B} : \mathcal{D} \to \mathcal{D}$ is said to be a *κ-strict pseudocontraction* [5] on $\mathcal{D}$ if

$$\|\mathcal{B}u - \mathcal{B}v\|^2 \leq \|u - v\|^2 + \kappa\|(u - \mathcal{B}u) - (v - \mathcal{B}v)\|^2, \ \forall\, u, v \in \mathcal{D};$$

($\mathcal{B}$3) $\mathcal{B} : \mathcal{E} \to \mathcal{E}$ is said to be *weakly sequentially continuous*, i.e., $\{\mathcal{B}(u_n)\}$ weakly converges to $\mathcal{B}(u)$ for every sequence $\{u_n\}$ converges weakly to u.

Corollary 4.1. *Assume that an operator $\mathcal{A} : \mathcal{D} \to \mathcal{E}$ satisfies the conditions ($\mathcal{A}$1)–($\mathcal{A}$4) and the solution set $VI(\mathcal{A}, \mathcal{D}) \neq \emptyset$. Choose $u_0, u_1 \in \mathcal{D}$, $\phi > 0$, $0 < \sigma < \min\left\{1, \frac{1}{L}\right\}$, $\mu \in (0, 1)$, $\rho_1 > 0$. Moreover, select $\{\psi_n\} \subset (0, 1)$ meet the conditions, i.e.,*

$$\lim_{n\to+\infty} \psi_n = 0 \text{ and } \sum_{n=1}^{+\infty} \psi_n = +\infty.$$

(i) *Compute*

$$\chi_n = u_n + \phi_n(u_n - u_{n-1}) - \psi_n[u_n + \phi_n(u_n - u_{n-1})],$$

where ϕ_n modified on each iteration as follows:

$$0 \leq \phi_n \leq \hat{\phi}_n \quad \text{and} \quad \hat{\phi}_n = \begin{cases} \min\left\{\frac{\phi}{2}, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \frac{\phi}{2} & \text{otherwise}, \end{cases}$$

where $\epsilon_n = \circ(\psi_n)$ is a positive sequence such that $\lim_{n\to+\infty} \frac{\epsilon_n}{\psi_n} = 0$.
(ii) *Compute*

$$\begin{cases} v_n = P_{\mathcal{D}}(\chi_n - \rho_n\mathcal{A}(\chi_n)), \\ u_{n+1} = P_{\mathcal{E}_n}(\chi_n - \rho_n\mathcal{A}(v_n)), \end{cases}$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle \chi_n - \rho_n\mathcal{A}(\chi_n) - v_n, z - v_n\rangle \leq 0\}.$$

(iii) *Compute*

$$\rho_{n+1} = \begin{cases} \min\left\{\sigma, \frac{\mu\langle \mathcal{A}v_n, u_{n+1} - v_n\rangle}{\langle \mathcal{A}\chi_n, u_{n+1} - v_n\rangle - \frac{L}{2}\|\chi_n - v_n\|^2 - \frac{L}{2}\|u_{n+1} - v_n\|^2 + 1}\right\}, \\ \quad \text{if} \quad \frac{\mu\langle \mathcal{A}v_n, u_{n+1} - v_n\rangle}{\langle \mathcal{A}\chi_n, u_{n+1} - v_n\rangle - \frac{L}{2}\|\chi_n - v_n\|^2 - \frac{L}{2}\|u_{n+1} - v_n\|^2 + 1} > 0, \\ \sigma \quad \text{otherwise}. \end{cases}$$

Then, the sequences $\{u_n\}$ converge strongly to $\zeta^* \in VI(\mathcal{A}, \mathcal{D})$.

Corollary 4.2. *Assume that $\mathcal{B} : \mathcal{D} \to \mathcal{D}$ is a κ-strict pseudocontraction and weakly continuous with solution set $Fix(\mathcal{B}, \mathcal{D}) \neq \emptyset$. Choose $u_0, u_1 \in \mathcal{D}$, $\phi > 0$, $0 < \sigma < \min\left\{1, \frac{1-\kappa}{3-2\kappa}\right\}$, $\mu \in (0,1)$, $\rho_1 > 0$. Moreover, select $\{\psi_n\} \subset (0,1)$ meet the conditions, i.e.,*

$$\lim_{n\to+\infty} \psi_n = 0 \text{ and } \sum_{n=1}^{+\infty} \psi_n = +\infty.$$

(i) *Compute*

$$\chi_n = u_n + \phi_n(u_n - u_{n-1}) - \psi_n[u_n + \phi_n(u_n - u_{n-1})],$$

where ϕ_n modified one each iteration as follows:

$$0 \le \phi_n \le \hat{\phi}_n \quad \text{and} \quad \hat{\phi}_n = \begin{cases} \min\left\{\frac{\phi}{2}, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \frac{\phi}{2} & \text{otherwise}. \end{cases}$$

where $\epsilon_n = \circ(\psi_n)$ is a positive sequence such that $\lim_{n\to+\infty} \frac{\epsilon_n}{\psi_n} = 0$.
(ii) *Compute*

$$\begin{cases} v_n = P_{\mathcal{D}}[\chi_n - \rho_n(\chi_n - \mathcal{B}(\chi_n))], \\ u_{n+1} = P_{\mathcal{E}_n}[\chi_n - \rho_n(v_n - \mathcal{B}(v_n))], \end{cases}$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle (1-\rho_n)\chi_n + \rho_n \mathcal{B}(\chi_n) - v_n, z - v_n\rangle \le 0\}.$$

(iii) *Evaluate stepsize rule for next iteration is evaluated as follows:*

$$\rho_{n+1} = \begin{cases} \min\left\{\sigma, \frac{\mu\langle v_n - \mathcal{B}v_n, u_{n+1} - v_n\rangle}{\langle \chi_n - \mathcal{B}(\chi_n), u_{n+1} - v_n\rangle - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|\chi_n - v_n\|^2 - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|u_{n+1} - v_n\|^2 + 1}\right\}, \\ \quad \text{if} \quad \frac{\mu\langle v_n - \mathcal{B}v_n, u_{n+1} - v_n\rangle}{\langle \chi_n - \mathcal{B}(\chi_n), u_{n+1} - v_n\rangle - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|\chi_n - v_n\|^2 - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|u_{n+1} - v_n\|^2 + 1} > 0, \\ \sigma \quad \text{otherwise}. \end{cases}$$

Then, the sequence $\{u_n\}$ converges strongly to $\zeta^* \in Fix(\mathcal{B}, \mathcal{D})$.

5. Numerical illustrations

In this section, the numerical performance of the proposed method is described in contrast with some similar works in the literature.

Example 5.1. The test problem here taken from the Nash-Cournot Oligo-polistic equilibrium model in [9, 21]. Suppose that the set $\mathcal{D}$ is defined by

$$\mathcal{D} := \{u \in \mathbb{R}^N : -10 \le u_i \le 10\}$$

and $f : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ is defined as follows

$$f(u, v) = \langle Mu + Nv + r, v - u\rangle, \quad \forall\, u, v \in \mathcal{D},$$

where $r \in \mathbb{R}^N$ and M, N matrices of order N. The matrix M is symmetric positive semi-definite and the matrix $N - M$ is symmetric negative semi-definite with Lipschitz-type criteria $k_1 = k_2 = \frac{1}{2}\|M - N\|$

(see [21] for details). Two matrices M, N are taken randomly [Two diagonal matrices randomly A_1 and A_2 with elements from $[0, 2]$ and $[-2, 0]$, respectively. Two random orthogonal matrices $O_1 = RandOrthMat(N)$ and $O_2 = RandOrthMat(N)$ are generated. Thus, a positive semi-definite matrix $B_1 = O_1 A_1 O_1^T$ and a negative semi-definite matrix $B_2 = O_2 A_2 O_2^T$ is obtained. Finally, set $N = B_1 + B_1^T$, $S = B_2 + B_2^T$ and $M = N - S$].

Experiment 1: In the first experiment, we take into account the numerical efficiency of the Algorithm 1 using different starting point choices. This experiment helps the reader see how much the starting points influenced the efficiency of the Algorithm 1. For these numerical results, we have use $N = 20$, $u_0 = u_1$, $\rho_1 = 0.50$, $\sigma = \frac{1}{2.2k_1}$, $\mu = 0.22$, $\phi = 1.00$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\psi_n = \frac{1}{20(n+2)}$, $D_n = Error = \|u_{n+1} - u_n\|$ for Algorithm 1 (**Alg-4**). Figures 1 and 2 demonstrate the numerical efficacy of the proposed mehod.

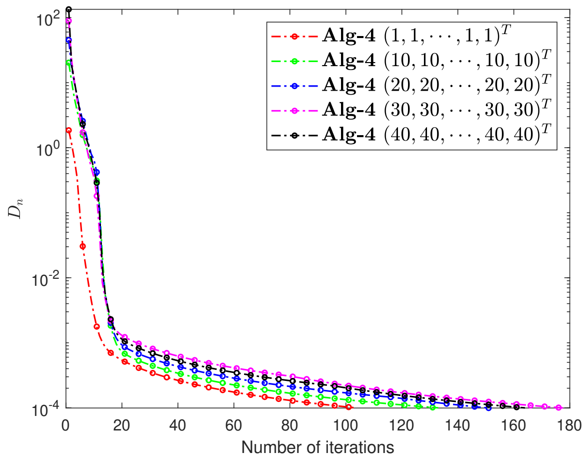

Figure 1. Numerical illustration of Algorithm 1 for different starting points while $N = 20$ and the number of iterations are 104, 133, 151, 178, 164, respectively.

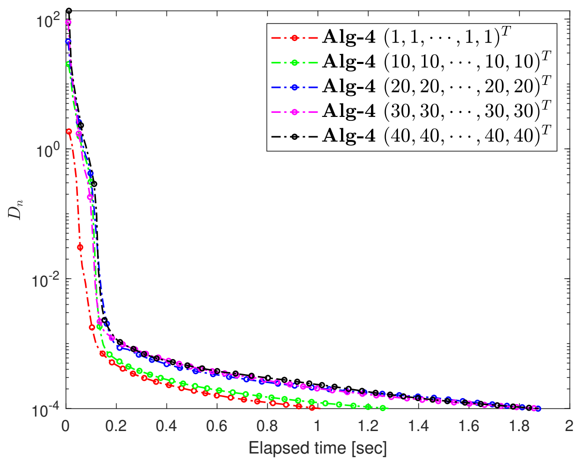

Figure 2. Numerical illustration of Algorithm 1 for different starting points while $N = 20$ and the execution time are 1.0100, 1.2801, 1.8741, 1.8852, 1.8766, respectively.

Experiment 2: In the second experiment, we consider the numerical efficiency of the Algorithm 1 using different inertial parameter ϕ choices. This experiment helps the reader see how much the inertial parameter ϕ influenced the efficiency of the Algorithm 1. For these numerical results, we have use $N = 20$, $u_0 = u_1 = (1, 1, \cdots, 1, 1)$, $\rho_1 = 0.30$, $\sigma = \frac{1}{2.5k_1}$, $\mu = 0.33$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\psi_n = \frac{1}{10(n+2)}$, $D_n = Error = \|u_{n+1} - u_n\|$ for Algorithm 1 (**Alg-4**). Figures 3 and 4 demonstrate the numerical efficacy of the proposed method.

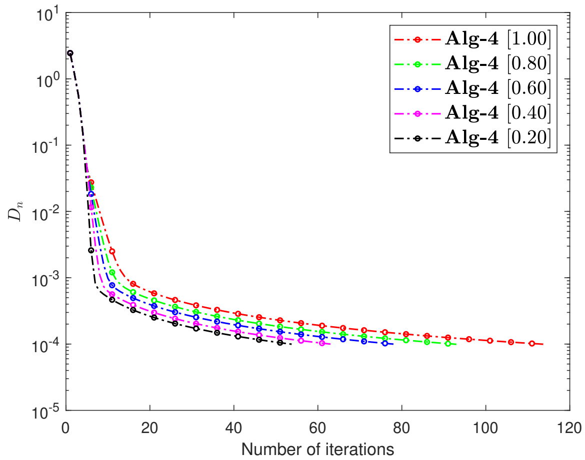

Figure 3. Numerical illustration of Algorithm 1 for $\phi = 1,\ 0.8,\ 0.6,\ 0.4,\ 0.2$ and the number of iterations are 114, 93, 78, 63, 54, respectively.

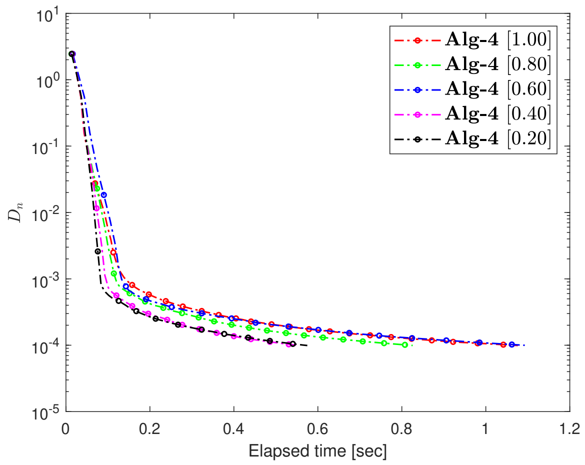

Figure 4. Numerical illustration of Algorithm 1 for $\phi = 1,\ 0.8,\ 0.6,\ 0.4,\ 0.2$ and the execution time are 1.0763, 0.8257, 1.0927, 0.5489, 0.5748, respectively.

Experiment 3: In third experiment, we provide the numerical comparison of Algorithm 1 with Algorithm 1 in [28] and Algorithm 3.1 in [11] and Algorithm 3 in [31]. For these numerical studies we have assumed that starting points are $u_0 = u_1 = v_0 = (1, 1, \cdots, 1)$, $N = 5, 10, 40, 100$ and error term $D_n = \|u_{n+1} - u_n\|$. Figures 5–12 have shown a number of results for Tolerance=10^{-4}. Information regarding the control parameters shall be considered as described in the following:
(i) $\rho_n = \frac{1}{4k_1}$, $\phi_n = \frac{1}{(n+1)^{0.5}}$, and $Error = \|u_{n+1} - u_n\|$ for Algorithm 3.1 in [11] (**Alg-1**).

(ii) $\rho = \frac{1}{4k_1}$, $\theta = 0.50$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\gamma_n = \frac{1}{20(n+2)}$, $\beta_n = \frac{7}{10}(1-\gamma_n)$, $Error = \|u_{n+1} - u_n\|$ for Algorithm 3 in [31] (**Alg-2**).
(iii) $\rho = \frac{1}{5k_1}$, $\phi_n = \frac{1}{100(n+2)}$, $f(u) = \frac{u}{2}$ and $Error = \|u_{n+1} - u_n\|$ for Algorithm 1 (**Alg-3**) in [28].
(iv) $\rho_1 = 0.50$, $\sigma = \frac{1}{2.2k_1}$, $\mu = 0.22$, $\phi = 1.00$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\psi_n = \frac{1}{20(n+2)}$, $Error = \|u_{n+1} - u_n\|$ for Algorithm 1 (**Alg-4**).

Then numerical results are reported in Table 1.

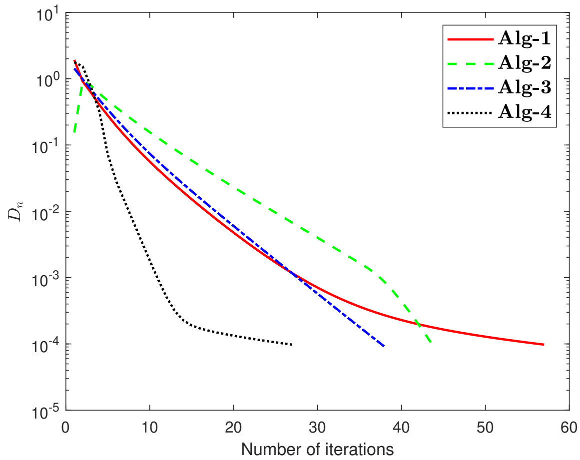

Figure 5. Numerical illustration of Algorithm 1 with Algorithm 1 in [28] and Algorithm 3.1 in [11] and Algorithm 3 in [31] for N=5.

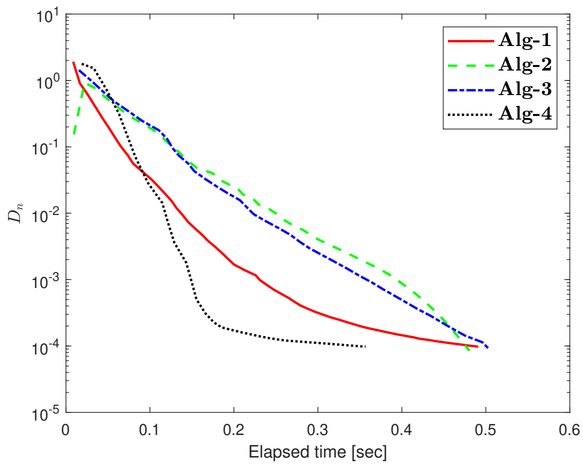

Figure 6. Numerical illustration of Algorithm 1 with Algorithm 1 in [28] and Algorithm 3.1 in [11] and Algorithm 3 in [31] for N=5.

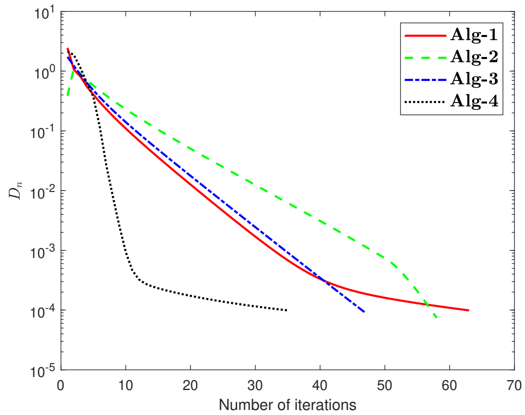

Figure 7. Numerical illustration of Algorithm 1 with Algorithm 1 in [28] and Algorithm 3.1 in [11] and Algorithm 3 in [31] for N=10.

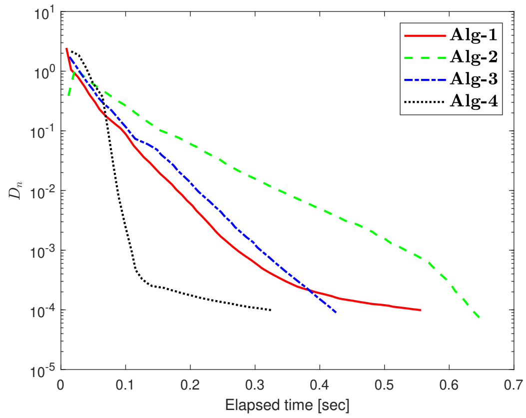

Figure 8. Numerical illustration of Algorithm 1 with Algorithm 1 in [28] and Algorithm 3.1 in [11] and Algorithm 3 in [31] for N=10.

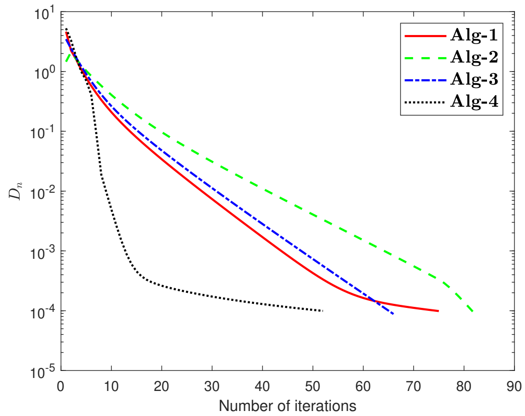

Figure 9. Numerical illustration of Algorithm 1 with Algorithm 1 in [28] and Algorithm 3.1 in [11] and Algorithm 3 in [31] for N=40.

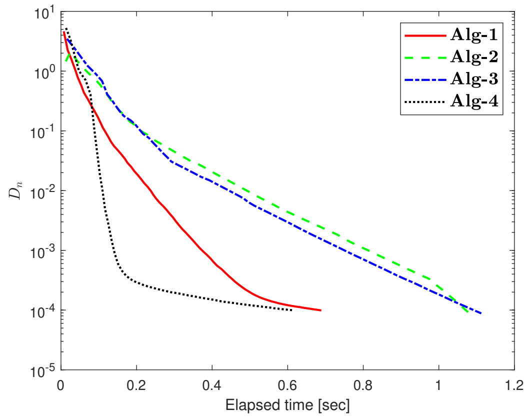

Figure 10. Numerical illustration of Algorithm 1 with Algorithm 1 in [28] and Algorithm 3.1 in [11] and Algorithm 3 in [31] for N=40.

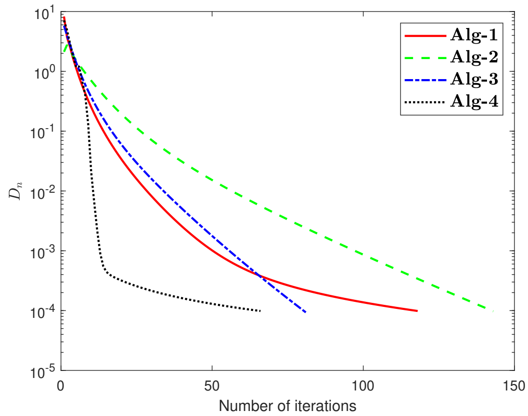

Figure 11. Numerical illustration of Algorithm 1 with Algorithm 1 in [28] and Algorithm 3.1 in [11] and Algorithm 3 in [31] for N=100.

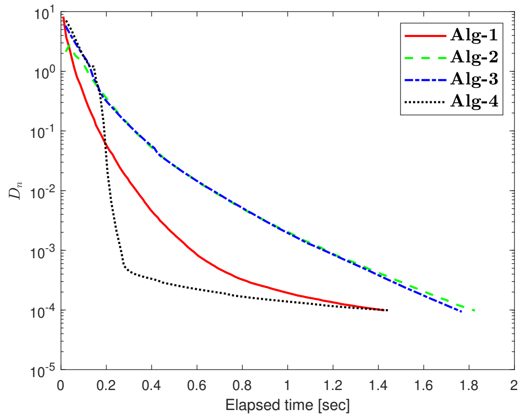

Figure 12. Numerical illustration of Algorithm 1 with Algorithm 1 in [28] and Algorithm 3.1 in [11] and Algorithm 3 in [31] for N=100.

Table 1. Numerical finding and their values for Figures 5–12.

Number of iterations					Elapsed time in seconds			
N	Alg-1	Alg-2	Alg-3	Alg-4	Alg-1	Alg-2	Alg-3	Alg-4
5	57	44	38	27	0.4910193	0.4805127	0.5035203	0.3568740
10	63	58	47	35	0.5564982	0.6466944	0.4253435	0.3249474
40	75	82	66	52	0.6889176	1.0781367	1.1124397	0.6130995
100	118	143	81	66	1.4267609	1.8261292	1.767970	1.448753

6. Conclusions

We constructed an explicit, inertial extragradient-type method to find a numerical solution to the pseudomonotone equilibrium problems in a real Hilbert space. This method is seen as a modification of the two-step gradient method. A strongly convergent result is well-proven, corresponding to the proposed algorithm. Numerical findings were presented to demonstrate our algorithm's numerical superiority over existing methods. These computational findings have indicated that the variable stepsize rule continues to increase the performance of the iterative sequence in this context.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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