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Research article

Statistical connections on decomposable Riemann manifold

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Abstract: Let (M, g, φ) be an n-dimensional locally decomposable Riemann manifold, that is, $g(\varphi X, Y) = g(X, \varphi Y)$ and $\nabla\varphi = 0$, where ∇ is Riemann (Levi-Civita) connection of metric g. In this paper, we construct a new connection on locally decomposable Riemann manifold, whose name is statistical (α, φ)-connection. A statistical α-connection is a torsion-free connection such that $\overline{\nabla}g = \alpha C$, where C is a completely symmetric $(0, 3)$-type cubic form. The aim of this article is to use connection $\overline{\nabla}$ and product structure φ in the same equation, which is possible by writing the cubic form C in terms of the product structure φ. We examine some curvature properties of the new connection and give examples of it.

Keywords: statistical manifold; α-connection; pure tensors; Tachibana operator
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1. Introduction

Statistical structures in modern differential geometry have studied by many authors in recent years. One of them is Lauritzen. In [1], Lauritzen created a statistical manifold by defining a totally symmetric tensor field C (cubic form) of type $(0, 3)$ on a Riemann manifold (M_n, g). He has shown that there is a torsion-free linear connection ${}^{(\alpha)}\nabla$ such that ${}^{(\alpha)}\nabla g = \alpha C$, where g is Riemann metric and $\alpha = \pm 1$. Then, he examined some properties of the curvature tensor field and defined the dual connection of this connection. Also, he showed the relationship between curvature and dual curvature tensor field of that connection and presented examples on the statistical manifold.

In this paper, we create a special connection inspired by statistical manifold on locally product Riemann manifold (M_n, φ, g). We call this new connection as statistical (α, φ)-connection. We investigate the decomposable condition for cubic form C expressed by the product structure φ. Then, we calculate the curvature tensor field of that connection and examine its some properties. We give two examples that support this connection. Finally, we define the dual of the new connection and investigate its curvature tensor field.

2. Preliminaries

Let M_n be a n-dimensional manifold. Throughout this article, all tensor fields, linear connections, and manifolds will always be regarded as differentiable of class C^∞. The class of (p,q)-type tensor fields will also be denoted $\Im_q^p(M_n)$. For example if the tensor field V is of type $(1,2)$, then $V \in \Im_2^1(M_n)$.

The tensor field K of type $(0,q)$ is called pure with respect to the φ if the following equation holds:

$$\begin{aligned} K(\varphi Y_1, Y_2, ..., Y_q) &= K(Y_1, \varphi Y_2, ..., Y_q) \\ &= ... \\ &= K(Y_1, Y_2, ..., \varphi Y_q), \end{aligned}$$

where φ is endomorphism, namely, $\varphi \in \Im_1^1(M_n)$ and $Y_1, Y_2, ..., Y_q \in \Im_0^1(M_n)$ [2, p.208]. Then, the Φ operator (or Tachibana operator) applied to pure tensor field K of type $(0,q)$ is given by

$$\begin{aligned} (\Phi_{\varphi X} K)(Y_1, Y_2, ..., Y_q) &= (\varphi X)(K(Y_1, Y_2, ..., Y_q)) \\ &\quad - X(K(\varphi Y_1, Y_2, ..., Y_q)) \\ &\quad + \sum_{i=1}^{q} K(Y_1, ..., (L_{Y_i}\varphi)X, ..., Y_q), \end{aligned} \tag{2.1}$$

where L_Y is the Lie differentiation according to a vector field Y [2, p.211].

In the equation (2.1), if $\Phi_\varphi K = 0$, then K is named Φ-tensor field. Especially, if φ is product structure, that is, $\varphi^2 = I$ and $\Phi_\varphi K = 0$, then K is called a decomposable tensor field [2, p.214].

The almost product Riemann manifold (M_n, φ, g) is a manifold that satisfies

$$g(\varphi X, Y) = g(X, \varphi Y)$$

and $\varphi^2 = I$, where g is Riemann metric. In [3], the authors (in Theorem 1) show that in almost product Riemann manifold, if $\Phi_\varphi g = 0$, then φ is integrable. Then, it is clear that the condition $\Phi_\varphi g = 0$ is equivalent $\nabla\varphi = 0$, where ∇ is Riemann (or Levi-Civita) connection of Riemann metric g. It is well-known that if φ is integrable, then the triplet (M_n, φ, g) is named locally product Riemann manifold. Besides, locally product Riemann manifold (M_n, φ, g) is a locally decomposable if and only if the product structure φ is parallel according to the Riemann connection ∇, in other words, $\nabla\varphi = 0$ [4, p.420]. Thus, it is easily said that the (M_n, φ, g) is a locally decomposable Riemann manifold if and only if $\Phi_\varphi g = 0$ [3] (in Theorem 2).

In adition, in [5], the authors (in Proposition 4.2) examined properties of the Riemann curvature tensor field R of the locally product Riemann manifold (M_n, φ, g) and showed that $\Phi_\varphi R = 0$, that is, the Riemann curvature tensor field R is decomposable.

3. Statistical (α, φ)-connections

Let (M_n, g, φ) be a locally decomposable Riemann manifold and ${}^{(\alpha)}\nabla$ be a torsion-free linear connection on this manifold that provides the following equation:

$${}^{(\alpha)}\nabla_X Y = \nabla_X Y - \frac{\alpha}{2}\overline{C}(X, Y). \tag{3.1}$$

Then, the connection ${}^{(\alpha)}\nabla$ is named statistical α-connection If $({}^{(\alpha)}\nabla_X g)(Y,Z) = \alpha C(X,Y,Z)$ is satisfied and C is totally symmetric, where $\alpha = \pm 1$, C is a the cubic form such that $C(X,Y,Z) = g(\overline{C}(X,Y),Z)$ and ∇ is Riemann connection of metric g [1, p.179–180].

In this paper, we will study a special version of cubic form $\overline{C}$, which is expressed as follows:

$$\begin{aligned} \overline{C}(X,Y) &= \eta(X)Y + \eta(Y)X + g(X,Y)U \\ &\quad +\eta(\varphi X)(\varphi Y) + \eta(\varphi Y)(\varphi X) + g(\varphi X,Y)(\varphi U), \end{aligned} \tag{3.2}$$

where φ is a product structure, that is, $\varphi^2(X) = I(X)$, η is a covector field (or 1-form) and U is a vector field such that $U = g^\sharp(\eta) = \eta^\sharp$, where $g^\sharp : \mathfrak{I}_1^0(M_n) \longrightarrow \mathfrak{I}_0^1(M_n)$, that is $g^\sharp$ is a musical isomorphisms. From the equation (3.2) and $C(X,Y,Z) = g(\overline{C}(X,Y),Z)$, we have

$$\begin{aligned} C(X,Y,Z) &= \eta(X)g(Y,Z) + \eta(Y)g(X,Z) + \eta(Z)g(X,Y) \\ &\quad +\eta(\varphi X)g(\varphi Y,Z) + \eta(\varphi Y)g(\varphi X,Z) + \eta(\varphi Z)g(\varphi X,Y). \end{aligned} \tag{3.3}$$

Then, it is clear that

$$({}^{(\alpha)}\nabla_X g)(Y,Z) = \alpha C(X,Y,Z)$$

and $C(X,Y,Z)$ is completely symmetric, that is,

$$\begin{aligned} C(X,Y,Z) &= C(X,Z,Y) = C(Z,X,Y) \\ &= C(Z,Y,X) = C(Y,X,Z) = C(Y,Z,X). \end{aligned}$$

From the (3.2), we get

Proposition 3.1. *The product structure φ is parallel according to the α-connection ${}^{(\alpha)}\nabla$ on locally decomposable Riemann manifold (M_n, g, φ), that is, ${}^{(\alpha)}\nabla\varphi = 0$.*

Throughout this paper, the α-connection ${}^{(\alpha)}\nabla$ on locally decomposable Riemann manifold (M_n, g, φ) is called "statistical (α, φ)-connection".

We easily say that the cubic form C is pure with regard to the product structure φ, that is,

$$C(\varphi X, Y, Z) = C(X, \varphi Y, Z) = C(X, Y, \varphi Z).$$

Therefore, we get

$${}^{(\alpha)}\nabla_{(\varphi X)}Y = {}^{(\alpha)}\nabla_X(\varphi Y) = \varphi({}^{(\alpha)}\nabla_X Y),$$

i.e., the connection ${}^{(\alpha)}\nabla$ is pure according to product structure φ. Also, in [6, p.19] , the author has already shown that any φ-connection $\overline{\nabla}$ is pure if and only if its torsion tensor is pure. Then, we write

Theorem 3.1. *In locally decomposable Riemann manifold (M_n, g, φ), if the covector field η in (3.3) is a decomposable tensor field, then the cubic form C is a decomposable tensor field.*

Proof. For the cubic form C given by the (3.3), from the (2.1), we obtain

$$(\Phi_{\varphi X}C)(Y_1, Y_2, Y_3) = (\nabla_{\varphi X}C)(Y_1, Y_2, Y_3) - (\nabla_X C)(\varphi Y_1, Y_2, Y_3). \tag{3.4}$$

Substituting (3.3) into the last equation, we get

$$\begin{aligned}(\Phi_{\varphi X}C)(Y_1,Y_2,Y_3) &= \left[(\nabla_{\varphi X}\eta)(Y_1)-(\nabla_X\eta)(\varphi Y_1)\right]g(Y_2,Y_3)\\ &\quad+\left[(\nabla_{\varphi X}\eta)(Y_2)-(\nabla_X\eta)(\varphi Y_2)\right]g(Y_1,Y_3)\\ &\quad+\left[(\nabla_{\varphi X}\eta)(Y_3)-(\nabla_X\eta)(\varphi Y_3)\right]g(Y_1,Y_2)\\ &\quad+\left[(\nabla_{\varphi X}\eta)(\varphi Y_1)-(\nabla_X\eta)(Y_1)\right]g(\varphi Y_2,Y_3)\\ &\quad+\left[(\nabla_{\varphi X}\eta)(\varphi Y_2)-(\nabla_X\eta)(Y_2)\right]g(\varphi Y_1,Y_3)\\ &\quad+\left[(\nabla_{\varphi X}\eta)(\varphi Y_3)-(\nabla_X\eta)(Y_3)\right]g(\varphi Y_1,Y_2).\end{aligned} \tag{3.5}$$

and for covector field η, we have

$$(\Phi_{\varphi X}\eta)(Y)=(\nabla_{\varphi X}\eta)(Y)-(\nabla_X\eta)(\varphi Y). \tag{3.6}$$

From the last two equations, we get

$$\begin{aligned}(\Phi_\varphi C)(X,Y_1,Y_2,Y_3) &= (\Phi_{\varphi X}\eta)(Y_1)g(Y_2,Y_3)+(\Phi_{\varphi X}\eta)(Y_2)g(Y_1,Y_3)\\ &\quad+(\Phi_{\varphi X}\eta)(Y_3)g(Y_1,Y_2)+(\Phi_{\varphi X}\eta)(\varphi Y_1)g(\varphi Y_2,Y_3)\\ &\quad+(\Phi_{\varphi X}\eta)(\varphi Y_2)g(\varphi Y_1,Y_3)+(\Phi_{\varphi X}\eta)(\varphi Y_3)g(\varphi Y_1,Y_2).\end{aligned}$$

It is clear that if $\Phi_\varphi\eta=0$, then $\Phi_\varphi C=0$. □

Corollary 3.1. *From the equation (3.4) in Theorem 3.1, we can write*

$$\begin{aligned}(\nabla_{\varphi X}C)(Y_1,Y_2,Y_3) &= (\nabla_X C)(\varphi Y_1,Y_2,Y_3)\\ &= (\nabla_X C)(Y_1,\varphi Y_2,Y_3)\\ &= (\nabla_X C)(Y_1,Y_2,\varphi Y_3),\end{aligned}$$

that is, the covariant derivation of the cubic form C is pure with respect to product structure φ.

In the following sections of the paper, we will assume that the covector field η is decomposable tensor field, i.e., the following equation always applies:

$$(\nabla_{\varphi X}\eta)(Y)-(\nabla_X\eta)(\varphi Y)=0.$$

The α-curvature tensor field ${}^{(\alpha)}\overline{R}$ of the statistical (α,φ)-connection ${}^{(\alpha)}\nabla$ is given by

$${}^{(\alpha)}\overline{R}(X,Y,Z)=({}^{(\alpha)}\nabla_X\,{}^{(\alpha)}\nabla_Y-{}^{(\alpha)}\nabla_Y\,{}^{(\alpha)}\nabla_X-{}^{(\alpha)}\nabla_{[X,Y]})Z.$$

Substituting (3.1) into the last equation, we obtain

$$\begin{aligned}{}^{(\alpha)}R(X,Y,Z,W) &= R(X,Y,Z,W)\\ &\quad-g(Y,W)\rho(X,Z)+g(X,W)\rho(Y,Z)\\ &\quad-g(Y,Z)q(X,W)+g(X,Z)q(Y,W)\\ &\quad-g(\varphi Y,W)\rho(X,\varphi Z)+g(\varphi X,W)\rho(Y,\varphi Z)\end{aligned} \tag{3.7}$$

$$-g(\varphi Y, Z)q(X, \varphi W) + g(\varphi X, Z)q(Y, \varphi W)$$
$$-g(Z, W)[\rho(X, Y) - \rho(Y, X)]$$
$$+g(\varphi Z, W)[\rho(X, \varphi Y) - \rho(\varphi Y, X)],$$

where $g({}^{(\alpha)}\overline{R}(X, Y, Z), W) = {}^{(\alpha)}R(X, Y, Z, W)$ and R is Riemann curvature tensor field of Riemann metric g,

$$\begin{aligned} \rho(X, Y) &= \frac{\alpha}{2}(\nabla_X \eta)Y + \frac{\alpha^2}{4}\eta(X)\eta(Y) + \frac{\alpha^2}{8}\eta(U)g(X, Y) \\ &\quad + \frac{\alpha^2}{4}\eta(\varphi X)\eta(\varphi Y) + \frac{\alpha^2}{8}\eta(\varphi U)g(\varphi X, Y) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} q(X, Y) &= \frac{\alpha}{2}(\nabla_X \eta)Y - \frac{\alpha^2}{4}\eta(X)\eta(Y) - \frac{\alpha^2}{8}\eta(U)g(X, Y) \\ &\quad - \frac{\alpha^2}{4}\eta(\varphi X)\eta(\varphi Y) - \frac{\alpha^2}{8}\eta(\varphi U)g(\varphi X, Y). \end{aligned} \tag{3.9}$$

From the last two equations, we get

$$\begin{aligned} \rho(X, Y) - \rho(Y, X) &= q(X, Y) - q(Y, X) \\ &= \frac{\alpha}{2}[(\nabla_X \eta)Y - (\nabla_Y \eta)X] \\ &= \frac{\alpha}{2}[(\nabla_X \eta)Y) - (\nabla_Y \eta)X + \eta(\nabla_X Y - \nabla_Y X) - \eta([X, Y])] \\ &= \frac{\alpha}{2}[(\nabla_X \eta)Y + \eta(\nabla_X Y) - (\nabla_Y \eta)X - \eta(\nabla_Y X) - \eta([X, Y])] \\ &= \frac{\alpha}{2}[X\eta(Y) - Y\eta(X) - \eta([X, Y])] \\ &= \alpha(d\eta)(X, Y), \end{aligned}$$

where d is exterior derivate operator applied to the covector field η. Then, we can write the following proposition and corollary.

Proposition 3.2. *The covector field η is closed, that is, $d\eta = 0$ if and only if*

$$\begin{aligned} \rho(X, Y) - \rho(Y, X) &= q(X, Y) - q(Y, X) \\ &= 0. \end{aligned} \tag{3.10}$$

Corollary 3.2. *For the differentiable function f on locally decomposable Riemann manifold (M_n, g, φ), it is well-known that $d^2 f = 0$. So, if $\eta = df = \frac{\partial f}{\partial x^i}dx^i$, then $d\eta = 0$ is directly obtained and we can write the equation (3.10).*

The tensor fields ρ and q are given by Eqs (3.8) and (3.9), respectively, is pure according to the product structure φ. Then, we write

$$\rho(X, \varphi Y) - \rho(\varphi X, Y) = q(X, \varphi Y) - q(\varphi X, Y)$$

$$
\begin{aligned}
&= \frac{\alpha}{2}\left[(\nabla_X\eta)(\varphi Y) - (\nabla_{\varphi X}\eta)(Y)\right] \\
&= 0.
\end{aligned}
$$

In addition, from the Eqs (2.1) and (3.8), we get

$$
(\Phi_{\varphi X}\rho)(Y_1, Y_2) = (\nabla_{\varphi X}\rho)(Y_1, Y_2) - (\nabla_X\rho)(\varphi Y_1, Y_2). \tag{3.11}
$$

Substituting (3.8) into the Eq (3.11), we obtain

$$
(\Phi_\varphi\rho)(X, Y_1, Y_2) = \frac{\alpha}{2}\left[\left(\nabla_{\varphi X}\nabla_{Y_1}\eta\right)(Y_2) - \left(\nabla_X\nabla_{\varphi Y_1}\eta\right)(Y_2)\right].
$$

For the Ricci identity of the covector field η, we have

$$
\left(\nabla_{\varphi X}\nabla_{Y_1}\eta\right)(Y_2) = \left(\nabla_{Y_1}\nabla_{\varphi X}\eta\right)(Y_2) - \frac{1}{2}\eta(\overline{R}(\varphi X, Y_1, Y_2)) \tag{3.12}
$$

and

$$
\left(\nabla_X\nabla_{Y_1}\eta\right)(\varphi Y_2) = \left(\nabla_{Y_1}\nabla_X\eta\right)(\varphi Y_2) - \frac{1}{2}\eta(\overline{R}(X, Y_1, \varphi Y_2)). \tag{3.13}
$$

From the last equations, we write

$$
\begin{aligned}
(\Phi_{\varphi X}\rho)(Y_1, Y_2) &= -\frac{1}{2}\eta(\overline{R}(\varphi X, Y_1, Y_2) - \overline{R}(X, Y_1, \varphi Y_2)) \\
&= 0
\end{aligned}
$$

and in the same way $(\Phi_{\varphi X}q) = 0$. Then, we have

Proposition 3.3. *The tensor fields ρ and q are given by Eqs (3.8) and (3.9), respectively are a decomposable tensor fields and because of the equation (3.11), we can write*

$$
(\nabla_{\varphi X}\rho)(Y, Z) = (\nabla_X\rho)(\varphi Y, Z) = (\nabla_X\rho)(Y, \varphi Z)
$$

and

$$
(\nabla_{\varphi X}q)(Y, Z) = (\nabla_X q)(\varphi Y, Z) = (\nabla_X q)(Y, \varphi Z),
$$

that is, the covariant derivation of the tensor fields ρ and q are pure with respect to the product structure φ.

With the simple calculation, we can say that the α-curvature tensor field ${}^{(\alpha)}R$ is pure with regard to the product structure φ, namely,

$$
\begin{aligned}
{}^{(\alpha)}R(\varphi Y_1, Y_2, Y_3, Y_4) &= {}^{(\alpha)}R(Y_1, \varphi Y_2, Y_3, Y_4) \\
&= {}^{(\alpha)}R(Y_1, Y_2, \varphi Y_3, Y_4) \\
&= {}^{(\alpha)}R(Y_1, Y_2, Y_3, \varphi Y_4).
\end{aligned}
$$

Then, from the Eq (2.1), we have

$$
(\Phi_{\varphi X}{}^{(\alpha)}R)(Y_1, Y_2, Y_3, Y_4) = (\nabla_{\varphi X}{}^{(\alpha)}R)(Y_1, Y_2, Y_3, Y_4) \tag{3.14}
$$

$$-(\nabla_X{}^{(\alpha)}R)(\varphi Y_1, Y_2, Y_3, Y_4).$$

If the expression of the α-curvature tensor field ${}^{(\alpha)}R$ is written in the last equation, then we obtain

$$\begin{aligned}(\Phi_{\varphi X}{}^{(\alpha)}R)(Y_1, Y_2, Y_3, Y_4) &= (\Phi_{\varphi X}R)(Y_1, Y_2, Y_3, Y_4)\\ &-[(\nabla_{\varphi X}\rho)(Y_1, Y_3) - (\nabla_X\rho)(Y_1, \varphi Y_3)]g(Y_2, Y_4)\\ &+[(\nabla_{\varphi X}\rho)(Y_2, Y_3) - (\nabla_X\rho)(Y_2, \varphi Y_3)]g(Y_1, Y_4)\\ &-[(\nabla_{\varphi X}q)(Y_1, Y_4) - (\nabla_X q)(Y_1, \varphi Y_4)]g(Y_2, Y_3)\\ &+[(\nabla_{\varphi X}q)(Y_2, Y_4) - (\nabla_X q)(Y_2, \varphi Y_4)]g(Y_1, Y_3)\\ &-[(\nabla_{\varphi X}\rho)(Y_1, \varphi Y_3) - (\nabla_X\rho)(Y_1, Y_3)]g(Y_2, \varphi Y_4)\\ &+[(\nabla_{\varphi X}\rho)(Y_2, \varphi Y_3) - (\nabla_X\rho)(Y_2, Y_3)]g(Y_1, \varphi Y_4)\\ &-[(\nabla_{\varphi X}q)(Y_1, \varphi Y_4) - (\nabla_X q)(Y_1, Y_4)]g(Y_2, \varphi Y_3)\\ &+[(\nabla_{\varphi X}q)(Y_2, \varphi Y_4) - (\nabla_X q)(Y_2, Y_4)]g(Y_1, \varphi Y_3)\\ &-[(\nabla_{\varphi X}\rho)(Y_1, Y_2) - (\nabla_{\varphi X}\rho)(Y_2, Y_1)\\ &\quad -((\nabla_X\rho)(Y_1, \varphi Y_2) - (\nabla_X\rho)(\varphi Y_2, Y_1))]g(Y_3, Y_4)\\ &+[(\nabla_{\varphi X}\rho)(Y_1, \varphi Y_2) - (\nabla_{\varphi X}\rho)(\varphi Y_2, Y_1)\\ &\quad -((\nabla_X\rho)(Y_1, Y_2) - (\nabla_X\rho)(Y_2, Y_1))]g(\varphi Y_3, Y_4).\end{aligned}$$

Furthermore, from Proposition 3.3, the last equation becomes the following form:

$$\begin{aligned}&(\Phi_{\varphi X}{}^{(\alpha)}R)(Y_1, Y_2, Y_3, Y_4)\\ =\ &(\Phi_{\varphi X}R)(Y_1, Y_2, Y_3, Y_4)\\ &-(\Phi_{\varphi X}\rho)(Y_1, Y_3)g(Y_2, Y_4) + (\Phi_{\varphi X}\rho)(Y_2, Y_3)g(Y_1, Y_4)\\ &-(\Phi_{\varphi X}q)(Y_1, Y_4)g(Y_2, Y_3) + (\Phi_{\varphi X}q)(Y_2, Y_4)g(Y_1, Y_3)\\ &-(\Phi_{\varphi X}\rho)(Y_1, \varphi Y_3)g(Y_2, \varphi Y_4) + (\Phi_{\varphi X}\rho)(Y_2, \varphi Y_3)g(Y_1, \varphi Y_4)\\ &-(\Phi_{\varphi X}q)(Y_1, \varphi Y_4)g(Y_2, \varphi Y_3) + (\Phi_{\varphi X}q)(Y_2, \varphi Y_4)g(Y_1, \varphi Y_3)\\ &-[(\Phi_{\varphi X}\rho)(Y_1, Y_2) - (\Phi_{\varphi X}\rho)(Y_2, Y_1)]g(Y_3, Y_4)\\ &+[(\Phi_{\varphi X}\rho)(Y_1, \varphi Y_2) - (\Phi_{\varphi X}\rho)(\varphi Y_2, Y_1)]g(\varphi Y_3, Y_4)\end{aligned}$$

and

$$(\Phi_{\varphi X}{}^{(\alpha)}R)(Y_1, Y_2, Y_3, Y_4) = 0.$$

Then, we obtain

Theorem 3.2. *The α-curvature tensor field ${}^{(\alpha)}R$ of the statistical (α, φ)-connection ${}^{(\alpha)}\nabla$ is decomposable tensor field and due to the equation (3.14), we can say that*

$$\begin{aligned}(\nabla_{\varphi X}{}^{(\alpha)}R)(Y_1, Y_2, Y_3, Y_4) &= (\nabla_X{}^{(\alpha)}R)(\varphi Y_1, Y_2, Y_3, Y_4)\\ &= (\nabla_X{}^{(\alpha)}R)(Y_1, \varphi Y_2, Y_3, Y_4)\\ &= (\nabla_X{}^{(\alpha)}R)(Y_1, Y_2, \varphi Y_3, Y_4)\\ &= (\nabla_X{}^{(\alpha)}R)(Y_1, Y_2, Y_3, \varphi Y_4),\end{aligned}$$

namely, the covariant derivation of the α-curvature tensor field ${}^{(\alpha)}R$-is pure with respect to the product structure φ.

4. Some example for Statistical (α, φ)-connections

Example 4.1. *Let $M_2 = \{(x,y) \in R^2, x > 0\}$ be a manifold with the metric g such that*

$$\left[g_{ij}(x,y)\right] = \begin{bmatrix} \frac{1}{x} & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, (M_2, g) is a Riemann manifold. The component of the Riemann connection of this manifold is as follow:

$$\Gamma^1_{11}(x,y) = -\frac{1}{2x}$$

and the others are zero. In addition, we say that (M_2, g) is a flat manifold, that is, Riemann curvature tensor R of that manifold is vanishing. The equation system satisfying the conditions $\varphi^m_i g_{mj} = \varphi^m_j g_{im}$ (purity) and $\varphi^m_i \varphi^j_m = \delta^j_i$ (product structure) is

$$\begin{cases} a^2 + bc = 1 \\ b(a+d) = 0 \\ c(a+d) = 0 \\ d^2 + bc = 1 \\ c = \frac{1}{x} b \end{cases}, \tag{4.1}$$

where

$$\left[\varphi^j_i(x,y)\right] = \begin{bmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{bmatrix}. \tag{4.2}$$

Then, a general solution of the equation system (4.1) is

$$\left[a = -d, b = x\sqrt{-\frac{1}{x}(d-1)(d+1)}, c = \sqrt{-\frac{1}{x}(d-1)(d+1)}\right]. \tag{4.3}$$

In the last equation, a special solution for d = 0 is as follow:

$$\left[\varphi^j_i(x,y)\right] = \begin{bmatrix} 0 & \sqrt{x} \\ \frac{1}{\sqrt{x}} & 0 \end{bmatrix}.$$

Here, because of $\nabla\varphi = 0$, the triplet (M_2, g, φ) is locally decomposable Riemann manifold.

The expression of the cubic form in local coordinates given by (3.2) is

$$\overline{C}^k_{ij} = \eta_i \delta^k_j + \eta_i \delta^k_j + \eta^k g_{ij} + \eta_t \varphi^t_i \varphi^k_j + \eta_t \varphi^t_j \varphi^k_i + \eta^t \varphi^k_t \varphi_{ij},$$

where $\eta^k = \eta_i g^{ik}$ and $\varphi_{ij} = \varphi^k_i g_{kj}$. For $\eta(x,y) = (\eta_1(x,y), \eta_2(x,y))$, the matrix shape of the cubic form is as follows:

$$\left[\overline{C}^1_{ij}(x,y)\right] = \begin{bmatrix} 3\eta_1 & 3\eta_2 \\ 3\eta_2 & 3x\eta_1 \end{bmatrix},$$

$$\left[\overline{C}^2_{ij}(x,y)\right] = \begin{bmatrix} \frac{3}{x}\eta_2 & 3\eta_1 \\ 3\eta_1 & 3\eta_2 \end{bmatrix},$$

for example, $\overline{C}_{11}^{1} = 3\eta_1$, $\overline{C}_{11}^{2} = \frac{3}{x}\eta_2$,..., *etc. In adition, the statistical* (α, φ)*-connection is given by*

$$^{(\alpha)}\Gamma_{ij}^{k}(x,y) = \Gamma_{ij}^{k}(x,y) - \frac{\alpha}{2}\overline{C}_{ij}^{\,k}(x,y)$$

and these components are

$$\left[^{(\alpha)}\Gamma_{ij}^{1}(x,y)\right] = \begin{bmatrix} -\frac{1}{2x} - \frac{3\alpha}{2}\eta_1 & -\frac{3\alpha}{2}\eta_2 \\ -\frac{3\alpha}{2}\eta_2 & -\frac{3\alpha}{2}x\eta_1 \end{bmatrix},$$

$$\left[^{(\alpha)}\Gamma_{ij}^{2}(x,y)\right] = \begin{bmatrix} -\frac{3\alpha}{2x}\eta_2 & -\frac{3\alpha}{2}\eta_1 \\ -\frac{3\alpha}{2}\eta_1 & -\frac{3\alpha}{2}\eta_2 \end{bmatrix}.$$

For $i, j, m = 1, 2$*, the components of the* $\Phi_\varphi\eta$ *are*

$$(\Phi_\varphi\eta)_{ij} = \varphi_i^m(\frac{\partial}{\partial x_m}\eta_j) - \varphi_j^m(\frac{\partial}{\partial x_i}\eta_m),$$

where $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x}$ *and* $\frac{\partial}{\partial x_2} = \frac{\partial}{\partial y}$*. Then, we have*

$$\begin{aligned} (\Phi_\varphi\eta)_{11} &= -\frac{1}{x}(\Phi_\varphi\eta)_{22} = \frac{1}{\sqrt{x}}(\frac{\partial}{\partial y}\eta_1 - \frac{\partial}{\partial x}\eta_2), \\ (\Phi_\varphi\eta)_{12} &= -(\Phi_\varphi\eta)_{21} = \frac{1}{\sqrt{x}}(\frac{\partial}{\partial y}\eta_2 - x\frac{\partial}{\partial x}\eta_1) \end{aligned}$$

and because of $\Phi_\varphi\eta = 0$*, we obtain*

$$\begin{aligned} \frac{\partial}{\partial y}\eta_1 &= \frac{\partial}{\partial x}\eta_2, \\ \frac{\partial}{\partial y}\eta_2 &= x\frac{\partial}{\partial x}\eta_1. \end{aligned}$$

Then, the components of the α*-curvature tensor field* $^{(\alpha)}R$ *are*

$$^{(\alpha)}R_{1212}(x,y) = {}^{(\alpha)}R_{1221}(x,y) = \frac{1}{2x}\eta_1.$$

Example 4.2. *Let* $N_2 = \{(x,y) \in R^2, x < 0\}$ *be a manifold with the metric* g *such that*

$$\left[g_{ij}(x,y)\right] = \begin{bmatrix} 1 + x^2 & -x \\ -x & 1 \end{bmatrix}.$$

Then, (N_2, g) *is a Riemann manifold and the component of the Riemann connection of this manifold is the following form:*

$$\Gamma_{11}^{2}(x,y) = -1$$

and the others are zero. Also, (N_2, g) *is a flat manifold. The equation system satisfying the conditions* $\varphi_i^m g_{mj} = \varphi_j^m g_{im}$ *and* $\varphi_i^m \varphi_m^j = \delta_i^j$ *is as follow:*

$$\begin{cases} a^2 + bc = 1 \\ b(a+d) = 0 \\ c(a+d) = 0 \\ d^2 + bc = 1 \\ x(d-a) + c = b(1+x^2) \end{cases},$$

where

$$\left[\varphi_i^j(x,y)\right] = \begin{bmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{bmatrix}.$$

Then, a general solution of the equation system (4.2) is

$$\left[a = -d, b = \frac{1}{x^2+1}\left(\sqrt{-d^2+x^2+1}+dx\right), c = \sqrt{-d^2+x^2+1}-dx\right]. \tag{4.4}$$

In the equation (4.4), a special solution for $d = 1$ is

$$\left[\varphi_i^j(x,y)\right] = \begin{cases} \begin{bmatrix} -1 & \frac{2x}{1+x^2} \\ 0 & 1 \end{bmatrix} & , if \;\; x > 0 \\ \begin{bmatrix} -1 & 0 \\ -2x & 1 \end{bmatrix} & , if \;\; x < 0 \end{cases},$$

where for $x < 0$, (N_2, g, φ) is locally decomposable Riemann manifold because of $\nabla\varphi = 0$. Then, we get

$$\left[\overline{C}_{ij}^1(x,y)\right] = \begin{bmatrix} 6(\eta_1 + x\eta_2) & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left[\overline{C}_{ij}^2(x,y)\right] = \begin{bmatrix} 6x(\eta_1 + 2x\eta_2) & -6x\eta_2 \\ -6x\eta_2 & 6\eta_2 \end{bmatrix}$$

and

$$\left[{}^{(\alpha)}\Gamma_{ij}^1(x,y)\right] = \begin{bmatrix} -3\alpha(\eta_1 + x\eta_2) & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left[{}^{(\alpha)}\Gamma_{ij}^2(x,y)\right] = \begin{bmatrix} -3\alpha x(\eta_1 + 2x\eta_2) - 1 & 3\alpha x\eta_2 \\ 3\alpha x\eta_2 & -3\alpha\eta_2 \end{bmatrix}.$$

From the equation (3.6), we have

$$\begin{aligned} (\Phi_\varphi\eta)_{11} &= 2x(\frac{\partial}{\partial x}\eta_2 - \frac{\partial}{\partial y}\eta_1), \\ (\Phi_\varphi\eta)_{12} &= 2(x\frac{\partial}{\partial y}\eta_2 - \frac{\partial}{\partial x}\eta_2), \\ (\Phi_\varphi\eta)_{21} &= 2(\frac{\partial}{\partial y}\eta_1 + \frac{\partial}{\partial y}\eta_2), \\ (\Phi_\varphi\eta)_{22} &= 0. \end{aligned}$$

and because of $\Phi_\varphi\eta = 0$,

$$\frac{\partial}{\partial x}\eta_2 = \frac{\partial}{\partial y}\eta_1 = -x\frac{\partial}{\partial y}\eta_2.$$

Then, we easily say that the components of the α-curvature tensor field ${}^{(\alpha)}R$ are

$${}^{(\alpha)}R_{1221}(x,y) = -x{}^{(\alpha)}R_{1222}(x,y) = -6\alpha x(\frac{\partial}{\partial x}\eta_2).$$

Then, we get

Corollary 4.1. *The α-curvature tensor field ${}^{(\alpha)}R$ of (N_2, g, φ) is vanishing, that is, ${}^{(\alpha)}R = 0$ if and only if*

$$\eta = (\eta_1(x,y), \eta_2(x,y)) = (\eta_1(x,c_1), \eta_2(c_2,c_3)),$$

where c_1, c_2, and c_3 are scalars.

5. Dual statistical (α, φ)-connections

In [1, p.181], the author defined the dual of the α-connection ${}^{(\alpha)}\nabla$ of given by the equation (3.1) as follows:

$${}^{D(\alpha)}\nabla_X Y = \nabla_X Y + \frac{\alpha}{2}\overline{C}(X,Y).$$

We easily say that ${}^{D(\alpha)}\nabla = {}^{-(\alpha)}\nabla$. Furthermore,

$$\begin{aligned}({}^{D(\alpha)}\nabla_X g)(Y,Z) &= -({}^{(\alpha)}\nabla_X g)(Y,Z) \\ &= -\alpha C(X,Y,Z)\end{aligned}$$

and

$$({}^{D(\alpha)}\nabla_X F)(Y) = 0,$$

that is, the dual α-connection ${}^{D(\alpha)}\nabla$ is statistical (α, φ)-connection and is named "dual statistical (α, φ)-connection". Also, in [1, p.182] (in Proposition 3.5), the author shows that the dual α-curvature tensor field ${}^{D(\alpha)}R$ of ${}^{D(\alpha)}\nabla$ is as follow:

$${}^{(\alpha)}R(X,Y,Z,W) = -\,{}^{D(\alpha)}R(X,Y,W,Z).$$

Then, we have

Theorem 5.1. *The dual α-curvature tensor field ${}^{D(\alpha)}R$ is a decomposable tensor field, i.e.,*

$$\Phi_\varphi\, {}^{(\alpha)}R = -(\Phi_\varphi\, {}^{D(\alpha)}R) = 0.$$

6. Conclusions

In this study, we define a special connection using the cubic form C on locally product Riemann manifold. We name this new connection as statistical (α, φ)-connection. We examine the curvature properties and give examples of this new connection. However, the cubic form C customized for the locally product Riemann manifold is made only for the product structure. This cubic form can also be studied on different special Riemann manifolds such that Kahler (or Anti-Kahler) manifold with complex structure E, $E^2 = -I$, Tangent (dual) manifold with tangent structure F, $F^2 = 0$ and Golden Riemann manifold with golden structure φ, $\varphi^2 = \varphi + I$, which is the most interesting structure lately.

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Conflict of interest

The author declares no conflict of interest in this paper.

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