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Research article

Lipschitz stability of an inverse problem for the Kawahara equation with damping

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Abstract: The aim of this paper is to establish a stability result regarding the inverse problem of retrieving the damping coefficient in Kawahara equation. We first establish an internal Carleman estimate for the linearized problem with the help of Dirichlet-Neumann type boundary conditions. Using the obtained Carleman estimate and the regularity of solutions for the Kawahara equation, we prove the Lipschitz type stability and uniqueness of the considered inverse problems.

Keywords: stability; inverse problems; Kawahara equation; Carleman estimate
Mathematics Subject Classification: 35K05, 35R30, 35Q53

1. Introduction

Let $\Omega := (0, 1)$, $T > 0$ be a fixed final time and $Q := \Omega \times (0, T)$. In this paper, we consider the following Kawahara equation [12, 15]

$$\begin{cases} y_t + y_{xxx} + \gamma y_{xxxxx} + yy_x + a(x)y = 0, & (x, t) \in Q, \\ y(x, 0) = y_0(x), & x \in \Omega, \\ y(0, t) = y(1, t) = y_x(0, t) = y_x(1, t) = 0, & t \in (0, T), \\ y_{xx}(1, t) = 0 & t \in (0, T), \end{cases} \tag{1.1}$$

where $\gamma < 0$, y_0 is the initial data and $a(x)$ is the damping coefficient. For simplicity we assume $\gamma = -1$. The Kawahara equation was first introduced by Kawahara [12] in 1972 to study the propagation of small-amplitude long waves that occur in various applications like, fluid dynamics, plasma physics, biology, and sociology. Further note that it is necessary to introduce the higher order effect of dispersion in order to balance the nonlinear effect when the coefficient of the third-order derivative in the KdV equation becomes zero or very small. The term $y_{xxx} + \gamma y_{xxxxx}$ is denoted as the conservative dispersive effect. The term $a(x)y$ plays a significant role in the designing of feedback damping mechanism and proving the exponential energy decay of solutions. Further, when the damping is localized, that is, it is effective only on a bounded subset of the given domain, the stabilization of the problem is more interesting (see, [15]). Note that we have also omitted the drift term u_x in the model (1.1) since it doesn't play any essential role in the study of Carleman estimate and inverse problems of the given system. For more detailed applications of this model, one may refer to [1, 7, 15].

In this paper, we focus on the reconstruction of a space-dependent damping coefficient of a zeroth order term based on Carleman estimate for fifth-order PDEs. This estimate was introduced by Carleman in 1939 for proofs of uniqueness results for ill-posed Cauchy problems. Carleman estimate and its theory were first proposed in the study of inverse problems, by Bukhgeim and Klibanov [4, 5]. In recent years, due to applications, inverse problems for higher-order partial differential equations have attained much more attention. In the literature, we could find numerous articles that deal with the inverse problems/controllability results for higher-order PDEs. For example, Glass et al. [8] established the local controllability to the trajectories of fifth-order Korteweg-de Vries equation (KdV) using the boundary controls. Gao [10] established a global Carleman estimate for the Kawahara equation by internal measurements and applied it to prove the unique continuation property and exponential decay of solutions. Recently, Mo Chen [7] studied the controllability to the trajectories for the Kawahara equation via Carleman estimate. Baudoin et al. [3] investigated the nonlinear inverse problem of retrieving the anti-diffusion coefficient from Kuramoto-Sivashinsky (K-S) equation. The same authors [2] established the inverse problem of reconstructing the principal coefficient of generalized Korteweg de Vries (KdV) equation. One may also look at [6] for the Carleman inequality of the classical KdV equation which is applied to study the null controllability problem. Gao [9] studied the inverse problem and the controllability to the trajectories for the K-S equation using internal measurement. A recent paper by Meléndez [13] discussed the inverse problem of retrieving the main coefficient in the K-S equation using the boundary measurement. For some of the other related results and applications of Carleman estimate, one can also refer to [16].

Apart from some of the literature mentioned above for inverse problems and controllability of higher-order systems, to the best of our knowledge, in the direction of inverse problems, there is no article available in the literature for recovering the unknown coefficient or source in the Kawahara equation. Thus we made an attempt to establish such a result in this paper.

Notations and Assumptions:

The following notations, function spaces and assumptions are often used throughout this article:

$$W^{5,\infty}(\Omega) := \{y \in L^\infty(\Omega); y_x, y_{xx}, y_{xxx}, y_{xxxx}, y_{xxxxx} \in L^\infty(\Omega)\},$$

$$W^{1,\infty}(0,T; W^{1,\infty}(\Omega)) = \{y, y_t, y_x, y_{xt} \in L^\infty(0,T; L^\infty(\Omega))\}$$

and

$$\mathcal{R}_T^k := C([0,T]; H^k(\Omega)) \text{ for } k \geq 0$$

$$\mathcal{P}_{0,T} := \mathcal{R}_T^0 \cap L^2(0,T; H^2(\Omega)),$$

with norm

$$\|y\|_{\mathcal{P}_{0,T}} := \|y\|_{\mathcal{R}_T^0} + \|y\|_{L^2(0,T;H^2(\Omega))}.$$

We also use function space $H^1(0,T;L^2(\Omega)) := \{y, y_t \in L^2(0,T;L^2(\Omega))\}$.

Assumption: 1.1.

(i) Let $\mathbb{K}_M$ be an M-bounded set in $W^{5,\infty}(\Omega)$, which is defined by

$$\mathbb{K}_M = \{a(x) \in W^{5,\infty}(\Omega) : \|a(x)\|_{W^{5,\infty}(\Omega)} \le M\}, \tag{1.2}$$

where M is any positive constant.

(ii) The initial data y_0 satisfies the compatibility conditions $y_0^{(i)}(x) = 0$ on $\partial\Omega$ for all $i = 0, 1, \cdots, 12$, where $y_0^{(i)} = \dfrac{d^i y_0}{dx^i}$.

The condition $y_0^{(i)}(x) = 0$ on $\partial\Omega$ for $i = 0, 1, 2$ is sufficient for the compatibility of the boundary conditions in (1.1) and the rest of the cases are needed only to prove the regularity of solutions given by Theorem 3.2.

2. Main results

The main part of our paper deals with the stability result for an inverse problem of the Kawahara equation which is explained in the following manner. Let $y(x,t)$ and $\tilde{y}(x,t)$ be the solutions to Kawahara equation (1.1) corresponding to the unknown coefficients $a(x)$ and $\tilde{a}(x)$ respectively. Define $u(x,t) := y(x,t) - \tilde{y}(x,t)$ and $f(x) := a(x) - \tilde{a}(x)$ so that u solves the equation

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + p(x,t)u_x + q(x,t)u = g(x,t), & (x,t) \in Q, \\ u(x,0) = 0, & x \in (0,1), \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = u_{xx}(1,t) = 0, & t \in (0,T), \end{cases} \tag{2.1}$$

where $p(x,t) := y(x,t)$, $q(x,t) = (a(x) + \tilde{y}_x(x,t))$, $r(x,t) := -\tilde{y}(x,t)$ and $g(x,t) = f(x)r(x,t)$.

The inverse problem can be stated as follows: *Is it possible to retrieve the damping coefficient $a = a(x)$ from the measurement of the solution u of Kawahara equation, namely certain spatial derivatives of u at time $\theta = T/2$ and the solution u on the subdomain $\omega \subset \Omega$?*

The answer is the following main result concerning the stability estimate of the Kawahara equation (1.1) in terms of internal measurements:

Theorem: 2.1. *Let $a \in \mathbb{K}_M$, $y_0 \in H^{10}(\Omega)$ and $\omega \subset \Omega$. Assume that $\inf_{x\in\Omega} |\tilde{y}(x,\theta)| \ge b > 0$, for some fixed $\theta \in (0,T)$. Then there exists a constant $C > 0$ depending on Ω, ω, T, M, b and y_0 such that*

$$\begin{aligned} \|a - \tilde{a}\|^2_{L^2(\Omega)} \le \; & C\Big(\|y - \tilde{y}\|^2_{H^1(0,T;L^2(\omega))} + \|y_{xxxx} - \tilde{y}_{xxxx}\|^2_{H^1(0,T;L^2(\omega))} \\ & + \|y(\cdot,\theta) - \tilde{y}(\cdot,\theta)\|^2_{H^5(0,1)}\Big). \end{aligned} \tag{2.2}$$

Remark: 2.1. *If we assume that $y(x,\theta) = \tilde{y}(x,\theta)$, for a.e. $x \in \Omega$, the internal stability estimate becomes*

$$\|a - \tilde{a}\|^2_{L^2(\Omega)} \le C\Big(\|y - \tilde{y}\|^2_{H^1(0,T;L^2(\omega))} + \|y_{xxxx} - \tilde{y}_{xxxx}\|^2_{H^1(0,T;L^2(\omega))}\Big).$$

The regularity of solutions for the direct problem (1.1) which is proved in Section 5 and a Carleman estimate stated in [10] (see, Theorem 1.1) play a vital role to prove Theorem 2.1. We will prove in Section 3 that if the initial condition $y_0 \in H^{10}(\Omega)$, then the solution of the main equation (1.1), namely $y \in C([0,T]; H^5(\Omega))$ and $y_t \in C([0,T]; H^2(\Omega)) \cap L^2(0,T; H^5(\Omega))$. This is mainly attained from the appropriate regularity of solution for the Kawahara equation. Since the right-hand side of (2.2) can be estimated as $y \in H^1(0,T; H^5(\Omega))$, we observe that the trajectories of solutions satisfying Theorem 2.1 is not an empty set.

The outline of the article is as follows: In Section 3, we establish the existence, uniqueness, and regularity of solutions of the Kawahara equation. In Section 4, we state a global Carleman estimate with internal observations for the fifth-order operator with Dirichlet-Neumann type boundary conditions. In Section 5, we establish a stability result for the Kawahara equation.

3. Existence, uniqueness and regularity of solutions

In this section, we establish the existence and uniqueness of solution of the Kawahara equation. Let $F \in L^1(0,T; H_0^1(\Omega))$. Consider the auxiliary problem

$$\begin{cases} y_t + y_{xxx} - y_{xxxxx} = F(x,t), & (x,t) \in Q, \\ y(x,0) = y_0(x), & x \in (0,1), \\ y(0,t) = y(1,t) = y_x(0,t) = 0, & t \in (0,T), \\ y_x(1,t) = y_{xx}(1,t) = 0, & t \in (0,T). \end{cases} \tag{3.1}$$

The existence and uniqueness of solution for the linear problem (3.1) can be obtained by using semigroup theory (see, [10, 15]).

Proposition: 3.1. *([15], Lemma 2.1) Let $y_0 \in L^2(\Omega)$ and $F \in L^1(0,T; H_0^1(\Omega))$. Then, the linearized equation* (3.1) *has a unique generalized solution $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$. Moreover, the solution satisfies*

$$\max_{t\in[0,T]} \|y(t)\|_{L^2(\Omega)} + \|y\|_{L^2(0,T;H^2(\Omega))} \le C\Big(\|F\|_{L^1(0,T;H_0^1(\Omega))} + \|y_0\|_{L^2(\Omega)}\Big). \tag{3.2}$$

The well-posedness of Eq (1.1) is obtained by using Proposition 3.1 and fixed point argument. The energy estimate (3.2) related to upper bound of y, which is obtained in terms of $F \in L^1(0,T; H_0^1(\Omega))$ plays a major role.

Theorem: 3.1. *Assume that $a(x) \in L^\infty(\Omega)$ and let $T > 0$ be given. Then for any $y_0 \in L^2(\Omega)$, there exists a unique solution $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$ of (1.1) satisfying*

$$\|y\|_{C([0,T];L^2(\Omega))\cap L^2(0,T;H^2(\Omega))} \le C(M,T)\Big(1 + \|y_0\|^2_{L^2(\Omega)}\Big). \tag{3.3}$$

Proof. The existence and uniqueness is proved in Theorem 2.1, [15] (see also [10]). Now, we only prove the global estimate (3.3).

Multiplying (1.1) by y, integrating over Ω and using Grönwall's inequality, we obtain

$$\max_{0\le t\le T} \|y(t)\|^2_{L^2(\Omega)} \le e^{CMT} \|y_0\|^2_{L^2(\Omega)}. \tag{3.4}$$

Similarly, multiplying (1.1) by xy, integrating over Ω, we get

$$\frac{d}{dt}\|x^{1/2}y(t)\|^2_{L^2(\Omega)} + 3\|y_x(t)\|^2_{L^2(\Omega)} + 5\|y_{xx}(t)\|^2_{L^2(\Omega)} \le \frac{2}{3}\|y(t)\|^3_{L^3(\Omega)} + 2M\|y(t)\|^2_{L^2(\Omega)}. \tag{3.5}$$

Using Agmon's inequality and Poincaré's inequality, we obtain

$$\begin{aligned} \|y(t)\|^3_{L^3(\Omega)} &\le \|y(t)\|_{L^\infty(\Omega)}\|y(t)\|^2_{L^2(\Omega)} \le \|y_x(t)\|^{1/2}_{L^2(\Omega)}\|y(t)\|^{1/2}_{L^2(\Omega)}\|y(t)\|^2_{L^2(\Omega)} \\ &\le C\|y_x(t)\|_{L^2(\Omega)}\|y(t)\|^2_{L^2(\Omega)} \le 3\|y_x(t)\|^2_{L^2(\Omega)} + C\|y(t)\|^4_{L^2(\Omega)}. \end{aligned} \tag{3.6}$$

Replacing (3.6) in (3.5) and using (3.4), we have

$$\frac{d}{dt}\|x^{1/2}y(t)\|^2_{L^2(\Omega)} + \|y_x(t)\|^2_{L^2(\Omega)} + 5\|y_{xx}(t)\|^2_{L^2(\Omega)} \le C(M,T)\left(\|y_0\|^2_{L^2(\Omega)} + \|y_0\|^4_{L^2(\Omega)}\right).$$

Integrating over $(0,t)$ to obtain

$$\|x^{1/2}y(t)\|^2_{L^2(\Omega)} + \|y_x\|^2_{L^2(0,T;L^2(\Omega))} + 5\|y_{xx}\|^2_{L^2(0,T;L^2(\Omega))} \le C(M,T)\left(\|y_0\|^2_{L^2(\Omega)} + \|y_0\|^4_{L^2(\Omega)}\right). \tag{3.7}$$

Note that from (3.7) and Poincaré's inequality, we also have

$$\|y\|^2_{L^2(0,T;L^2(\Omega))} \le C(M,T)\left(1 + \|y_0\|^4_{L^2(\Omega)}\right). \tag{3.8}$$

Adding (3.4), (3.7) and (3.8), one can complete the proof. □

To complete the main result of this paper, we prove the following regularity result.

Theorem: 3.2. *Assume that $a(x) \in L^\infty(\Omega)$ and let $T > 0$ be given. Further assume that Assumption 1.1-(ii) holds true. If $y_0 \in H^{10}(\Omega)$, then the solution $y \in C([0,T];H^5(\Omega))$, $y_t \in C([0,T];H^2(\Omega)) \cap L^2(0,T;H^5(\Omega))$ and satisfies the following:*

$$\|y\|_{C([0,T];H^5(\Omega))} \le \exp\Big\{\exp\Big\{C(M,T)\Big(1 + \|y_0\|^2_{H^{10}(\Omega)}\Big)\Big\}\Big\}, \tag{3.9}$$

and

$$\|y_t\|_{C([0,T];H^2(\Omega))\cap L^2(0,T;H^5(\Omega))} \le \exp\Big\{\exp\Big\{C(M,T)\Big(1 + \|y_0\|^2_{H^{10}(\Omega)}\Big)\Big\}\Big\}. \tag{3.10}$$

Proof. From (1.1), we have

$$\begin{aligned} &\|y_{xxxxx}\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad\le 4\left(\|y_t\|^2_{L^2(0,T;L^2(\Omega))} + \|yy_x\|^2_{L^2(0,T;L^2(\Omega))} + \|y_{xxx}\|^2_{L^2(0,T;L^2(\Omega))} + M^2\|y\|^2_{L^2(0,T;L^2(\Omega))}\right). \end{aligned} \tag{3.11}$$

Observe that by (3.11), we have

$$\|y\|^2_{L^2(0,T;H^5(\Omega))} \le C(M)\left(\|y_t\|^2_{L^2(0,T;L^2(\Omega))} + \|yy_x\|^2_{L^2(0,T;L^2(\Omega))} + \|y\|^2_{L^2(0,T;H^4(\Omega))}\right). \tag{3.12}$$

Using the continuous injection $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$, we get

$$\|yy_x\|^2_{L^2(0,T;L^2(\Omega))} \le C\|y\|^2_{C([0,T];L^2(\Omega))}\|y\|^2_{L^2(0,T;H^2(\Omega))}. \tag{3.13}$$

By Ehrling's lemma (see Theorem 7.30, [14]) for any $\epsilon > 0$ and $y \in L^2(0, T; H^5(\Omega))$, we obtain

$$\|y\|^2_{L^2(0,T;H^4(\Omega))} \leq \epsilon\|y\|^2_{L^2(0,T;H^5(\Omega))} + C(\epsilon)\|y\|^2_{L^2(0,T;L^2(\Omega))}. \tag{3.14}$$

Substituting (3.13) and (3.14) in (3.12) and choosing $\epsilon = 1/2C$, we get

$$\|y\|^2_{L^2(0,T;H^5(\Omega))} \leq C\left(\|y_t\|^2_{L^2(0,T;L^2(\Omega))} + \|y\|^2_{C([0,T];L^2(\Omega))}\|y\|^2_{L^2(0,T;H^2(\Omega))} + \|y\|^2_{L^2(0,T;L^2(\Omega))}\right). \tag{3.15}$$

Taking derivative of (1.1) with respect to time, we also have

$$\begin{aligned}\|y_{xxxxxt}\|^2_{L^2(0,T;L^2(\Omega))} \leq\ & 5\left(\|y_{tt}\|^2_{L^2(0,T;L^2(\Omega))} + \|yy_{xt}\|^2_{L^2(0,T;L^2(\Omega))} + \|y_t y_x\|^2_{L^2(0,T;L^2(\Omega))}\right.\\ &\left. + \|y_{xxxt}\|^2_{L^2(0,T;L^2(\Omega))} + M^2\|y_t\|^2_{L^2(0,T;L^2(\Omega))}\right).\end{aligned} \tag{3.16}$$

Applying the continuous injection $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$, we obtain

$$\|y_t y_x\|^2_{L^2(0,T;L^2(\Omega))} \leq C\|y_t\|^2_{C([0,T];L^2(\Omega))}\|y\|^2_{L^2(0,T;H^2(\Omega))}, \tag{3.17}$$

and

$$\|yy_{xt}\|^2_{L^2(0,T;L^2(\Omega))} \leq C\|y\|^2_{C([0,T];L^2(\Omega))}\|y_t\|^2_{L^2(0,T;H^2(\Omega))}. \tag{3.18}$$

By Ehrling's lemma, we get

$$\|y_t\|^2_{L^2(0,T;H^4(\Omega))} \leq \epsilon\|y_t\|^2_{L^2(0,T;H^5(\Omega))} + C(\epsilon)\|y_t\|^2_{L^2(0,T;L^2(\Omega))}, \tag{3.19}$$

for any $\epsilon > 0$. Once again follow the same process as we did above and using (3.17), (3.18) and (3.19), we obtain

$$\begin{aligned}\|y_t\|^2_{L^2(0,T;H^5(\Omega))} \leq\ & C(M)\left(\|y_{tt}\|^2_{L^2(0,T;L^2(\Omega))} + \|y_t\|^2_{C([0,T];L^2(\Omega))}\|y\|^2_{L^2(0,T;H^2(\Omega))}\right.\\ &\left. + \|y\|^2_{C([0,T];L^2(\Omega))}\|y_t\|^2_{L^2(0,T;H^2(\Omega))} + \|y_t\|^2_{L^2(0,T;L^2(\Omega))}\right).\end{aligned} \tag{3.20}$$

To estimate the right side of the above two inequalites (3.15) and (3.20), we need to establish the following auxiliary lemmas.

Lemma: 3.1. *Assume that $a(x) \in L^\infty(\Omega)$ and let $T > 0$ be given. If $y_0 \in H^5(\Omega)$, then the solution $y \in \mathcal{P}_{0,T}$ of (1.1) has time regularity such that $y_t \in \mathcal{P}_{0,T}$ and satisfies the following:*

$$\|y_t\|_{\mathcal{P}_{0,T}} \leq \exp\left\{C(M,T)\left(1 + \|y_0\|^2_{H^5(\Omega)}\right)\right\}. \tag{3.21}$$

Proof. Let us take a derivative of (1.1) with respect to time and set $\eta := y_t$. Then η satisfies the equation

$$\begin{cases}\eta_t + \eta_{xxx} - \eta_{xxxxx} + \eta y_x + y\eta_x + a(x)\eta = 0, & (x,t) \in Q,\\ \eta(x,0) = \eta_0(x), & x \in \Omega,\\ \eta(0,t) = \eta(1,t) = \eta_x(0,t) = \eta_x(1,t) = \eta_{xx}(1,t) = 0, & t \in (0,T),\end{cases} \tag{3.22}$$

where $\eta_0(x) := y_t(x,0)$. Notice that from (1.1), we have $y_t(x,0) = -y_0^{(3)}(x) + y_0^{(5)}(x) - y_0(x)y_0'(x) - a(x)y_0(x)$.
Multiplying (3.22) by η, integrating over Ω and apply Cauchy's inequality to obtain

$$\frac{d}{dt}\|\eta(t)\|^2_{L^2(\Omega)} + \eta^2_{xx}(0,t) \le C(M)\Big(\|y_x\|_{L^\infty(\Omega)} + 1\Big)\|\eta(t)\|^2_{L^2(\Omega)}. \tag{3.23}$$

Using Grönwall's inequality, the continuous injection $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and by Theorem 3.1, we attain

$$\max_{0\le t\le T}\|\eta(t)\|^2_{L^2(\Omega)} \le \exp\Big\{C(M,T)\Big(\|y_0\|^2_{L^2(\Omega)} + 1\Big)\Big\}\|\eta_0\|^2_{L^2(\Omega)}. \tag{3.24}$$

From the initial condition and the continuous injection $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we get

$$\begin{aligned}\|\eta_0\|_{L^2(\Omega)} &\le C(M)\Big(\|y_0\|_{H^3(\Omega)} + \|y_0\|_{H^5(\Omega)} + \|y_0\|_{H^2(\Omega)}\|y_0\|_{H^1(\Omega)} + \|y_0\|_{L^2(\Omega)}\Big)\\ &\le C(M)\Big(\|y_0\|_{H^5(\Omega)} + \|y_0\|^2_{H^5(\Omega)}\Big) \le C(M)\exp\Big\{\|y_0\|_{H^5(\Omega)}\Big\}.\end{aligned} \tag{3.25}$$

Replacing (3.25) in (3.24), we obtain

$$\|\eta\|^2_{C([0,T];L^2(\Omega))} \le \exp\Big\{C(M,T)\Big(1 + \|y_0\|^2_{H^5(\Omega)}\Big)\Big\}. \tag{3.26}$$

Once again multiplying (3.22) by $x\eta$ and integrating over Ω, we obtain

$$\begin{aligned}&\frac{d}{dt}\|x^{1/2}\eta(t)\|^2_{L^2(\Omega)} + 3\|\eta_x(t)\|^2_{L^2(\Omega)} + 5\|\eta_{xx}(t)\|^2_{L^2(\Omega)}\\ &\qquad\le C(M)\Big(\|y_x\|_{L^\infty(\Omega)} + \|y\|_{L^\infty(\Omega)} + 1\Big)\|\eta(t)\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.27}$$

Integrating (3.27) over $(0,t)$ and applying Theorem 3.1, (3.25) and (3.26), we get

$$\|x^{1/2}\eta(t)\|^2_{L^2(\Omega)} + \|\eta_x\|^2_{L^2(0,T;L^2(\Omega))} + \|\eta_{xx}\|^2_{L^2(0,T;L^2(\Omega))} \le \exp\Big\{C(M,T)\Big(1 + \|y_0\|^2_{H^5(\Omega)}\Big)\Big\}. \tag{3.28}$$

Therefore, putting together (3.26) and (3.28), one can get (3.21).

Finally, we show that the solution η of (3.22) is equal to y_t, where y is the solution of (1.1).
Let us define

$$y(x,t) = \int_0^t \eta(x,\tau)d\tau + y_0(x). \tag{3.29}$$

It leads to the following:

$$\begin{aligned}&y_t(x,t) + y_{xxx}(x,t) - y_{xxxxx}(x,t) + y(x,t)y_x(x,t) + a(x)y(x,t)\\ &= \eta(x,t) + \int_0^t \Big(\eta_{xxx}(x,\tau) - \eta_{xxxxx}(x,\tau) + \eta(x,\tau)y_x(x,\tau) + y(x,\tau)\eta_x(x,\tau) + a(x)\eta(x,\tau)\Big)d\tau\\ &\quad + y_0^{(3)}(x) - y_0^{(5)}(x) + y_0(x)y_0^{(1)}(x) + a(x)y_0(x)\\ &= \eta(x,t) - \int_0^t \eta_\tau(x,\tau)d\tau + y_0^{(3)}(x) - y_0^{(5)}(x) + y_0(x)y_0^{(1)}(x) + a(x)y_0(x)\\ &= \eta(x,0) + y_0^{(3)}(x) - y_0^{(5)}(x) + y_0(x)y_0^{(1)}(x) + a(x)y_0(x) = 0.\end{aligned}$$

Notice that when η solves the system (3.22), y solves (1.1). Moreover from (3.29), we get the initial condition $y(x,0) = y_0(x)$ and using the compatibility condition, we obtain

$$y(0,t) = \int_0^t \eta(0,\tau)d\tau + y_0(0) = 0.$$

Similarly, one can get the remaining boundary conditions $y(1,t) = 0$, $y_x(0,t) = 0$, $y_x(1,t) = 0$ and $y_{xx}(1,t) = 0$. This completes the proof. □

Lemma: 3.2. *Assume that $a(x) \in L^\infty(\Omega)$ and let $T > 0$ be given. If $y_0 \in H^{10}(\Omega)$, then the solution $y \in \mathcal{P}_{0,T}$ of (1.1) has time regularity such that $y_{tt} \in \mathcal{P}_{0,T}$ and satisfies the following:*

$$\|y_{tt}\|_{\mathcal{P}_{0,T}} \leq \exp\Big\{\exp\Big\{C(M,T)\Big(1 + \|y_0\|^2_{H^{10}(\Omega)}\Big)\Big\}\Big\}. \tag{3.30}$$

Proof. Take a derivative of (3.22) with respect to time and let us set $z := \eta_t$. Then, we have

$$\begin{cases} z_t + z_{xxx} - z_{xxxxx} + yz_x + y_x z + a(x)z = -2\eta\eta_x, & (x,t) \in Q, \\ z(x,0) = z_0(x), & x \in \Omega, \\ z(0,t) = z(1,t) = z_x(0,t) = z_x(1,t) = z_{xx}(1,t) = 0, & t \in (0,T), \end{cases} \tag{3.31}$$

where $z_0(x) := \eta_t(x,0)$. Observe that from (3.22), we have $\eta_t(x,0) = -\eta_0^{(3)}(x) + \eta_0^{(5)}(x) - \eta_0(x)y_0'(x) - y_0(x)\eta_0'(x) - a(x)\eta_0(x)$.

Multiplying (3.31) by z, integrating over Ω and applying Cauchy's inequality, we obtain

$$\frac{d}{dt}\|z(t)\|^2_{L^2(\Omega)} + z^2_{xx}(0,t) \leq C(M)\Big(\|y_x\|_{L^\infty(\Omega)} + \|\eta_x\|^2_{L^\infty(\Omega)} + 1\Big)\|z(t)\|^2_{L^2(\Omega)} + \|\eta(t)\|^2_{L^2(\Omega)}. \tag{3.32}$$

Using Grönwall's inequality, the continuous injection $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$, by Theorem 3.1 and Lemma 3.1, we attain

$$\max_{0\leq t\leq T} \|z(t)\|^2_{L^2(\Omega)} \leq \exp\Big\{\exp\Big\{C(M,T)\Big(1 + \|y_0\|^2_{H^{10}(\Omega)}\Big)\Big\}\Big\}, \tag{3.33}$$

where we have used

$$\begin{aligned} \|z_0\|_{L^2(\Omega)} &\leq C(M)\Big(\|\eta_0\|_{H^5(\Omega)} + \|\eta_0\|_{L^2(\Omega)}\|y_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\Omega)}\|y_0\|_{H^1(\Omega)}\Big) \\ &\leq C(M)\Big(\|\eta_0\|_{H^5(\Omega)} + \|\eta_0\|^2_{H^5(\Omega)} + \|y_0\|^2_{H^2(\Omega)}\Big) \end{aligned} \tag{3.34}$$

and

$$\|\eta_0\|_{H^5(\Omega)} \leq C(M)\Big(\|y_0\|_{H^{10}(\Omega)} + \|y_0\|^2_{H^{10}(\Omega)}\Big) \leq C(M)\exp\Big\{\|y_0\|_{H^{10}(\Omega)}\Big\}. \tag{3.35}$$

Multiplying (3.31) by xz, integrating over Ω and using Cauchy's inequality to get

$$\begin{aligned} &\frac{d}{dt}\|x^{1/2}z(t)\|^2_{L^2(\Omega)} + 3\|z_x(t)\|^2_{L^2(\Omega)} + 5\|z_{xx}(t)\|^2_{L^2(\Omega)} \\ &\quad \leq C(M)\Big(\|\eta_x\|^2_{L^\infty(\Omega)} + \|y_x\|_{L^\infty(\Omega)} + \|y\|_{L^\infty(\Omega)} + 1\Big)\|z(t)\|^2_{L^2(\Omega)} + \|\eta(t)\|^2_{L^2(\Omega)}. \end{aligned}$$

Integrating over $(0,t)$ to obtain

$$\begin{aligned}&\|x^{1/2}z(t)\|^2_{L^2(\Omega)} + \|z_x\|^2_{L^2(0,T;L^2(\Omega))} + \|z_{xx}\|^2_{L^2(0,T;L^2(\Omega))}\\&\quad\le C(M)\Big(\|\eta\|^2_{L^2(0,T;H^2(\Omega))} + \sqrt{T}\|y\|_{L^2(0,T;H^2(\Omega))} + T\Big)\|z\|^2_{C([0,T];L^2(\Omega))} + \|\eta\|^2_{L^2(0,T;L^2(\Omega))} + \|z_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Using Theorem 3.1, Lemma 3.1, (3.33), (3.34) and $x \le e^x$ for all $x \in \mathbb{R}$, we get

$$\begin{aligned}&\|x^{1/2}z(t)\|^2_{L^2(\Omega)} + \|z_x\|^2_{L^2(0,T;L^2(\Omega))} + \|z_{xx}\|^2_{L^2(0,T;L^2(\Omega))}\\&\qquad\le \exp\Big\{\exp\Big\{C(M,T)\Big(1+\|y_0\|^2_{H^{10}(\Omega)}\Big)\Big\}\Big\}.\end{aligned}\tag{3.36}$$

Similar to Lemma 3.1, we can show the equality of solutions z and y_{tt}. The proof is thus completed. □

Use Theorem 3.1 and Lemma 3.1 in (3.15) to obtain

$$\|y\|^2_{L^2(0,T;H^5(\Omega))} \le \exp\Big\{C(M,T)\Big(1+\|y_0\|^2_{H^5(\Omega)}\Big)\Big\}.\tag{3.37}$$

Similarly, applying Theorem 3.1, Lemma 3.1 and Lemma 3.2 in (3.20), we get

$$\|y_t\|^2_{L^2(0,T;H^5(\Omega))} \le \exp\Big\{\exp\Big\{C(M,T)\Big(1+\|y_0\|^2_{H^{10}(\Omega)}\Big)\Big\}\Big\}.\tag{3.38}$$

Now, adding (3.37) and (3.38) to get

$$\|y\|^2_{H^1(0,T;H^5(\Omega))} \le \exp\Big\{\exp\Big\{C(M,T)\Big(1+\|y_0\|^2_{H^{10}(\Omega)}\Big)\Big\}\Big\}.\tag{3.39}$$

Using (3.39) and the continuous injection $H^1(0,T;H^5(\Omega)) \hookrightarrow C([0,T];H^5(\Omega))$, one can obtain (3.9).

Multiplying (3.22) by $-\eta_{xt}$ and integrating over Ω, we get

$$\begin{aligned}\frac{d}{dt}\|\eta_{xx}(t)\|^2_{L^2(\Omega)} \;\le\;& C(M)\Big(\Big(\|y_x(t)\|^2_{L^\infty(\Omega)}+1\Big)\|\eta(t)\|^2_{L^2(\Omega)} + \|y(t)\|^2_{L^2(\Omega)}\|\eta_x(t)\|^2_{L^\infty(\Omega)}\\&+\|\eta_{xt}(t)\|^2_{L^2(\Omega)} + \|\eta_{xxxxx}(t)\|^2_{L^2(\Omega)}\Big).\end{aligned}$$

Using the continuous embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$, integrating over $(0,t)$, applying Theorem 3.1, Lemma 3.1, Lemma 3.2 and (3.38), we obtain

$$\|\eta_{xx}(t)\|^2_{L^2(\Omega)} \le \exp\Big\{\exp\Big\{C(M,T)\Big(1+\|y_0\|^2_{H^{10}(\Omega)}\Big)\Big\}\Big\}.$$

By the help of Poincaré's inequality, we also estimate $\|\eta_x(t)\|^2_{L^2(\Omega)}$ and again by Lemma 3.1 together with (3.38), we get (3.10). This completes the proof of Theorem 3.2. □

4. Carleman estimate

In this section, we state an internal type Carleman estimate for the linearized Kawahara equation. We need to introduce suitable weight functions which will be useful for obtaining the Carleman estimates.

4.1. Internal Carleman estimate

Let $\omega \subset \Omega$ be an arbitrary open subset. Let us assume $\psi = \psi(x) \in C^\infty(\bar{\Omega})$ be the function that satisfies

$$\begin{cases} \psi(x) > 0 \ \ \forall x \in \Omega, \ \ \psi(x) = 0 \ \ \forall x \in \partial\Omega, \ \ \|\psi\|_{C(\bar{\Omega})} = 1 \\ |\psi_x(x)| > 0 \ \ \ \forall x \in \bar{\Omega} \setminus \omega \text{ and } \psi_x(0) > 0, \psi_x(1) < 0. \end{cases} \tag{4.1}$$

For the existence of such a function ψ one can refer to [10]. Let λ be a sufficiently large positive constant. We define ϕ and α for any $\lambda > 1$ and $t \in (0, T)$:

$$\phi(x,t) = e^{\lambda(\psi(x)+3)}/\beta(t), \qquad \alpha(x,t) = \left(e^{\lambda(\psi(x)+3)} - e^{5\lambda}\right)/\beta(t), \tag{4.2}$$

where $\beta(t) = t(T-t)$. Furthermore, in order to prove the main results, we may need the following estimates for the functions ϕ and α :

$$|\phi_t| \le CT\phi^2, \ \ |\alpha_t| \le CT\phi^2 \text{ and } |\alpha_{tt}| \le CT^2\phi^3. \tag{4.3}$$

Further note that

$$\phi_x = \lambda\phi\psi_x, \quad \alpha_x = \lambda\phi\psi_x \text{ and } \phi^{-1} \le (T/2)^2. \tag{4.4}$$

Now let us state an interior Carleman estimate for the linear operator $\mathcal{L}y := y_t - y_{xxxxx}$. Here, we set

$$J := \Big\{y \in L^2(0,T;H^5(\Omega)) \mid y(0,t) = y(1,t) = y_x(0,t) = y_x(1,t) = y_{xx}(1,t) = 0, \\ \mathcal{L}y \in L^2(0,T;L^2(\Omega))\Big\}.$$

Proposition: 4.1. *([10], Theorem 1.1) Let ψ, ϕ and α be defined as in (4.1)-(4.2). Then there exist four constants $s_0 > 0$, $\lambda_0 > 1$, $C_0 > 0$ and $C_1 > 0$, such that for $\lambda = \lambda_0$ and for all $s \ge s_0 := C_0(T + T^2)$, the following inequality holds for all $y \in J$:*

$$\begin{aligned} & s^9 \iint_Q \phi^9 e^{2s\alpha}|y|^2 dxdt + s^7 \iint_Q \phi^7 e^{2s\alpha}|y_x|^2 dxdt + s^5 \iint_Q \phi^5 e^{2s\alpha}|y_{xx}|^2 dxdt \\ & \quad + s^3 \iint_Q \phi^3 e^{2s\alpha}|y_{xxx}|^2 dxdt + s \iint_Q \phi e^{2s\alpha}|y_{xxxx}|^2 dxdt \\ & \le \ C_1 \left(\iint_Q e^{2s\alpha}|\mathcal{L}y|^2 dxdt + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|y|^2 dxdt + s \iint_{Q_\omega} \phi e^{2s\alpha}|y_{xxxx}|^2 dxdt \right), \end{aligned} \tag{4.5}$$

where $Q_\omega := \omega \times (0, T)$.

Now we apply the above Carleman estimate to get the similar estimate for the linearized equation (2.1) by using same weight functions as defined in (4.1)-(4.2).

Theorem: 4.1. *Let ψ, ϕ and α be defined as in (4.1)-(4.2). Then there exist constants $s_0^* > 0$, $\lambda_0 > 1$, and $C > 0$, such that for $\lambda \ge \lambda_0$ and for all $s \ge s_0^*$, the following inequality holds for all $u \in J$:*

$$\mathcal{M}(u) \le C\left(\iint_Q e^{2s\alpha}|g|^2 dxdt + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|u|^2 dxdt + s \iint_{Q_\omega} \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt \right), \tag{4.6}$$

where u is the solution of the system (2.1) *and*

$$\begin{aligned}\mathcal{M}(u) \;&:=\; s^{-1}\iint_Q \phi^{-1}e^{2s\alpha}|\mathcal{L}_1 u|^2 dxdt + s^9\iint_Q \phi^9 e^{2s\alpha}|u|^2 dxdt + s^7\iint_Q \phi^7 e^{2s\alpha}|u_x|^2 dxdt\\ &\quad + s^5\iint_Q \phi^5 e^{2s\alpha}|u_{xx}|^2 dxdt + s^3\iint_Q \phi^3 e^{2s\alpha}|u_{xxx}|^2 dxdt + s\iint_Q \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt.\end{aligned}$$

Proof. Let us define the unknown function $u(x,t) = e^{-s\alpha}w(x,t)$ for the operator $\mathcal{L}u = u_t - u_{xxxxx}$ to write the following resulting equation

$$\mathcal{L}_1 w + \mathcal{L}_2 w + \mathcal{L}_3 w = e^{s\alpha}\mathcal{L}(e^{-s\alpha}w) \tag{4.7}$$

where

$$\begin{aligned}\mathcal{L}_1 w &= w_t - w_{xxxxx} - 10s^2\alpha_x^2 w_{xxx} - 30s^2\alpha_x\alpha_{xx}w_{xx} - 5s^4\alpha_x^4 w_x - 6s^4\alpha_x^3\alpha_{xx}w\\ \mathcal{L}_2 w &= s^5\alpha_x^5 w + 10s^3\alpha_x^3 w_{xx} + 5s\alpha_x w_{xxxx}\\ \mathcal{L}_3 w &= s\alpha_t w - 10s^3\alpha_x^2\alpha_{xxx}w - 15s^3\alpha_x\alpha_{xx}^2 w - 10s^2\alpha_{xx}\alpha_{xxx}w - 5s^2\alpha_x\alpha_{xxxx}w\\ &\quad - s\alpha_{xxxxx}w - 4s^4\alpha_x^3\alpha_{xx}w - 30s^3\alpha_x^2\alpha_{xx}w_x - 20s^2\alpha_x\alpha_{xxx}w_x - 15s^2\alpha_{xx}^2 w_x\\ &\quad - 5s\alpha_{xxxx}w_x - 10s\alpha_{xxx}w_{xx} - 10s\alpha_{xx}w_{xxx}.\end{aligned}$$

The relation (4.7) will play a major role to get the following Carleman estimate for the operator $\mathcal{L}$ defined above. By Proposition 4.1 and the operator $\mathcal{L}u = u_t - u_{xxxxx} = g - u_{xxx} - pu_x - qu$ from (2.1), we get

$$\begin{aligned} & s^9\iint_Q \phi^9 e^{2s\alpha}|u|^2 dxdt + s^7\iint_Q \phi^7 e^{2s\alpha}|u_x|^2 dxdt + s^5\iint_Q \phi^5 e^{2s\alpha}|u_{xx}|^2 dxdt\\ &\quad + s^3\iint_Q \phi^3 e^{2s\alpha}|u_{xxx}|^2 dxdt + s\iint_Q \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt\\ &\le C_1\left(\iint_Q e^{2s\alpha}|g - u_{xxx} - pu_x - qu|^2 dxdt + s^9\iint_{Q_\omega} \phi^9 e^{2s\alpha}|u|^2 dxdt + s\iint_{Q_\omega} \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt\right).\end{aligned} \tag{4.8}$$

Now, we find the bound for the first term of the right side of (4.8): Use Theorem 3.2 and the Sobolev embedding $L^\infty(0,T;H^2(\Omega)) \hookrightarrow L^\infty(0,T;W^{1,\infty}(\Omega))$ to get the regularity results that $y \in L^\infty(0,T;W^{1,\infty}(\Omega))$ and $\tilde{y} \in L^\infty(0,T;W^{1,\infty}(\Omega))$. It shows that $||p||_{L^\infty(0,T;L^\infty(\Omega))}$ and $||q||_{L^\infty(0,T;L^\infty(\Omega))}$ are bounded by constants depending on Ω, T, M and y_0. It leads to the estimate

$$\begin{aligned}&\iint_Q e^{2s\alpha}|g - u_{xxx} - pu_x - qu|^2 dxdt\\ &\quad \le C\left(\iint_Q e^{2s\alpha}|g|^2 dxdt + \iint_Q e^{2s\alpha}\Big(|u|^2 + |u_x|^2 + |u_{xxx}|^2\Big)dxdt\right).\end{aligned} \tag{4.9}$$

Substitute (4.9) in (4.8), we have

$$s^9\iint_Q \phi^9 e^{2s\alpha}|u|^2 dxdt + s^7\iint_Q \phi^7 e^{2s\alpha}|u_x|^2 dxdt + s^5\iint_Q \phi^5 e^{2s\alpha}|u_{xx}|^2 dxdt$$

$$
\begin{aligned}
&+s^3 \iint_Q \phi^3 e^{2s\alpha}|u_{xxx}|^2 dxdt + s \iint_Q \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt \\
\leq\ & C\left(\iint_Q e^{2s\alpha}|g|^2 dxdt + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|u|^2 dxdt + s \iint_{Q_\omega} \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt\right),
\end{aligned} \tag{4.10}
$$

for any $s \geq s_1 := \max\{s_0, CT^2\}$.
Using (4.7) along with the estimates on weight functions (4.3) and (4.4), we have

$$
s^{-1} \iint_Q \phi^{-1}|\mathcal{L}_1 w|^2 dxdt \leq C s^{-1} \iint_Q \phi^{-1}\left(|e^{s\alpha}\mathcal{L}u|^2 + |\mathcal{L}_2 w|^2 + |\mathcal{L}_3 w|^2\right) dxdt, \tag{4.11}
$$

where

$$
\begin{aligned}
s^{-1} \iint_Q \phi^{-1}|\mathcal{L}_2 w|^2 dxdt \ \leq\ & C\left(s^9 \iint_Q \phi^9 |w|^2 dxdt + s^5 \iint_Q \phi^5 |w_{xx}|^2 dxdt\right. \\
&\left. + s \iint_Q \phi |w_{xxxx}|^2 dxdt\right),
\end{aligned}
$$

and

$$
\begin{aligned}
s^{-1} \iint_Q \phi^{-1}|\mathcal{L}_3 w|^2 dxdt \ \leq\ & s^9 \iint_Q \phi^9 |w|^2 dxdt + s^7 \iint_Q \phi^7 |w_x|^2 dxdt \\
& + s^5 \iint_Q \phi^5 |w_{xx}|^2 dxdt + s^3 \iint_Q \phi^3 |w_{xxx}|^2 dxdt,
\end{aligned}
$$

for any $s \geq s_2 = CT(T + T^{3/4})$. Now, we go back to the original variable using the transformation $w(x,t) = e^{s\alpha}u(x,t)$ as follows:

$$
\begin{aligned}
& s^{-1} \iint_Q \phi^{-1} e^{2s\alpha}|\mathcal{L}_1 u|^2 dxdt \\
\leq\ & C\left(s^{-1} \iint_Q \phi^{-1} e^{2s\alpha}|\mathcal{L}u|^2 dxdt + s^9 \iint_Q \phi^9 e^{2s\alpha}|u|^2 dxdt + s^7 \iint_Q \phi^7 e^{2s\alpha}|u_x|^2 dxdt\right. \\
&\left. + s^5 \iint_Q \phi^5 e^{2s\alpha}|u_{xx}|^2 dxdt + s^3 \iint_Q \phi^3 e^{2s\alpha}|u_{xxx}|^2 dxdt + s \iint_Q \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt\right),
\end{aligned} \tag{4.12}
$$

for any $s \geq s_2$. Using (4.9) and putting together (4.10) and (4.12), we obtain

$$
\mathcal{M}(u) \leq C\left(\iint_Q e^{2s\alpha}|g|^2 dxdt + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|u|^2 dxdt + s \iint_{Q_\omega} \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt\right), \tag{4.13}
$$

for any $s \geq s_0^* := \max\{s_1, s_2, CT^2\}$. This concludes the proof. □

5. Stability results

In this section, we establish a stability result for the Kawahara equation (1.1) which will be proved using the ideas of [4, 11]. To prove the stability result of this paper, we need to deduce a Carleman estimate for the variant of the operator $\mathcal{L}$ in Proposition 4.1. Here we use the data at some time $\theta = T/2$. Moreover, we write for simplicity, $p(x,\theta) := p(\theta)$, $q(x,\theta) := q(\theta)$, $r(x,\theta) := r(\theta)$ and $u(x,\theta) := u(\theta)$.

Proof of Theorem 2.1: First we take a differentiation of the Eq (2.1) with respect to time. The function, $v(x,t) := u_t(x,t)$ satisfies the following equation

$$\begin{cases} v_t + v_{xxx} - v_{xxxxx} + pv_x + qv = G(x,t), & (x,t) \in Q, \\ v(x,0) = f(x)r(x,0), & x \in \Omega, \\ v(0,t) = v(1,t) = v_x(0,t) = v_x(1,t) = v_{xx}(1,t) = 0, & t \in (0,T), \end{cases} \tag{5.1}$$

where $G(x,t) = fr_t - p_t u_x - q_t u$. Then we apply the Carleman estimate derived in Theorem 4.1 for (5.1) to obtain

$$\begin{aligned} &\mathcal{M}(v) \\ &\leq C\left(\iint_Q e^{2s\alpha}|G(x,t)|^2 dxdt + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|v|^2 dxdt + s\iint_{Q_\omega} \phi e^{2s\alpha}|v_{xxxx}|^2 dxdt\right), \end{aligned} \tag{5.2}$$

for all $v \in J$ and for any $s \geq s_0^*$, where $\mathcal{M}(\cdot)$ is defined in (4.6).

To estimate the first term on the right side of (5.2), note that by Theorem 3.2, $y, \tilde{y},\ y_t, \tilde{y}_t \in L^\infty(0,T;H^2(\Omega))$ and the Sobolev embedding $L^\infty(0,T;H^2(\Omega)) \hookrightarrow L^\infty(0,T;W^{1,\infty}(\Omega))$ we obtain the regularity results that $y \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$ and $\tilde{y} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$. It shows that $\|r_t\|_{L^\infty(0,T;L^\infty(\Omega))}$, $\|p_t\|_{L^\infty(0,T;L^\infty(\Omega))}$ and $\|q_t\|_{L^\infty(0,T;L^\infty(\Omega))}$ are bounded by constants depending on Ω, T, M and y_0, which lead to $G \in L^2(0,T,L^2(\Omega))$. Thus we have

$$\begin{aligned} &\iint_Q e^{2s\alpha}|G(x,t)|^2 dxdt \\ &\leq 3\left(\|r_t\|^2_{L^\infty(Q)} \iint_Q e^{2s\alpha}|f|^2 dxdt + \|q_t\|^2_{L^\infty(Q)} \iint_Q e^{2s\alpha}|u|^2 dxdt + \|p_t\|^2_{L^\infty(Q)} \iint_Q e^{2s\alpha}|u_x|^2 dxdt\right). \end{aligned} \tag{5.3}$$

We substitute (5.3) in (5.2) to obtain

$$\begin{aligned} \mathcal{M}(v) \leq\ & C\left(\iint_Q e^{2s\alpha}|f|^2 dxdt + s^9 \iint_Q \phi^9 e^{2s\alpha}|u|^2 dxdt + s^7 \iint_Q \phi^7 e^{2s\alpha}|u_x|^2 dxdt\right. \\ & \left. + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|v|^2 dxdt + s\iint_{Q_\omega} \phi e^{2s\alpha}|v_{xxxx}|^2 dxdt\right), \end{aligned} \tag{5.4}$$

for any sufficiently large $s \geq \tilde{s}_0 = \max\{CT^2, s_0^*\}$.

Here, we use Theorem 4.1 for the terms $s^9 \iint_Q \phi^9 e^{-2s\alpha}|u|^2 dxdt$ and $s^7 \iint_Q \phi^7 e^{-2s\alpha}|u_x|^2 dxdt$ which occur on the right side of (5.4) to get

$$\begin{aligned} \mathcal{M}(v) \leq\ & C\left(\iint_Q e^{2s\alpha}|f|^2 dxdt + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|u|^2 dxdt + s\iint_{Q_\omega} \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt\right. \\ & \left. + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|v|^2 dxdt + s\iint_{Q_\omega} \phi e^{2s\alpha}|v_{xxxx}|^2 dxdt\right), \end{aligned} \tag{5.5}$$

where we have used $\|r\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C$ and for any $s \geq \tilde{s}_0$.

Next we acquire an estimate for the solution v of the transformed equation (5.1) at $t = \theta$ by introducing an integral with θ and the operator $\mathcal{L}_1 w$ as follows:

$$\begin{aligned} I &= \int_0^\theta \int_\Omega (\mathcal{L}_1 w) w dxdt \\ &= \iint_{Q_\theta} (w_t - w_{xxxxx} - 10s^2\alpha_x^2 w_{xxx} - 30s^2\alpha_x\alpha_{xx}w_{xx} - 5s^4\alpha_x^4 w_x - 6s^4\alpha_x^3\alpha_{xx}w) w dxdt \\ &= \frac{1}{2}\int_\Omega |w(x,\theta)|^2 dx + \mathcal{N}, \end{aligned} \tag{5.6}$$

where $Q_\theta := \Omega \times (0,\theta)$. Here $\mathcal{L}_1 w$ is the operator that would arise if we derive Carleman estimate for $\mathcal{L}v = v_t - v_{xxxxx}$ as we have computed in (4.7) and

$$\begin{aligned} \mathcal{N} &:= \iint_{Q_\theta} \left(-ww_{xxxxx} - 10s^2\alpha_x^2 ww_{xxx} - 30s^2\alpha_x\alpha_{xx}ww_{xx} - 5s^4\alpha_x^4 ww_x - 6s^4\alpha_x^3\alpha_{xx}w^2\right) dxdt \\ &= -\iint_{Q_\theta} w_{xx}w_{xxx}dxdt - 10s^2\iint_{Q_\theta} \alpha_x\alpha_{xx}ww_{xx}dxdt - 10s^2\iint_{Q_\theta}\alpha_x\alpha_{xx}|w_x|^2dxdt \\ &\quad +4s^4\iint_{Q_\theta}\alpha_x^3\alpha_{xx}|w|^2dxdt. \end{aligned} \tag{5.7}$$

Using (4.3) and (4.4), we obtain

$$\begin{aligned} \mathcal{N} &\leq C\left(s^4\iint_{Q_\theta}\phi^4|w|^2dxdt + s^2\iint_{Q_\theta}\phi^2|w_x|^2dxdt + s^2\iint_{Q_\theta}\phi^2|w_{xx}|^2dxdt \right.\\ &\quad \left. + \iint_{Q_\theta}|w_{xxx}|^2dxdt\right), \end{aligned} \tag{5.8}$$

for any $s \geq CT^2$.

Next, taking $\mathcal{L}_1 w = \mathcal{L}_1(e^{s\alpha}v) = e^{s\alpha}\widehat{\mathcal{L}_1}v$ into account, we apply Hölder's inequality followed by Cauchy-Schwarz inequality to yield the following

$$\begin{aligned} |I| &= \left|\iint_{Q_\theta} e^{2s\alpha}(\widehat{\mathcal{L}_1}v)vdxdt\right| \\ &\leq C(\Omega)T^6s^{-3}\left[s^{-3}\iint_Q \phi^{-3}e^{2s\alpha}|\widehat{\mathcal{L}_1}v|^2dxdt + s^9\iint_Q\phi^9e^{2s\alpha}|v|^2dxdt\right], \end{aligned} \tag{5.9}$$

where a simple computation yields

$$\begin{aligned} \widehat{\mathcal{L}_1}v &= \mathcal{L}_1 v + \left(s\alpha_t - 16s^5\alpha_x^5 - 70s^4\alpha_x^3\alpha_{xx} - 20s^3\alpha_x^2\alpha_{xxx} - 45s^3\alpha_x\alpha_{xx}^2\right.\\ &\quad \left.-10s^2\alpha_{xx}\alpha_{xxx} - 5s^2\alpha_x\alpha_{xxxx} - s\alpha_{xxxxx}\right)v - \left(35s^4\alpha_x^4 + 120s^3\alpha_x^2\alpha_{xx}\right.\\ &\quad \left.+20s^2\alpha_x\alpha_{xxx} + 15s^2\alpha_{xx}^2 + 5s\alpha_{xxxx}\right)v_x - \left(20s^3\alpha_x^3 + 30s^2\alpha_x\alpha_{xxx}\right.\\ &\quad \left.+10s\alpha_{xxx}\right)v_{xx} - 10\left(s^2\alpha_x^2 + s\alpha_{xx}\right)v_{xxx} - 5s\alpha_x v_{xxxx}. \end{aligned}$$

We also have

$$
\begin{aligned}
& s^{-3} \iint_Q \phi^{-3} e^{2s\alpha} |\widehat{\mathcal{L}_1} v|^2 dxdt \\
& \quad \leq C \left(s^{-3} T^4 \iint_Q \phi^{-1} e^{2s\alpha} |\mathcal{L}_1 v|^2 dxdt + s^9 \iint_Q \phi^9 e^{2s\alpha} |v|^2 dxdt + s^7 \iint_Q \phi^7 e^{2s\alpha} |v_x|^2 dxdt \right. \\
& \quad \left. + s^5 \iint_Q \phi^5 e^{2s\alpha} |v_{xx}|^2 dxdt + s^3 \iint_Q \phi^3 e^{2s\alpha} |v_{xxx}|^2 + s \iint_Q \phi e^{2s\alpha} |v_{xxxx}|^2 dxdt \right),
\end{aligned}
$$

for any $s \geq \tilde{s}_1 = CT(T + T^{3/4})$.

Now we are ready to obtain the stability estimate for the unknown coefficient $a(x)$ using the relation $w(x,t) = e^{s\alpha} v(x,t)$, (5.6), (5.8) and (5.9) as follows

$$
\begin{aligned}
& \int_\Omega |v(x,\theta)|^2 e^{2s\alpha(x,\theta)} dx = 2I - 2\mathcal{N} \\
\leq \; & C(\Omega) T^6 s^{-3} \left[s^{-3} T^4 \iint_Q \phi^{-1} e^{2s\alpha} |\mathcal{L}_1 v|^2 dxdt + s^9 \iint_Q \phi^9 e^{2s\alpha} |v|^2 dxdt + s^7 \iint_Q \phi^7 e^{2s\alpha} |v_x|^2 dxdt \right. \\
& \left. + s^5 \iint_Q \phi^5 e^{2s\alpha} |v_{xx}|^2 dxdt + s^3 \iint_Q \phi^3 e^{2s\alpha} |v_{xxx}|^2 dxdt + s \iint_Q \phi e^{2s\alpha} |v_{xxxx}|^2 dxdt \right],
\end{aligned}
$$

for any $s \geq CT^2$. From (5.5), we have

$$
\begin{aligned}
& \int_\Omega |v(x,\theta)|^2 e^{2s\alpha(x,\theta)} dx \\
\leq \; & C(\Omega) T^6 s^{-3} \left(\iint_Q e^{2s\alpha} |f|^2 dxdt + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha} |u|^2 dxdt + s \iint_{Q_\omega} \phi e^{2s\alpha} |u_{xxxx}|^2 dxdt \right. \\
& \left. + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha} |v|^2 dxdt + s \iint_{Q_\omega} \phi e^{2s\alpha} |v_{xxxx}|^2 dxdt \right),
\end{aligned} \tag{5.10}
$$

for any $s \geq CT^2$. From (2.1), we get that

$$
f(x) r(\theta) = v(\theta) + u_{xxx}(\theta) - u_{xxxxx}(\theta) + p(\theta) u_x(\theta) + q(\theta) u(\theta),
$$

and from (5.10), we have

$$
\begin{aligned}
& \int_\Omega |f(x) r(\theta)|^2 e^{2s\alpha(\theta)} dx \\
\leq \; & C(\Omega, y_0) T^6 s^{-3} \left(\iint_Q e^{2s\alpha} |f|^2 dxdt + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha} |u|^2 dxdt + s \iint_{Q_\omega} \phi e^{2s\alpha} |u_{xxxx}|^2 dxdt \right. \\
& \left. + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha} |v|^2 dxdt + s \iint_{Q_\omega} \phi e^{2s\alpha} |v_{xxxx}|^2 dxdt \right) \\
& + C \left(\int_\Omega e^{2s\alpha(\theta)} (|u_{xxxxx}(\theta)|^2 + |u_{xxx}(\theta)|^2 + |u_x(\theta)|^2 + |u(\theta)|^2) dx \right),
\end{aligned} \tag{5.11}
$$

where we have used $||p||_{L^\infty(0,T;L^\infty(\Omega))} \leq C$ and $||q||_{L^\infty(0,T;L^\infty(\Omega))} \leq C$. By assumption, $\inf_{x\in\Omega} |r(x,\theta)| \geq b > 0$ and Carleman weight function satisfies the inequality $e^{2s\alpha} \leq e^{2s\alpha(\theta)}$ for all $(x,t) \in Q$, we obtain

$$\begin{aligned}
&\int_\Omega |f(x)|^2 \left(b^2 - \frac{CT^7}{s^3}\right) e^{2s\alpha(\theta)} dx \\
&\leq \ C\left(s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|u|^2 dxdt + s\iint_{Q_\omega} \phi e^{2s\alpha}|u_{xxxx}|^2 dxdt + s^9 \iint_{Q_\omega} \phi^9 e^{2s\alpha}|v|^2 dxdt \right. \\
&\qquad \left. + s\iint_{Q_\omega} \phi e^{2s\alpha}|v_{xxxx}|^2 dxdt + \int_\Omega e^{2s\alpha(\theta)} \left(|u_{xxxxx}(\theta)|^2 + |u_{xxx}(\theta)|^2 + |u_x(\theta)|^2 + |u(\theta)|^2\right) dx\right),
\end{aligned}$$

for any $s \geq \tilde{s}_2 = \max\{\tilde{s}_0, \tilde{s}_1\}$. For the choice of s large enough such that $s \geq \tilde{s}_3 = \max\left\{\tilde{s}_2, \frac{CT^{\frac{7}{3}}}{b^{\frac{2}{3}}}\right\}$ and using the fact that $\inf_{x\in\Omega} e^{2\tilde{s}_3\alpha(\theta)} \geq b_0 > 0, \ \forall x \in \bar{\Omega}$ and $e^{2\tilde{s}_3\alpha}\phi^k \leq C < \infty$, for any $k \in \mathbb{R}$, we have

$$||f||^2_{L^2(\Omega)} \leq C\left(||u||^2_{H^1(0,T;L^2(\omega))} + ||u_{xxxx}||^2_{H^1(0,T;L^2(\omega))} + ||u(\theta)||^2_{H^5(\Omega)}\right), \tag{5.12}$$

where C is depending on Ω, T, M, b, b_0 and y_0. This completes the proof of the stability result. □

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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