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Research article

Some results on p-adic valuations of Stirling numbers of the second kind

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Abstract: Let n and k be nonnegative integers. The Stirling number of the second kind, denoted by $S(n,k)$, is defined as the number of ways to partition a set of n elements into exactly k nonempty subsets and we have

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

Let p be a prime and $v_p(n)$ stand for the p-adic valuation of n, i.e., $v_p(n)$ is the biggest nonnegative integer r with p^r dividing n. Divisibility properties of Stirling numbers of the second kind have been studied from a number of different perspectives. In this paper, we present a formula to calculate the exact value of p-adic valuation of $S(n,n-k)$, where $n \geq k+1$ and $1 \leq k \leq 7$. From this, for any odd prime p, we prove that $v_p((n-k)!S(n,n-k)) < n$ if $n \geq k+1$ and $0 \leq k \leq 7$. It confirms partially Clarke's conjecture proposed in 1995. We also give some results on $v_p(S(ap^n, ap^n-k))$, where a and n are positive integers with $(a,p)=1$ and $1 \leq k \leq 7$.

Keywords: Stirling number of the second kind; p-adic valuation; Stirling-like numbers; r-associated Stirling number of the second kind
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1. Introduction

Let n and k be nonnegative integers. Let $(x)_k$ and $\langle x\rangle_k$ stand for the *falling factorial* and the *rising factorial*, which are defined by $(x)_k := x(x-1)(x-2)\ldots(x-k+1)$ if $k \geq 1$ and $(x)_0 := 1$, and $\langle x\rangle_k := x(x+1)\ldots(x+k-1)$ if $k \geq 1$ and $\langle x\rangle_0 := 1$, respectively. The *Stirling number of the first kind*, denoted by $s(n,k)$, counts the number of permutations of n elements with k disjoint cycles. One can also characterize $s(n,k)$ by

$$(x)_n = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k.$$

The *Stirling number of the second kind* $S(n,k)$ is defined as the number of ways to partition a set of n elements into exactly k nonempty subsets, and we have

$$S(n,k) = \frac{1}{k!}\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}i^n, \tag{1.1}$$

where $\binom{k}{i}$ represents the binomial coefficient, which is defined by

$$\binom{k}{i} = \frac{(k)_i}{i!} = \frac{k!}{i!(k-i)!} \ \ (k \geq i).$$

One can also characterize the Stirling number of the second kind by

$$x^n = \sum_{k=0}^{\infty} S(n,k)(x)_k.$$

Given a prime p and a nonzero integer m, there exist unique integers a and r, with $p \nmid a$ and $r \geq 0$, such that $m = ap^r$. The number r is called the *p-adic valuation* of m, denoted by $r = v_p(m)$. If $x = \frac{m_1}{m_2}$, where m_1 and m_2 are integers and $m_2 \neq 0$, then we define $v_p(x) := v_p(m_1) - v_p(m_2)$. Let $d_p(m)$ denote the base p digital sum of m. Many authors studied the divisibility properties of Stirling numbers of the second kind. For each given k, the sequence $\{S(n,k), n \geq k\}$ is known to be periodic modulo prime powers. Carlitz [1] and Kwong [2] have studied the length of this period. Chan and Manna [3] characterized $S(n,k)$ modulo prime powers in terms of binomial coefficients when k is a multiple of prime powers. Let p be a prime. For any positive integer n and k, define

$$T_p(n,k) := \sum_{\substack{i=0 \\ p\nmid i}}^{k}(-1)^{k-i}\binom{k}{i}i^n.$$

By formula (1.1) we can see that $k!S(n,k) - T_p(n,k)$ is divisible at least by p^n. In this sense, $T_p(n,k)$ is also known as *Stirling-like numbers* [4]. In 1990, Davis [5] gave a method for calculating $v_2(T_2(n,5))$ and $v_2(T_2(n,6))$. Then Clarke [4] generalized this result by applying Hensel's Lemma on the p-adic integers. He also conjectured that $v_p(k!S(n,k)) = v_p(T_p(n,k))$ if $n \geq k$. We note that Hong, Zhao and Zhao [6–8] have presented some important results on the divisibility properties of Stirling numbers of the second kind. In fact, they proved several conjectures proposed by Amdeberhan, Manna and Moll [9] and by Lengyel [10]. We refer the readers to [11–16] for other results on the divisibility properties of Stirling numbers of the both kinds.

Let n and k be positive integers. For any positive integer r, let $S_r(n,k)$ denote the *r-associated Stirling number of the second kind*, which is defined as the number of ways to partition a set of n elements into k nonempty subsets such that each of the k subsets has at least r elements (see, for example, [17]). Define $S_r(0,0) = 1, S_r(n,0) = S_r(0,k) = 0$ and $S_r(n,k) = 0$ if $n < rk$. Then $S_1(n,k) = S(n,k)$ and for $r \geq 2$ and $n \geq rk$ one has

$$S_r(n+1,k) = kS_r(n,k) + \binom{n}{r-1} S_r(n-r+1,k-1).$$

Note that

$$S_2(n+1,k) = kS_2(n,k) + nS_2(n-1,k-1),$$

and the first values of $S_2(n,k)$ can be listed as follows (see [17]):

$k\backslash n$	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	3	10	25	56	119	246	501	1012	2035	4082	8177
3	0	0	0	0	15	105	490	1918	6825	22935	74316	235092	731731
4	0	0	0	0	0	0	105	1260	9450	56980	302995	1487200	6914908
5	0	0	0	0	0	0	0	0	945	17325	190575	1636635	12122110
6	0	0	0	0	0	0	0	0	0	0	10395	270270	4099095
7	0	0	0	0	0	0	0	0	0	0	0	0	135135

We can now state the first main result of this paper as follows.

Theorem 1.1. *Let p be a prime number. For any positive integers n and k with $n \geq k+1$, we have*

$$\begin{aligned} v_p(S(n,n-k)) &= v_p\left(\binom{n}{k+1}\right) + t_p(n,k) \\ &= \sum_{i=0}^{k} v_p(n-i) - v_p((k+1)!) + t_p(n,k), \end{aligned}$$

where $t_p(n,k) := 0$ if $k = 1$ and $t_p(n,k) := v_p(f_k(n)) - v_p(\langle k+2\rangle_{k-1})$ if $k \geq 2$ with

$$f_k(n) := \langle k+2\rangle_{k-1} + \sum_{i=k+2}^{2k} \langle i+1\rangle_{2k-i} \cdot (n-k-1)_{i-k-1} S_2(i,i-k).$$

In particular, we have

$$t_p(n,k) = \begin{cases} v_p(3n-5) - v_p(4), & \text{if } k=2, \\ v_p(n^2-5n+6) - v_p(2), & \text{if } k=3, \\ v_p(15n^3-150n^2+485n-502) - v_p(48), & \text{if } k=4, \\ v_p(3n^4-50n^3+305n^2-802n+760) - v_p(16), & \text{if } k=5, \\ v_p(63n^5-1575n^4+15435n^3-73801n^2+171150n-152696) - v_p(576), & \text{if } k=6 \end{cases}$$

and

$$t_p(n,7) = v_p(9n^6-315n^5+4515n^4-33817n^3+139020n^2-295748n+252336) - v_p(144).$$

From Theorem 1.1, we can deduce the following result.

Theorem 1.2. *Let p be a prime and n be a positive integer.*

(i). If $n \geq 2$*, then*

$$v_p(S(n,n-1)) = v_p(n) + v_p(n-1) - v_p(2).$$

(ii). If $n \geq 3$*, then*

$$v_p(S(n,n-2)) = v_p(n) + v_p(n-1) + v_p(n-2) + v_p(3n-5) - v_p(4) - v_p(3!).$$

(iii). If $n \geq 4$*, then*

$$v_p(S(n,n-3)) = v_p(n) + v_p(n-1) + 2v_p(n-2) + 2v_p(n-3) - v_p(2) - v_p(4!).$$

(iv). If $n \geq 5$*, then*

$$v_p(S(n,n-4)) = \sum_{i=0}^{4} v_p(n-i) + v_p(15n^3 - 150n^2 + 485n - 502) - v_p(48) - v_p(5!).$$

(v). If $n \geq 6$*, then*

$$v_p(S(n,n-5)) = \sum_{i=0}^{5} v_p(n-i) + v_p(n-4) + v_p(n-5) + v_p(3n^2 - 23n + 38) - v_p(16) - v_p(6!).$$

(vi). If $n \geq 7$*, then*

$$\begin{aligned} &v_p(S(n,n-6)) \\ &= \sum_{i=0}^{6} v_p(n-i) + v_p(63n^5 - 1575n^4 + 15435n^3 - 73801n^2 + 171150n - 152696) \\ &\quad - v_p(576) - v_p(7!). \end{aligned}$$

(vii). If $n \geq 8$*, then*

$$\begin{aligned} v_p(S(n,n-7)) = &\sum_{i=0}^{7} v_p(n-i) + v_p(n-6) + v_p(n-7) \\ &+ v_p(9n^4 - 198n^3 + 1563n^2 - 5182n + 6008) - v_p(144) - v_p(8!). \end{aligned}$$

In 1995, Clarke proposed the following conjecture:

Conjecture 1.3. [4] *Let n and k be nonnegative integers. If* $n \geq k+1$ *and p is an odd prime, then*

$$v_p((n-k)!S(n,n-k)) < n.$$

Evidently, Conjecture 1.3 implies that $v_p((n-k)!S(n,n-k)) = v_p(T(n,n-k))$ holds if $n \geq k+1$ and p is an odd prime. For the cases that $1 \leq k \leq 4$, Clarke [4] checked the truth of above conjecture numerically. By using Theorem 1.2, we can derive the following result which proves Conjecture 1.3 for the cases $0 \leq k \leq 7$ and is the third main result of this paper.

Theorem 1.4. *Let p be an odd prime. Let k be an integer such that* $0 \leq k \leq 7$*. For any positive integer n with* $n \geq k+1$*, one has*

$$v_p((n-k)!S(n,n-k)) < n. \tag{1.2}$$

This paper is organized as follows. First of all, in the next section, we reveal some preliminaries. Subsequently, we prove Theorems 1.1 and 1.2 in Section 3. In Section 4, we give the proof of Theorem 1.4. Finally, we present some interesting consequences of Theorem 1.2 in Section 5.

2. Some preliminaries

In this section, we present several auxiliary lemmas that are needed.

As introduced before, $S_2(n,k)$ represents the 2-associated Stirling number of the second kind. The exact values of Stirling number of the second kind can be computed by the following result.

Lemma 2.1. [17] *For any positive integers n and k with $n \ge k$, we have*

$$S(n, n-k) = \sum_{i=k+1}^{2k} \binom{n}{i} S_2(i, i-k).$$

Lemma 2.2. *Let p be an odd prime and m be a positive integer. For positive real number x, we define the function $G_{m,p}$ as follows:*

$$G_{m,p}(x) := \frac{x^{m(p-1)}}{p^{x(p-2)}}.$$

Then $G_{m,p}$ is strictly monotonic decreasing on the interval $[2m, +\infty)$ if $p = 3$, and on the interval $[m, +\infty)$ if $p \ge 5$.

Proof. Since $p \ge 3$ and

$$\frac{d}{dx} G_{m,p}(x) = \frac{x^{m(p-1)-1}}{p^{x(p-2)}} (m(p-1) - x(p-2)\ln p),$$

one gets that $\frac{d}{dx} G_{m,p}(x) < 0$ if $m \le \frac{x}{2}$ with $p = 3$ or $m \le x$ with $p \ge 5$. □

For any given real number y, let $\lfloor y \rfloor$ stand for the largest integer no more than y. We have the following result.

Lemma 2.3. *Let p be an odd prime. Let n and k be positive integers such that $k \le \min\{n, 7\}$. We have*

$$\sum_{i=0}^{k} v_p(n-i) \le \min\left\{1, \left\lfloor \frac{k}{3} \right\rfloor\right\} + \max\left\{1, \left\lfloor \frac{k}{3} \right\rfloor\right\} \log_p n.$$

Proof. If $\sum_{i=0}^{k} v_p(n-i) = 0$, then Lemma 2.3 clearly holds. In the following we let

$$\sum_{i=0}^{k} v_p(n-i) \ge 1.$$

Then at least one element in the set $\{v_p(n), ..., v_p(n-k)\}$ is nonzero. We now divide rest of the proof into the following cases.

Case 1. $k \in \{1, 2\}$. In this case, one can easily see that only one of $\{v_p(n), ..., v_p(n-k)\}$ is nonzero. This infers that

$$\sum_{i=0}^{k} v_p(n-i) = \max\{v_p(n), ..., v_p(n-k)\} \le \log_p n.$$

Case 2. $k \in \{3, 4, 5\}$. We need to show that

$$\sum_{i=0}^{k} v_p(n-i) \le 1 + \log_p n. \tag{2.1}$$

Since $3 \leq k \leq 5$, one gets that no more than two terms of $\{v_p(n), ..., v_p(n-k)\}$ can be nonzero. If there is only one term being nonzero, then we must have

$$\sum_{i=0}^{k} v_p(n-i) = \max\{v_p(n), ..., v_p(n-k)\} \leq \log_p n.$$

So (2.1) holds. Otherwise, we can assume that $v_p(n-i) \geq 1$ and $v_p(n-j) \geq 1$ with $0 \leq i < j \leq k$. Then one can deduce that $v_p(j-i) \geq 1$ and so $p \in \{3, 5\}$. But $1 \leq j-i \leq k \leq 5$. Hence we must have $v_p(j-i) = 1$. By using the isosceles triangle principle (see, for example, [18]), we derive that one of $\{v_p(n-i), v_p(n-j)\}$ must be 1. Therefore

$$\sum_{i=0}^{k} v_p(n-i) = 1 + \max\{v_p(n), ..., v_p(n-k)\} \leq 1 + \log_p n.$$

Thus (2.1) is proved.

CASE 3. $k \in \{6, 7\}$. We need to show that

$$\sum_{i=0}^{k} v_p(n-i) \leq 1 + 2\log_p n. \tag{2.2}$$

Since $k = 6$ or $k = 7$, one gets that no more than three terms of $\{v_p(n), ..., v_p(n-k)\}$ can be nonzero. If there are at most two terms being nonzero, then

$$\sum_{i=0}^{k} v_p(n-i) \leq 2\log_p n.$$

So (2.2) holds.

If three terms of $\{v_p(n), ..., v_p(n-k)\}$ are nonzero, then we must have $p = 3$. Suppose that $v_3(n-i_1) \geq 1$, $v_3(n-i_2) \geq 1$ and $v_3(n-i_3) \geq 1$ with $0 \leq i_1 < i_2 < i_3 \leq k \leq 7$. Then one can derive that $v_3(i_3-i_2) = v_3(i_2-i_1) = 1$. From this, we can obtain that one of $\{v_3(n-i_1), v_3(n-i_2), v_3(n-i_3)\}$ must be 1. Hence one arrives at

$$\sum_{i=0}^{k} v_p(n-i) \leq 1 + 2\log_p n$$

as (2.2) desired.

This completes the proof of Lemma 2.3. □

3. Proofs of Theorems 1.1 and 1.2

This section is dedicated to the proofs of Theorems 1.1 and 1.2. We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. First of all, Theorem 1.1 is clearly true when $k = 1$, since

$$S(n, n-1) = \binom{n}{2}.$$

So in what follows, we let $n \geq k+1 \geq 3$. From Lemma 2.1, one knows that

$$\begin{aligned} S(n,n-k) &= \binom{n}{k+1} S_2(k+1,1) + \sum_{i=k+2}^{2k} \binom{n}{i} S_2(i,i-k) \\ &= \binom{n}{k+1} + \sum_{i=k+2}^{2k} \binom{n}{i} S_2(i,i-k). \end{aligned} \tag{3.1}$$

Let i be a positive integer with $k+2 \leq i \leq 2k$. It is easy to see that

$$\binom{n}{i} = \frac{(n)_i}{i!} = \frac{(n)_{k+1} \cdot (n-k-1)_{i-k-1}}{(k+1)! \cdot \langle k+2 \rangle_{i-k-1}} = \binom{n}{k+1} \frac{(n-k-1)_{i-k-1}}{\langle k+2 \rangle_{i-k-1}}.$$

Together with (3.1) give us that

$$\begin{aligned} & v_p(S(n,n-k)) \\ &= v_p\Big(\binom{n}{k+1} + \sum_{i=k+2}^{2k} \binom{n}{i} S_2(i,i-k)\Big) \\ &= v_p\Big(\binom{n}{k+1}\Big(1 + \sum_{i=k+2}^{2k} \frac{(n-k-1)_{i-k-1}}{\langle k+2 \rangle_{i-k-1}} S_2(i,i-k)\Big)\Big) \\ &= v_p\Big(\binom{n}{k+1}\Big) + v_p\Big(1 + \sum_{i=k+2}^{2k} \frac{(n-k-1)_{i-k-1}}{\langle k+2 \rangle_{i-k-1}} S_2(i,i-k)\Big). \end{aligned} \tag{3.2}$$

Note that $S_2(i,i-k)$ is an integer. Let

$$f_k(n) := \langle k+2 \rangle_{k-1} + \sum_{i=k+2}^{2k} \langle i+1 \rangle_{2k-i} \cdot (n-k-1)_{i-k-1} S_2(i,i-k)$$

and

$$t_p(n,k) := v_p(f_k(n)) - v_p(\langle k+2 \rangle_{k-1}).$$

Then $f_k(n)$ is a polynomial with integer coefficients and $\deg f_k = k-1$, and by (3.2) one derives that

$$v_p(S(n,n-k)) = v_p\Big(\binom{n}{k+1}\Big) + t_p(n,k).$$

In the following, we deal with $t_p(n,k)$ for $2 \leq k \leq 7$. We divide this into six cases.

Case 1. $k=2$. Since $S_2(4,2)=3$, by the definition of $f_2(n)$ we get that

$$f_2(n) = 4 + 3(n-3) = 3n-5,$$

and so

$$t_p(n,2) = v_p(f_2(n)) - v_p(4) = v_p(3n-5) - v_p(4).$$

Case 2. $k=3$. From $S_2(5,2)=10$ and $S_2(6,3)=15$ one deduces that

$$f_3(n) = 5 \cdot 6 + 6 \cdot 10(n-4) + 15(n-4)(n-5) = 15(n^2 - 5n + 6)$$

and

$$t_p(n,3) = v_p(f_3(n)) - v_p(5\cdot 6) = v_p(n^2-5n+6) - v_p(2).$$

Case 3. $k=4$. By $S_2(6,2)=25$, $S_2(7,3)=105$ and $S_2(8,4)=105$, we derive that

$$\begin{aligned} f_4(n) &= 6\cdot 7\cdot 8 + 7\cdot 8\cdot 25(n-5) + 8\cdot 105(n-5)(n-6) + 105(n-5)(n-6)(n-7) \\ &= 7(15n^3-150n^2+485n-502) \end{aligned}$$

and

$$\begin{aligned} t_p(n,4) &= v_p(f_4(n)) - v_p(6\cdot 7\cdot 8) \\ &= v_p(7) + v_p(15n^3-150n^2+485n-502) - v_p(6\cdot 7\cdot 8) \\ &= v_p(15n^3-150n^2+485n-502) - v_p(48). \end{aligned}$$

Case 4. $k=5$. It follows from $S_2(7,2)=56$, $S_2(8,3)=490$, $S_2(9,4)=1260$ and $S_2(10,5)=945$ that

$$\begin{aligned} f_5(n) &= 7\cdot 8\cdot 9\cdot 10 + 8\cdot 9\cdot 10\cdot 56(n-6) + 9\cdot 10\cdot 490(n-6)(n-7) \\ &\quad + 10\cdot 1260(n-6)(n-7)(n-8) + 945(n-6)(n-7)(n-8)(n-9) \\ &= 315(3n^4-50n^3+305n^2-802n+760) \end{aligned}$$

and

$$\begin{aligned} t_p(n,5) &= v_p(f_5(n)) - v_p(7\cdot 8\cdot 9\cdot 10) \\ &= v_p(315) + v_p(3n^4-50n^3+305n^2-802n+760) - v_p(7\cdot 8\cdot 9\cdot 10) \\ &= v_p(3n^4-50n^3+305n^2-802n+760) - v_p(16). \end{aligned}$$

Case 5. $k=6$. Since $S_2(8,2)=119$, $S_2(9,3)=1918$, $S_2(10,4)=9450$, $S_2(11,5)=17325$ and $S_2(12,6)=10395$, one deduces that

$$\begin{aligned} f_6(n) &= \langle 8\rangle_5 + \langle 9\rangle_4\cdot 119(n-7) + \langle 10\rangle_3\cdot 1918(n-7)(n-8) \\ &\quad + \langle 11\rangle_2\cdot 9450(n-7)(n-8)(n-9) + 12\cdot 17325(n-7)(n-8)(n-9)(n-10) \\ &\quad + 10395(n-7)(n-8)(n-9)(n-10)(n-11) \\ &= 165(63n^5-1575n^4+15435n^3-73801n^2+171150n-152696) \end{aligned}$$

and

$$\begin{aligned} t_p(n,6) &= v_p(f_6(n)) - v_p(\langle 8\rangle_5) \\ &= v_p(63n^5-1575n^4+15435n^3-73801n^2+171150n-152696) - v_p(576). \end{aligned}$$

Case 6. $k=7$. By $S_2(9,2)=246$, $S_2(10,3)=6825$, $S_2(11,4)=56980$, $S_2(12,5)=190575$, $S_2(13,6)=270270$ and $S_2(14,7)=135135$, we obtain that

$$f_7(n) = \langle 9\rangle_6 + \langle 10\rangle_5\cdot 246(n-8) + \langle 11\rangle_4\cdot 6825(n-8)(n-9)$$

$$
\begin{aligned}
&+ \langle 12\rangle_3 \cdot 56980(n-8)(n-9)(n-10)\\
&+ \langle 13\rangle_2 \cdot 190575(n-8)(n-9)(n-10)(n-11)\\
&+ 14 \cdot 270270(n-8)(n-9)(n-10)(n-11)(n-12)\\
&+ 135135(n-8)(n-9)(n-10)(n-11)(n-12)(n-13)\\
=\ & 15015(9n^6 - 315n^5 + 4515n^4 - 33817n^3 + 139020n^2 - 295748n + 252336)
\end{aligned}
$$

and

$$
\begin{aligned}
t_p(n,7) &= v_p(f_7(n)) - v_p(\langle 9\rangle_6)\\
&= v_p(9n^6 - 315n^5 + 4515n^4 - 33817n^3 + 139020n^2 - 295748n + 252336) - v_p(144).
\end{aligned}
$$

This concludes the proof of Theorem 1.1. □

From the proof of Theorem 1.1, one may see that for any positive integers n and k with $n \geq k+1$, $f_k(n)$ is a polynomial with integer coefficients and $\deg f_k = k-1$. We also note that the expression of $f_k(n)$ depends on the values of the 2-associated Stirling number of the second kind $S_2(n,k)$. However, the computation of the exact value of $S_2(n,k)$ becomes complicated when $k \geq 8$. Therefore we only present the formulas for $f_k(n)$ and $t_p(n,k)$ for the case $1 \leq k \leq 7$.

Subsequently, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. (i). Since $n \geq 2$, from Theorem 1.1 one knows that

$$v_p(S(n,n-1)) = v_p(n) + v_p(n-1) - v_p(2).$$

So part (i) of Theorem 1.2 is proved.

(ii). Let $n \geq 3$. By Theorem 1.1 we derive that

$$
\begin{aligned}
v_p(S(n,n-2)) &= v_p(n) + v_p(n-1) + v_p(n-2) + t_p(n,2) - v_p(3!)\\
&= v_p(n) + v_p(n-1) + v_p(n-2) + v_p(3n-5) - v_p(4) - v_p(3!)
\end{aligned}
$$

as (ii) desired. Hence part (ii) of Theorem 1.2 is proved.

(iii). Since $n \geq 4$ and $n^2 - 5n + 6 = (n-2)(n-3)$, it follows from Theorem 1.1 that

$$
\begin{aligned}
v_p(S(n,n-3)) &= \sum_{i=0}^{3} v_p(n-i) - v_p(4!) + t_p(n,3)\\
&= \sum_{i=0}^{3} v_p(n-i) - v_p(4!) + v_p(n^2-5n+6) - v_p(2)\\
&= v_p(n) + v_p(n-1) + 2v_p(n-2) + 2v_p(n-3) - v_p(2) - v_p(4!).
\end{aligned}
$$

This completes the proof of part (iii).

(iv). Let $n \geq 5$. By Theorem 1.1 one obtains that

$$v_p(S(n,n-4)) = \sum_{i=0}^{4} v_p(n-i) - v_p(5!) + t_p(n,4)$$

$$= \sum_{i=0}^{4} v_p(n-i) + v_p(15n^3 - 150n^2 + 485n - 502) - v_p(48) - v_p(5!)$$

as part (iv) asserted.

(v). Since $n \geq 6$ and $3n^4 - 50n^3 + 305n^2 - 802n + 760 = (n-4)(n-5)(3n^2 - 23n + 38)$, it follows from Theorem 1.1 that

$$v_p(S(n, n-5)) = \sum_{i=0}^{5} v_p(n-i) - v_p(6!) + t_p(n, 5) \tag{3.3}$$

and

$$\begin{aligned} t_p(n, 5) &= v_p(3n^4 - 50n^3 + 305n^2 - 802n + 760) - v_p(16) \\ &= v_p(3n^2 - 23n + 38) + v_p(n-4) + v_p(n-5) - v_p(16). \end{aligned} \tag{3.4}$$

Then by using (3.3) and (3.4), we deduce that

$$v_p(S(n, n-5)) = \sum_{i=0}^{5} v_p(n-i) + v_p(n-4) + v_p(n-5) + v_p(3n^2 - 23n + 38) - v_p(16) - v_p(6!).$$

So part (v) is proved.

(vi). Let $n \geq 7$. By Theorem 1.1 we deduce that

$$\begin{aligned} & v_p(S(n, n-6)) \\ = & \sum_{i=0}^{6} v_p(n-i) - v_p(7!) + t_p(n, 6) \\ = & \sum_{i=0}^{6} v_p(n-i) + v_p(63n^5 - 1575n^4 + 15435n^3 - 73801n^2 + 171150n - 152696) - v_p(576) - v_p(7!). \end{aligned}$$

Hence the proof of part (vi) is finished.

(vii). Since $n \geq 8$ and

$$\begin{aligned} & 9n^6 - 315n^5 + 4515n^4 - 33817n^3 + 139020n^2 - 295748n + 252336 \\ & = (n-6)(n-7)(9n^4 - 198n^3 + 1563n^2 - 5182n + 6008), \end{aligned} \tag{3.5}$$

it follows from Theorem 1.1 and (3.5) that

$$\begin{aligned} & v_p(S(n, n-7)) \\ = & \sum_{i=0}^{7} v_p(n-i) - v_p(8!) + t_p(n, 7) \\ = & \sum_{i=0}^{7} v_p(n-i) - v_p(8!) - v_p(144) \end{aligned}$$

$$
\begin{aligned}
&+ v_p(9n^6 - 315n^5 + 4515n^4 - 33817n^3 + 139020n^2 - 295748n + 252336) \\
= &\sum_{i=0}^{7} v_p(n-i) + v_p(n-6) + v_p(n-7) \\
&+ v_p(9n^4 - 198n^3 + 1563n^2 - 5182n + 6008) - v_p(144) - v_p(8!).
\end{aligned}
$$

Therefore part (vii) is proved. This completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.4

In this section, we present the proof of Theorem 1.4.

Proof of Theorem 1.4. Let p be an odd prime. First, since $S(n, n) = 1$ and

$$
v_p(n!) = \frac{n - d_p(n)}{p-1} < n,
$$

(1.2) is clearly true when $k = 0$. Now let $1 \le k \le 7$. Consider the following cases.

CASE 1. $k = 1$. Since $p \ge 3$, from Theorem 1.2 one knows that

$$
v_p(S(n, n-1)) = v_p(n) + v_p(n-1).
$$

It then follows from Lemma 2.3 that

$$
\begin{aligned}
&v_p((n-1)!S(n, n-1)) - n \\
&= \frac{n - 1 - d_p(n-1)}{p-1} + v_p(n) + v_p(n-1) - n \\
&\le \frac{1}{p-1}\Big(n - 2 + (p-1)(\log_p n - n)\Big) \\
&= \frac{1}{p-1}\Big(n(2-p) + (p-1)\log_p n - 2\Big) \\
&= \frac{1}{p-1}\Big(\log_p \frac{n^{p-1}}{p^{n(p-2)}} - 2\Big).
\end{aligned}
\tag{4.1}
$$

Note that $n \ge 2$. Hence we derive from Lemma 2.2 that

$$
\begin{aligned}
\log_p \frac{n^{p-1}}{p^{n(p-2)}} - 2 &= \log_p G_{1,p}(n) - 2 \\
&\le \log_p G_{1,p}(2) - 2 \\
&= \log_p \frac{2^{p-1}}{p^{2(p-2)}} - 2 \\
&= (p-1)\log_p \frac{2}{p^2} < 0.
\end{aligned}
\tag{4.2}
$$

By (4.1) and (4.2), one then obtains that $v_p((n-1)!S(n, n-1)) < n$ as desired.

Case 2. $k = 2$. Note that $n \geq 3$ and $p \geq 3$. It follows from Theorem 1.2 that

$$v_p(S(n, n-2)) = \sum_{i=0}^{2} v_p(n-i) + v_p(3n-5) - v_p(3!).$$

Thus one deduces that

$$\begin{aligned}
& v_p((n-2)!S(n, n-2)) - n \\
&= \frac{n-2-d_p(n-2)}{p-1} + \sum_{i=0}^{2} v_p(n-i) + v_p(3n-5) - v_p(3!) - n \\
&\leq \frac{1}{p-1}\Big(n-3+(p-1)\Big(\sum_{i=0}^{2} v_p(n-i) + v_p(3n-5) - v_p(3) - n\Big)\Big) := \frac{D_{2,p}(n)}{p-1} \qquad (4.3)
\end{aligned}$$

with

$$D_{2,p}(n) = (p-1)\Big(\sum_{i=0}^{2} v_p(n-i) + v_p(3n-5) - v_p(3)\Big) - n(p-2) - 3. \qquad (4.4)$$

If $v_p(3n-5) = 0$, then by Lemma 2.3 we obtain that

$$\sum_{i=0}^{2} v_p(n-i) + v_p(3n-5) - v_p(3) \leq \log_p n. \qquad (4.5)$$

If $v_p(3n-5) \geq 1$, then we must have $p \geq 5$ and $v_p(n-1) = v_p(n-2) = 0$. Also note that $v_p(n) = v_p(3n) = v_p(3n-5+5)$ and $v_p(3n-5) = v_p(\frac{3n-5}{3}) \leq \log_p n$. This implies that $\min\{v_p(n), v_p(3n-5)\} \leq 1$, and so

$$\begin{aligned}
& \sum_{i=0}^{2} v_p(n-i) + v_p(3n-5) - v_p(3) \\
&= v_p(n) + v_p(3n-5) \\
&\leq 1 + \max\{v_p(n), v_p(3n-5)\} \\
&\leq 1 + \log_p n. \qquad (4.6)
\end{aligned}$$

Since $p \geq 3$, we derive from Lemma 2.2 together with (4.4) to (4.6) that

$$\begin{aligned}
D_{2,p}(n) &\leq (p-1)(1+\log_p n) - n(p-2) - 3 \\
&= \log_p \frac{n^{p-1}}{p^{n(p-2)}} + p - 4 \\
&= \log_p G_{1,p}(n) + p - 4 \\
&\leq \log_p G_{1,p}(3) + p - 4 \\
&= (p-1)\log_p \frac{3}{p^2} < 0. \qquad (4.7)
\end{aligned}$$

So (4.3) and (4.7) imply that $v_p((n-2)!S(n,n-2)) < n$.

Case 3. $k = 3$. Since $p \geq 3$, by Theorem 1.2, we get that

$$\begin{aligned} v_p(S(n,n-3)) &= v_p(n) + v_p(n-1) + 2v_p(n-2) + 2v_p(n-3) - v_p(2) - v_p(4!) \\ &= \sum_{i=0}^{3} v_p(n-i) + v_p(n-2) + v_p(n-3) - v_p(3). \end{aligned}$$

It follows that

$$\begin{aligned} & v_p((n-3)!S(n,n-3)) - n \\ &= \frac{n-3-d_p(n-3)}{p-1} + v_p(S(n,n-3)) - n \\ &\leq \frac{1}{p-1}\Big(n-4+(p-1)(v_p(S(n,n-3))-n)\Big) := \frac{D_{3,p}(n)}{p-1} \end{aligned} \tag{4.8}$$

with

$$\begin{aligned} D_{3,p}(n) &= n-4+(p-1)\big(v_p(S(n,n-3))-n\big) \\ &= (p-1)\Big(\sum_{i=0}^{3} v_p(n-i) + v_p(n-2) + v_p(n-3) - v_p(3)\Big) - n(p-2) - 4. \end{aligned} \tag{4.9}$$

Note that $v_p(n-2) + v_p(n-3) \leq \log_p n$. Hence, by Lemma 2.3 one deduces that

$$\sum_{i=0}^{3} v_p(n-i) + v_p(n-2) + v_p(n-3) - v_p(3) \leq 1 + 2\log_p n.$$

It then follows from (4.9) that

$$\begin{aligned} D_{3,p}(n) &\leq (p-1)(1+2\log_p n) - n(p-2) - 4 \\ &= \log_p \frac{n^{2(p-1)}}{p^{n(p-2)}} + p - 5 \\ &= \log_p G_{2,p}(n) + p - 5 \\ &\leq \log_p G_{2,p}(4) + p - 5 \\ &= (p-1)\log_p \frac{4^2}{p^3} < 0 \end{aligned} \tag{4.10}$$

since $p \geq 3$ and $n \geq 4$. So (4.8) and (4.10) give us that $v_p((n-3)!S(n,n-3)) < n$.

Case 4. $k = 4$. Note that $n \geq 5$ and $p \geq 3$. From Theorem 1.2, we derive that

$$\begin{aligned} v_p(S(n,n-4)) &= \sum_{i=0}^{4} v_p(n-i) + v_p(15n^3 - 150n^2 + 485n - 502) - v_p(48) - v_p(5!) \\ &= \sum_{i=0}^{4} v_p(n-i) + v_p(15n^3 - 150n^2 + 485n - 502) - 2v_p(3) - v_p(5) \end{aligned}$$

and

$$\begin{aligned}
&v_p((n-4)!S(n,n-4)) - n \\
&= \frac{n-4-d_p(n-4)}{p-1} + v_p(S(n,n-4)) - n \\
&\leq \frac{1}{p-1}\Big(n-5+(p-1)(v_p(S(n,n-4))-n)\Big) := \frac{D_{4,p}(n)}{p-1} \qquad (4.11)
\end{aligned}$$

with

$$\begin{aligned}
D_{4,p}(n) &= (p-1)v_p(S(n,n-4)) - n(p-2) - 5 \\
&= (p-1)\Big(\sum_{i=0}^{4} v_p(n-i) + v_p(15n^3-150n^2+485n-502) - 2v_p(3) - v_p(5)\Big) - n(p-2) - 5.
\end{aligned}$$

First of all, it is easy to check that $D_{4,p}(n) < 0$ for $5 \leq n \leq 9$. In what follows, let $n \geq 10$.

If $v_p(15n^3 - 150n^2 + 485n - 502) = 0$, then by Lemma 2.3 we obtain that

$$\sum_{i=0}^{4} v_p(n-i) + v_p(15n^3-150n^2+485n-502) - 2v_p(3) - v_p(5) \leq 1 + \log_p n. \qquad (4.12)$$

Now let $v_p(15n^3 - 150n^2 + 485n - 502) \geq 1$. Note that

$$\begin{aligned}
15n^3-150n^2+485n-502 &= 15(n-1)^3 - 105(n-1)^2 + 230(n-1) - 152 \\
&= 15(n-2)^3 - 60(n-2)^2 + 65(n-2) - 12 \\
&= 15(n-3)^3 - 15(n-3)^2 - 10(n-3) + 8 \\
&= 15(n-4)^3 + 30(n-4)^2 + 5(n-4) - 2
\end{aligned}$$

and $502 = 2 \cdot 251$, $152 = 2^3 \cdot 19$, $12 = 2^2 \cdot 3$. Then we deduce that $p \neq 5$, $v_p(n-3) = v_p(n-4) = 0$ and

$$v_p(15n^3-150n^2+485n-502) \leq \begin{cases} 1 + \log_p n^3, & \text{if } p = 3, \\ \log_p n^3, & \text{if } p \geq 7 \end{cases}$$

since $v_p(15n^3-150n^2+485n-502) = v_p(\frac{15n^3-150n^2+485n-502}{15}) + v_p(15) \leq \log_p n^3 + v_p(15)$. If $p = 3$, then one gets that $v_3(n) = v_3(n-1) = 0$ and $v_3(n-2) \geq 1$. Also note that $\min\{v_3(n-2), v_3(15n^3-150n^2+485n-502)\} \leq 1$. It follows that

$$\begin{aligned}
&\sum_{i=0}^{4} v_3(n-i) + v_3(15n^3-150n^2+485n-502) - 2v_3(3) - v_3(5) \\
&= v_3(n-2) + v_3(15n^3-150n^2+485n-502) - 2 \\
&\leq \log_3 n^3. \qquad (4.13)
\end{aligned}$$

If $p \geq 7$, then one has $v_p(n-2) = 0$ and only one of $\{v_p(n), v_p(n-1)\}$ can be nonzero. Without loss of generality, assume that $v_p(n) = 0$ and $v_p(n-1) \geq 1$. Then it follows that $\min\{v_p(n-1), v_p(15n^3-150n^2+485n-502)\} \leq 1$. This gives us that

$$\sum_{i=0}^{4} v_p(n-i) + v_p(15n^3-150n^2+485n-502) - 2v_p(3) - v_p(5)$$

$$= v_p(n) + v_p(n-1) + v_p(15n^3 - 150n^2 + 485n - 502)$$
$$\le 1 + \log_p n^3. \tag{4.14}$$

Since $p \ge 3$ and $n \ge 10$, by (4.12) to (4.14) together with Lemma 2.2 one derives that

$$\begin{aligned} D_{4,p}(n) &\le (p-1)(1+\log_p n^3) - n(p-2) - 5 \\ &= \log_p \frac{n^{3(p-1)}}{p^{n(p-2)}} + p - 6 \\ &= \log_p G_{3,p}(n) + p - 6 \\ &\le \log_p G_{3,p}(10) + p - 6 \\ &= \log_p \frac{10^{3(p-1)}}{p^{9p-14}} < 0. \end{aligned} \tag{4.15}$$

Thus (4.11) and (4.15) tell us that $v_p((n-4)!S(n,n-4)) < n$.

Case 5. $k = 5$. Since $p \ge 3$, from Theorem 1.2, we obtain that

$$\begin{aligned} v_p(S(n,n-5)) &= \sum_{i=0}^{5} v_p(n-i) + v_p(n-4) + v_p(n-5) + v_p(3n^2 - 23n + 38) - v_p(16) - v_p(6!) \\ &= \sum_{i=0}^{5} v_p(n-i) + v_p(n-4) + v_p(n-5) + v_p(3n^2 - 23n + 38) - 2v_p(3) - v_p(5) \end{aligned}$$

and

$$\begin{aligned} &v_p((n-5)!S(n,n-5)) - n \\ &= \frac{n-5-d_p(n-5)}{p-1} + v_p(S(n,n-5)) - n \\ &\le \frac{1}{p-1}\Big(n - 6 + (p-1)(v_p(S(n,n-5)) - n)\Big) := \frac{D_{5,p}(n)}{p-1} \end{aligned} \tag{4.16}$$

with

$$\begin{aligned} D_{5,p}(n) &= (p-1)v_p(S(n,n-5)) - n(p-2) - 6 \\ &= (p-1)\Big(\sum_{i=0}^{5} v_p(n-i) + v_p(n-4) + v_p(n-5) \\ &\quad + v_p(3n^2 - 23n + 38) - 2v_p(3) - v_p(5)\Big) - n(p-2) - 6. \end{aligned} \tag{4.17}$$

In what follows, we show that $D_{5,p}(n) < 0$. If $v_p(3n^2 - 23n + 38) = 0$, then by Lemma 2.3 and $v_p(n-4) + v_p(n-5) \le \log_p n$ we deduce that

$$\sum_{i=0}^{5} v_p(n-i) + v_p(n-4) + v_p(n-5) + v_p(3n^2 - 23n + 38) - 2v_p(3) - v_p(5) \le 1 + 2\log_p n. \tag{4.18}$$

Now let $v_p(3n^2 - 23n + 38) \geq 1$. Since $38 = 2 \cdot 19$, $18 = 2 \cdot 3^2$ and

$$\begin{aligned}
&3n^2 - 23n + 38 \\
&= 3(n-1)^2 - 17(n-1) + 18 = 3(n-2)^2 - 11(n-2) + 4 \\
&= 3(n-3)^2 - 5(n-3) - 4 = 3(n-4)^2 + (n-4) - 6 = 3(n-5)^2 + 7(n-5) - 2,
\end{aligned}$$

one gets that $v_p(n-2) = v_p(n-3) = v_p(n-5) = 0$ and one of $v_p(n)$ and $v_p(n-1)$ must be zero. If $p = 3$, then we have $v_3(n) = 0$ and

$$\begin{cases} v_3(n-1) \leq \log_3 n \text{ and } v_3(3n^2 - 23n + 38) \leq 1 + \log_3 n^2, & \text{if } v_3(n-4) = 1, \\ v_3(n-1) = v_3(3n^2 - 23n + 38) = 1, & \text{if } v_3(n-4) \geq 2. \end{cases}$$

Hence one deduces that

$$\begin{aligned}
&\sum_{i=0}^{5} v_3(n-i) + v_3(n-4) + v_3(n-5) + v_3(3n^2 - 23n + 38) - 2v_3(3) - v_3(5) \\
&= v_3(n-1) + 2v_3(n-4) + v_3(3n^2 - 23n + 38) - 2 \leq 1 + \log_3 n^3. \qquad (4.19)
\end{aligned}$$

If $p \geq 5$, then one derives that $v_p(n-1) = v_p(n-4) = 0$. Thus

$$\begin{aligned}
&\sum_{i=0}^{5} v_p(n-i) + v_p(n-4) + v_p(n-5) + v_p(3n^2 - 23n + 38) - 2v_p(3) - v_p(5) \\
&= v_p(n) + v_p(3n^2 - 23n + 38) - v_p(5) \leq \log_p n^3. \qquad (4.20)
\end{aligned}$$

Hence (4.17) to (4.20) imply that

$$\begin{aligned}
D_{5,p}(n) &\leq (p-1)(1 + \log_p n^3) - n(p-2) - 6 \\
&= \log_p \frac{n^{3(p-1)}}{p^{n(p-2)}} + p - 7 \\
&= \log_p G_{3,p}(n) + p - 7 \\
&\leq \log_p G_{3,p}(6) + p - 7 \\
&= (p-1) \log_p \frac{6^3}{p^5} < 0. \qquad (4.21)
\end{aligned}$$

Thus (4.16) and (4.21) give us that $v_p((n-5)!S(n, n-5)) < n$.

Case 6. $k = 6$. Note that $n \geq 7$ and $p \geq 3$. By Theorem 1.2, we deduce that

$$\begin{aligned}
v_p(S(n, n-6)) &= \sum_{i=0}^{6} v_p(n-i) - v_p(576) - v_p(7!) \\
&\quad + v_p(63n^5 - 1575n^4 + 15435n^3 - 73801n^2 + 171150n - 152696) \\
&= \sum_{i=0}^{6} v_p(n-i) - 4v_p(3) - v_p(5) - v_p(7)
\end{aligned}$$

$$+ v_p(63n^5 - 1575n^4 + 15435n^3 - 73801n^2 + 171150n - 152696)$$

and

$$\begin{aligned} & v_p((n-6)!S(n,n-6)) - n \\ &= \frac{n-6-d_p(n-6)}{p-1} + v_p(S(n,n-6)) - n \\ &\le \frac{1}{p-1}\Big(n - 7 + (p-1)(v_p(S(n,n-6)) - n)\Big) := \frac{D_{6,p}(n)}{p-1} \end{aligned} \tag{4.22}$$

with

$$\begin{aligned} D_{6,p}(n) &= (p-1)v_p(S(n,n-6)) - n(p-2) - 7 \\ &= (p-1)\Big(\sum_{i=0}^{6} v_p(n-i) + v_p(E_{6,n}) - 4v_p(3) - v_p(5) - v_p(7)\Big) - n(p-2) - 7 \end{aligned} \tag{4.23}$$

and

$$E_{6,n} := 63n^5 - 1575n^4 + 15435n^3 - 73801n^2 + 171150n - 152696.$$

If $v_p(E_{6,n}) = 0$, then by Lemma 2.3 we derive that

$$\sum_{i=0}^{6} v_p(n-i) - 4v_p(3) - v_p(5) - v_p(7) + v_p(E_{6,n}) \le 1 + 2\log_p n. \tag{4.24}$$

From (4.23) and (4.24) one obtains that

$$\begin{aligned} D_{6,p}(n) &\le (p-1)(1 + 2\log_p n) - n(p-2) - 7 \\ &= \log_p \frac{n^{2(p-1)}}{p^{n(p-2)}} + p - 8 \\ &= \log_p G_{2,p}(n) + p - 8 \\ &\le \log_p G_{2,p}(7) + p - 8 \\ &= (p-1)\log_p \frac{7^2}{p^6} < 0. \end{aligned} \tag{4.25}$$

In what follows, we let $v_p(E_{6,n}) \ge 1$. Since

$$\begin{aligned} E_{6,n} &= 63n^5 - 1575n^4 + 15435n^3 - 73801n^2 + 171150n - 152696 \\ &= 63(n-1)^5 - 1260(n-1)^4 + 9765(n-1)^3 - 36316(n-1)^2 + 63868(n-1) - 41424 \\ &= 63(n-2)^5 - 945(n-2)^4 + 5355(n-2)^3 - 13951(n-2)^2 + 15806(n-2) - 5304 \\ &= 63(n-3)^5 - 630(n-3)^4 + 2205(n-3)^3 - 2926(n-3)^2 + 504(n-3) + 1024 \\ &= 63(n-4)^5 - 315(n-4)^4 + 315(n-4)^3 + 539(n-4)^2 - 938(n-4) + 240 \\ &= 63(n-5)^5 - 315(n-5)^3 + 224(n-5)^2 + 140(n-5) - 96 \\ &= 63(n-6)^5 + 315(n-6)^4 + 315(n-6)^3 - 91(n-6)^2 - 42(n-6) + 16 \end{aligned}$$

and $152696 = 2^3 \cdot 19087$, $41424 = 2^4 \cdot 3 \cdot 863$, $5304 = 2^3 \cdot 3 \cdot 13 \cdot 17$, $1024 = 2^{10}$, $240 = 2^4 \cdot 3 \cdot 5$, $96 = 2^5 \cdot 3$ and $16 = 2^4$, one can deduce that $v_p(n-3) = v_p(n-6) = 0$. If $p = 3$, then only two of $\{v_3(n), v_3(n-1), v_3(n-2), v_3(n-4), v_3(n-5)\}$ can be nonzero. Suppose that $v_3(n-i) \geq 1$ and $v_3(n-j) \geq 1$ for $0 \leq i < j \leq 6$. Then either $1 \leq v_3(E_{6,n}) \leq \log_3(n-5)^5 + v_3(63)$ with $v_3(n-i) = v_3(n-j) = 1$ or $v_3(E_{6,n}) = 1$ with $\max\{v_3(n-i), v_3(n-j)\} \geq 2$, this implies that

$$\begin{aligned}
&\sum_{i=0}^{6} v_3(n-i) + v_3(E_{6,n}) - 4v_3(3) - v_3(5) - v_3(7) \\
&= v_3(n-i) + v_3(n-j) + v_3(E_{6,n}) - 4 \\
&\leq 2 + \log_3(n-5)^5 + v_3(63) - 4 \\
&\leq \log_3(n-5)^5.
\end{aligned}$$

It is easy to check that $D_{6,3}(n) < 0$ when $7 \leq n \leq 14$. If $n \geq 15$, then by (4.23) one gets that

$$\begin{aligned}
D_{6,3}(n) &\leq 2\log_3(n-5)^5 - n - 7 \\
&= \log_3 \frac{(n-5)^{10}}{3^{n-5}} - 12 \\
&= \log_3 G_{5,3}(n-5) - 12 \\
&\leq \log_3 G_{5,3}(10) - 12 \\
&= \log_3 \frac{10^{10}}{3^{22}} < 0.
\end{aligned} \tag{4.26}$$

If $p \geq 5$, we can obtain that $v_p(n-5) = 0$ and only one of $\{v_p(n), v_p(n-1), v_p(n-2), v_p(n-4)\}$ can be nonzero. Without loss of generality, assume that $v_p(n) \geq 1$ and $v_p(n-1) = v_p(n-2) = v_p(n-4) = 0$. It follows that $\min\{v_p(n), v_p(E_{6,n}\} \leq 1$. Hence

$$\sum_{i=0}^{6} v_p(n-i) + v_p(E_{6,n}) - 4v_p(3) - v_p(5) - v_p(7) \leq 1 + \log_p n^5. \tag{4.27}$$

Obviously, one has $D_{6,p}(n) < 0$ if $n = 7$. Now let $n \geq 8$. Then by Lemma 2.2 together with (4.23) and (4.27), we arrive at

$$\begin{aligned}
D_{6,p}(n) &\leq (p-1)(1 + \log_p n^5) - n(p-2) - 7 \\
&= \log_p \frac{n^{5(p-1)}}{p^{n(p-2)}} + p - 8 \\
&= \log_p G_{5,p}(n) + p - 8 \\
&\leq \log_p G_{5,p}(8) + p - 8 \\
&= \log_p \frac{8^{5(p-1)}}{p^{7p-8}} < 0
\end{aligned} \tag{4.28}$$

as desired. Thus (4.22), (4.25), (4.26) and (4.28) tell us that $v_p((n-6)!S(n, n-6)) < n$.

Case 7. $k = 7$. From Theorem 1.2, we obtain that

$$v_p(S(n, n-7))$$

$$= \sum_{i=0}^{5} v_p(n-i) + 2v_p(n-6) + 2v_p(n-7)$$
$$+ v_p(9n^4 - 198n^3 + 1563n^2 - 5182n + 6008) - v_p(144) - v_p(8!)$$
$$= \sum_{i=0}^{7} v_p(n-i) + v_p(n-6) + v_p(n-7)$$
$$+ v_p(9n^4 - 198n^3 + 1563n^2 - 5182n + 6008) - 4v_p(3) - v_p(5) - v_p(7)$$

and

$$\begin{aligned} & v_p((n-7)!S(n,n-7)) - n \\ &= \frac{n-7-d_p(n-7)}{p-1} + v_p(S(n,n-7)) - n \\ &\le \frac{1}{p-1}\Big(n-8+(p-1)(v_p(S(n,n-7))-n)\Big) := \frac{D_{7,p}(n)}{p-1} \end{aligned} \tag{4.29}$$

with

$$\begin{aligned} D_{7,p}(n) &= (p-1)v_p(S(n,n-7)) - n(p-2) - 8 \\ &= -n(p-2) - 8 + (p-1)\Big(\sum_{i=0}^{7} v_p(n-i) + v_p(n-6) + v_p(n-7) \\ &\quad + v_p(E_{7,n}) - 4v_p(3) - v_p(5) - v_p(7)\Big) \end{aligned} \tag{4.30}$$

and

$$E_{7,n} := 9n^4 - 198n^3 + 1563n^2 - 5182n + 6008.$$

In what follows, we show that $D_{7,p}(n) < 0$. If $v_p(E_{7,n}) = 0$, then by Lemma 2.3 we derive that

$$\sum_{i=0}^{7} v_p(n-i) + v_p(n-6) + v_p(n-7) + v_p(E_{7,n}) - 4v_p(3) - v_p(5) - v_p(7) \le 1 + 3\log_p n$$

since $v_p(n-6) + v_p(n-7) \le \log_p n$. It follows from (4.30) that

$$\begin{aligned} D_{7,p}(n) &\le (p-1)(1+3\log_p n) - n(p-2) - 8 \\ &= \log_p \frac{n^{3(p-1)}}{p^{n(p-2)}} + p - 9 \\ &= \log_p G_{3,p}(n) + p - 9 \\ &\le \log_p G_{3,p}(8) + p - 9 \\ &= (p-1)\log_p \frac{8^3}{p^7} < 0. \end{aligned} \tag{4.31}$$

Now let $v_p(E_{7,n}) \ge 1$. Note that

$$E_{7,n} = 9n^4 - 198n^3 + 1563n^2 - 5182n + 6008$$

$$= 9(n-1)^4 - 162(n-1)^3 + 1023(n-1)^2 - 2614(n-1) + 2200$$
$$= 9(n-2)^4 - 126(n-2)^3 + 591(n-2)^2 - 1018(n-2) + 456$$
$$= 9(n-3)^4 - 90(n-3)^3 + 267(n-3)^2 - 178(n-3) - 88$$
$$= 9(n-4)^4 - 54(n-4)^3 + 51(n-4)^2 + 122(n-4) - 80$$
$$= 9(n-5)^4 - 18(n-5)^3 - 57(n-5)^2 + 98(n-5) + 48$$
$$= 9(n-6)^4 + 18(n-6)^3 - 57(n-6)^2 - 34(n-6) + 80$$
$$= 9(n-7)^4 + 54(n-7)^3 + 51(n-7)^2 - 58(n-7) + 16$$

and $6008 = 2^3 \cdot 751$, $2200 = 2^3 \cdot 5^2 \cdot 11$, $456 = 2^3 \cdot 3 \cdot 19$, $88 = 2^3 \cdot 11$, $80 = 2^4 \cdot 5$, $48 = 2^4 \cdot 3$, $16 = 2^4$. Thus one deduces that $v_p(n-7) = 0$. We divide rest of the proof into the following subcases.

Case 7.1. $p = 3$. Then we have $v_3(n) = v_3(n-1) = v_3(n-3) = v_3(n-4) = v_3(n-6) = 0$. Moreover, one can derive that either $1 \le v_3(E_{7,n}) \le 2 + \log_3 n^4$ with $v_3(n-2) = v_3(n-5) = 1$ or $v_3(E_{7,n}) = 1$ with $\max\{v_3(n-2), v_3(n-5)\} \ge 2$ and $\min\{v_3(n-2), v_3(n-5)\} = 1$. This implies that

$$\begin{aligned}&\sum_{i=0}^{5} v_3(n-i) + 2v_3(n-6) + 2v_3(n-7) + v_3(E_{7,n})\\ &= v_3(n-2) + v_3(n-5) + v_3(E_{7,n})\\ &\le 4 + \log_3 n^4. \end{aligned} \tag{4.32}$$

It follows from (4.30) and (4.32) that

$$\begin{aligned} D_{7,3}(n) &\le 2(4 + \log_3 n^4 - 4) - n - 8\\ &= \log_3 \frac{n^8}{3^{n+8}}\\ &= \log_3 G_{4,3}(n) - 8\\ &\le \log_3 G_{4,3}(8) - 8\\ &= \log_3 \frac{8^8}{3^{16}} < 0. \end{aligned} \tag{4.33}$$

Case 7.2. $p = 5$. One notes that $v_5(n) = v_5(n-2) = v_5(n-3) = v_5(n-5) = 0$ and at most two terms of $\{v_5(n-1), v_5(n-4), v_5(n-6)\}$ can be nonzero. Moreover, if $v_5(n-4) \ge 1$ and $v_5(n-1) = v_5(n-6) = 0$, then we can deduce that

$$v_5(n-1) + v_5(n-4) + 2v_5(n-6) = v_5(n-4) \le \log_5(n-4) \le 2 + 2\log_5(n-5);$$

and if $v_5(n-4) = 0$ with $v_5(n-1) \ge 1$ and $v_5(n-6) \ge 1$, then we get that $\min\{v_5(n-1), v_5(n-6)\} = 1$, which infers that

$$v_5(n-1) + v_5(n-4) + 2v_5(n-6) = v_5(n-1) + 2v_5(n-6) \le 2 + 2\log_5(n-5).$$

Together with $v_5(E_{7,n}) \le \log_5(n-5)^4$, one obtains that

$$\sum_{i=0}^{5} v_5(n-i) + 2v_5(n-6) + 2v_5(n-7) + v_5(E_{7,n})$$

$$
\begin{aligned}
&= v_5(n-1) + v_5(n-4) + 2v_5(n-6) + v_5(E_{7,n})\\
&\le 2 + \log_5(n-5)^6.
\end{aligned}
$$

It is easy to check that $D_{7,5}(n) \le 4(2+\log_5(n-5)^6) - 3n - 8 = 4\log_5(n-5)^6 - 3n < 0$ for $8 \le n \le 10$. Now let $n \ge 11$. Then it follows from Lemma 2.2 that

$$
\begin{aligned}
D_{7,5}(n) &\le 4\Big(2 + \log_5(n-5)^6\Big) - 3n - 8\\
&= \log_5 \frac{(n-5)^{24}}{5^{3(n-5)}} - 15\\
&= \log_5 G_{6,5}(n-5) - 15\\
&\le \log_5 G_{6,5}(6) - 15\\
&= \log_5 \frac{6^{24}}{5^{33}} < 0.
\end{aligned} \tag{4.34}
$$

Case 7.3. $p \ge 7$. One has $v_p(n-4) = v_p(n-5) = v_p(n-6) = 0$ and only one of $\{v_p(n), v_p(n-1), v_p(n-2), v_p(n-3)\}$ can be nonzero. This infers that

$$
\begin{aligned}
&\sum_{i=0}^{5} v_p(n-i) + 2v_p(n-6) + 2v_p(n-7) + v_p(E_{7,n})\\
&= \sum_{i=0}^{3} v_p(n-i) + v_p(E_{7,n})\\
&\le \log_p n^5.
\end{aligned} \tag{4.35}
$$

It follows from (4.30) and (4.35) that

$$
\begin{aligned}
D_{7,p}(n) &\le (p-1)\log_p n^5 - n(p-2) - 8\\
&= \log_p \frac{n^{5(p-1)}}{p^{n(p-2)}} - 8\\
&= \log_p G_{5,p}(n) - 8\\
&\le \log_p G_{5,p}(8) - 8\\
&= (p-1)\log_p \frac{8^5}{p^8} < 0.
\end{aligned} \tag{4.36}
$$

Hence (4.29), (4.31), (4.33), (4.34) and (4.36) give us that $v_p((n-7)!S(n, n-7)) < n$.
This completes the proof of Theorem 1.4. □

5. More results and proofs

From Theorem 1.2, we deduce the following interesting result.

Proposition 5.1. *Let p be a prime. Let n and a be positive integers with* $(a, p) = 1$.
(i). One has

$$
v_p(S(ap^n, ap^n - 1)) = \begin{cases} n-1, & \text{if } p = 2,\\ n, & \text{if } p > 2.\end{cases}
$$

(ii). If $n \geq 2$, then

$$v_p(S(ap^n, ap^n - 2)) = \begin{cases} n-2, & \text{if } p = 2, \\ n-1, & \text{if } p = 3, \\ n+1, & \text{if } p = 5, \\ n, & \text{if } p > 5 \end{cases}$$

and

$$v_p(S(ap^n, ap^n - 3)) = \begin{cases} n-2, & \text{if } p = 2, \\ n+1, & \text{if } p = 3, \\ n, & \text{if } p > 3. \end{cases}$$

(iii). If $n \geq 3$, then

$$v_p(S(ap^n, ap^n - 4)) = \begin{cases} n-3, & \text{if } p = 2, \\ n-1, & \text{if } p \in \{3,5\}, \\ n+1, & \text{if } p = 251, \\ n, & \text{otherwise}, \end{cases} \tag{5.1}$$

$$v_p(S(ap^n, ap^n - 5)) = \begin{cases} n-2, & \text{if } p = 2, \\ n-1, & \text{if } p = 3, \\ n+1, & \text{if } p \in \{5,19\}, \\ n, & \text{otherwise}, \end{cases} \tag{5.2}$$

$$v_p(S(ap^n, ap^n - 6)) = \begin{cases} n-3, & \text{if } p = 2, \\ n-2, & \text{if } p = 3, \\ n-1, & \text{if } p = 7, \\ n+1, & \text{if } p = 19087, \\ n, & \text{otherwise} \end{cases} \tag{5.3}$$

and

$$v_p(S(ap^n, ap^n - 7)) = \begin{cases} n-3, & \text{if } p = 2, \\ n-1, & \text{if } p = 3, \\ n+1, & \text{if } p \in \{7,751\}, \\ n, & \text{otherwise}. \end{cases} \tag{5.4}$$

Proof. Since $n \geq 1$ and $(a, p) = 1$, one may deduce that $v_p(ap^n) = n$ and $v_p(ap^n - 1) = 0$.

(i). From Theorem 1.2, we derive that

$$v_p(S(ap^n, ap^n - 1)) = v_p(ap^n) + v_p(ap^n - 1) - v_p(2) = n - v_p(2).$$

Thus the truth of (i) follows immediately.

(ii). Let $n \geq 2$. One gets that $v_p(ap^n - 2) = v_p(2)$ and $v_p(ap^n - 3) = v_p(3)$. Note that $v_p(3ap^n - 5) = v_p(5)$. By Theorem 1.2, we then obtain that

$$v_p(S(ap^n, ap^n - 2)) = \sum_{i=0}^{2} v_p(ap^n - i) + v_p(3ap^n - 5) - v_p(4) - v_p(3!)$$

$$= n + v_p(2) + v_p(5) - v_p(4) - v_p(3!)$$
$$= n + v_p(5) - v_p(4) - v_p(3) \quad (5.5)$$

and

$$\begin{aligned} v_p(S(ap^n, ap^n - 3)) &= \sum_{i=0}^{3} v_p(ap^n - i) + v_p(ap^n - 2) + v_p(ap^n - 3) - v_p(2) - v_p(4!) \\ &= n + 2v_p(2) + 2v_p(3) - v_p(2) - v_p(4!) \\ &= n + v_p(3) - v_p(4). \end{aligned} \quad (5.6)$$

Thus the truth of (ii) follows immediately from (5.5) and (5.6).

(iii). Let $n \geq 3$. It is easy to see that $v_p(ap^n - 4) = v_p(4)$, $v_p(ap^n - 5) = v_p(5)$, $v_p(ap^n - 6) = v_p(6)$ and $v_p(ap^n - 7) = v_p(7)$. From Theorem 1.2, one can deduce that

$$\begin{aligned} &v_p(S(ap^n, ap^n - 4)) \\ &= \sum_{i=0}^{4} v_p(ap^n - i) + v_p(15a^3p^{3n} - 150a^2p^{2n} + 485ap^n - 502) - v_p(48) - v_p(5!) \\ &= n + v_p(2) + v_p(3) + v_p(4) - v_p(48) - v_p(5!) + v_p(15a^3p^{3n} - 150a^2p^{2n} + 485ap^n - 502) \\ &= n - v_p(48) - v_p(5) + v_p(15a^3p^{3n} - 150a^2p^{2n} + 485ap^n - 502). \end{aligned} \quad (5.7)$$

Note that $48 = 2^4 \cdot 3$ and $502 = 2 \cdot 251$. Thus we derive that

$$v_p(15a^3p^{3n} - 150a^2p^{2n} + 485ap^n - 502) = \begin{cases} 1, & \text{if } p \in \{2, 251\}, \\ 0, & \text{otherwise.} \end{cases}$$

Together with (5.7), one then gets the truth of (5.1).

Subsequently, by Theorem 1.2 we obtain that

$$\begin{aligned} &v_p(S(ap^n, ap^n - 5)) \\ &= \sum_{i=0}^{5} v_p(ap^n - i) + v_p(ap^n - 4) + v_p(ap^n - 5) \\ &\quad + v_p(3a^2p^{2n} - 23ap^n + 38) - v_p(16) - v_p(6!) \\ &= n + v_p(2) + v_p(3) + 2v_p(4) + 2v_p(5) - v_p(16) - v_p(6!) + v_p(3a^2p^{2n} - 23ap^n + 38) \\ &= n + v_p(5) - v_p(6) - v_p(4) + v_p(3a^2p^{2n} - 23ap^n + 38). \end{aligned} \quad (5.8)$$

Note that $38 = 2 \cdot 19$. It follows that

$$v_p(3a^2p^{2n} - 23ap^n + 38) = \begin{cases} 1, & \text{if } p \in \{2, 19\}, \\ 0, & \text{otherwise.} \end{cases}$$

Together with (5.8), one then obtains the truth of (5.2).

Furthermore, from Theorem 1.2 we get that

$$v_p(S(ap^n, ap^n - 6))$$

$$\begin{aligned}
&= \sum_{i=0}^{6} v_p(ap^n - i) - v_p(576) - v_p(7!) \\
&\quad + v_p(63a^5p^{5n} - 1575a^4p^{4n} + 15435a^3p^{3n} - 73801a^2p^{2n} + 171150ap^n - 152696) \\
&= n - v_p(576) - v_p(7) \\
&\quad + v_p(63a^5p^{5n} - 1575a^4p^{4n} + 15435a^3p^{3n} - 73801a^2p^{2n} + 171150ap^n - 152696). \qquad (5.9)
\end{aligned}$$

Since $152696 = 2^3 \cdot 19087$ and $v_2(171150) = 1$ and $n \geq 3$, one has

$$v_p(63a^5p^{5n} - 1575a^4p^{4n} + 15435a^3p^{3n} - 73801a^2p^{2n} + 171150ap^n - 152696) = \begin{cases} 3, & \text{if } p = 2, \\ 1, & \text{if } p = 19087, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $576 = 2^6 \cdot 3^2$. Together with (5.9), one derives the truth of (5.3).

Finally, by Theorem 1.2 we can deduce that

$$\begin{aligned}
&v_p(S(ap^n, ap^n - 7)) \\
&= \sum_{i=0}^{7} v_p(ap^n - i) + v_p(ap^n - 6) + v_p(ap^n - 7) - v_p(144) - v_p(8!) \\
&\quad + v_p(9a^4p^{4n} - 198a^3p^{3n} + 1563a^2p^{2n} - 5182ap^n + 6008) \\
&= n + v_p(7) - v_p(24) - v_p(8) + v_p(9a^4p^{4n} - 198a^3p^{3n} + 1563a^2p^{2n} - 5182ap^n + 6008). \qquad (5.10)
\end{aligned}$$

Since $6008 = 2^3 \cdot 751$ and $v_2(5182) = 1$ and $n \geq 3$, one has

$$v_p(9a^4p^{4n} - 198a^3p^{3n} + 1563a^2p^{2n} - 5182ap^n + 6008) = \begin{cases} 3, & \text{if } p = 2, \\ 1, & \text{if } p = 751, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $24 = 2^3 \cdot 3$. Together with (5.10), one gets the truth of (5.4).

This completes the proof of Proposition 5.1. □

Moreover, we can obtain the following result.

Proposition 5.2. *Let p be a prime.*

(i). One has

$$v_p(S(p, p-1)) = \begin{cases} 0, & \textit{if } p = 2, \\ 1, & \textit{if } p > 2. \end{cases}$$

(ii). If $p \geq 3$, then

$$v_p(S(p, p-2)) = \begin{cases} 0, & \textit{if } p = 3, \\ 2, & \textit{if } p = 5, \\ 1, & \textit{if } p > 5. \end{cases}$$

(iii). If $p \geq 5$, then

$$v_p(S(p, p-3)) = 1$$

and

$$v_p(S(p,p-4)) = \begin{cases} 0, & \text{if } p = 5, \\ 2, & \text{if } p = 251, \\ 1, & \text{otherwise}. \end{cases}$$

(iv). If $p \geq 7$, then

$$v_p(S(p,p-5)) = \begin{cases} 2, & \text{if } p = 19, \\ 1, & \text{if } p \neq 19 \end{cases}$$

and

$$v_p(S(p,p-6)) = \begin{cases} 0, & \text{if } p = 7, \\ 2, & \text{if } p = 19087, \\ 1, & \text{otherwise}. \end{cases}$$

(v). If $p \geq 11$, then

$$v_p(S(p,p-7)) = \begin{cases} 2, & \text{if } p = 751, \\ 1, & \text{if } p \neq 751. \end{cases}$$

(vi). For any odd prime p, one has

$$v_p(S(p^2,p^2-4)) = \begin{cases} 1, & \text{if } p \in \{3,5\}, \\ 3, & \text{if } p = 251, \\ 2, & \text{otherwise}, \end{cases}$$

$$v_p(S(p^2,p^2-5)) = \begin{cases} 1, & \text{if } p = 3, \\ 3, & \text{if } p \in \{5,19\}, \\ 2, & \text{otherwise}, \end{cases}$$

$$v_p(S(p^2,p^2-6)) = \begin{cases} 0, & \text{if } p = 3, \\ 1, & \text{if } p = 7, \\ 3, & \text{if } p = 19087, \\ 2, & \text{otherwise} \end{cases}$$

and

$$v_p(S(p^2,p^2-7)) = \begin{cases} 1, & \text{if } p = 3, \\ 3, & \text{if } p \in \{7,751\}, \\ 2, & \text{otherwise}. \end{cases}$$

Proof. It is easy to check that parts (i) and (ii) are true, so we omit the details.

(iii). Since $p \geq 5$, by Theorem 1.2, we get that

$$v_p(S(p,p-3)) = v_p(p) + v_p(p-1) + 2v_p(p-2) + 2v_p(p-3) - v_p(2) - v_p(4!) = 1$$

and

$$\begin{aligned} v_p(S(p,p-4)) &= \sum_{i=0}^{4} v_p(p-i) + v_p(15p^3 - 150p^2 + 485p - 502) - v_p(48) - v_p(5!) \\ &= 1 + v_p(15p^3 - 150p^2 + 485p - 502) - v_p(5). \end{aligned} \tag{5.11}$$

Note that $502 = 2 \cdot 251$ and $485 \not\equiv 2 \pmod{251}$. So one has

$$v_p(15p^3 - 150p^2 + 485p - 502) = \begin{cases} 1, & \text{if } p = 251, \\ 0, & \text{otherwise.} \end{cases}$$

Together with (5.11) we arrive at the truth of part (iii).

(iv). Let $p \geq 7$. Since $576 = 2^6 \cdot 3^2$, from Theorem 1.2, one gets that

$$\begin{aligned} v_p(S(p, p-5)) &= \sum_{i=0}^{5} v_p(p-i) + v_p(p-4) + v_p(p-5) + v_p(3p^2 - 23p + 38) - v_p(16) - v_p(6!) \\ &= 1 + v_p(3p^2 - 23p + 38) \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} &v_p(S(p, p-6)) \\ &= \sum_{i=0}^{6} v_p(p-i) + v_p(63p^5 - 1575p^4 + 15435p^3 - 73801p^2 + 171150p - 152696) \\ &\quad - v_p(576) - v_p(7!) \\ &= 1 - v_p(7) + v_p(63p^5 - 1575p^4 + 15435p^3 - 73801p^2 + 171150p - 152696). \end{aligned} \tag{5.13}$$

Note that $38 = 2 \cdot 19$ and $23 \not\equiv 2 \pmod{19}$, $152696 = 2^3 \cdot 19087$ and $171150 \not\equiv 2^3 \pmod{19087}$. It infers that

$$v_p(3p^2 - 23p + 38) = \begin{cases} 1, & \text{if } p = 19, \\ 0, & \text{otherwise} \end{cases}$$

and

$$v_p(63p^5 - 1575p^4 + 15435p^3 - 73801p^2 + 171150p - 152696) = \begin{cases} 1, & \text{if } p = 19087, \\ 0, & \text{otherwise.} \end{cases}$$

By (5.12) and (5.13), we can deduce the truth of part (iv).

(v). Since $p \geq 11$ and $144 = 2^4 \cdot 3^2$, it follows from Theorem 1.2 that

$$\begin{aligned} v_p(S(p, p-7)) &= \sum_{i=0}^{7} v_p(p-i) + v_p(p-6) + v_p(p-7) \\ &\quad + v_p(9p^4 - 198p^3 + 1563p^2 - 5182p + 6008) - v_p(144) - v_p(8!) \\ &= 1 + v_p(9p^4 - 198p^3 + 1563p^2 - 5182p + 6008). \end{aligned} \tag{5.14}$$

By $6008 = 2^3 \cdot 751$ and $5182 \not\equiv 2^3 \pmod{751}$, one knows that

$$v_p(9p^4 - 198p^3 + 1563p^2 - 5182p + 6008) = \begin{cases} 1, & \text{if } p = 751, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the truth of part (v) follows immediately from (5.14).

(vi). Let $p \geq 3$. By Theorem 1.2, we get that

$$\begin{aligned} v_p(S(p^2, p^2-4)) &= \sum_{i=0}^{4} v_p(p^2-i) + v_p(15p^6 - 150p^4 + 485p^2 - 502) - v_p(48) - v_p(5!) \\ &= 2 + v_p(251) - v_p(3) - v_p(5), \end{aligned} \tag{5.15}$$

$$\begin{aligned} v_p(S(p^2, p^2-5)) &= \sum_{i=0}^{5} v_p(p^2-i) + v_p(p^2-4) + v_p(p^2-5) + v_p(3p^4 - 23p^2 + 38) - v_p(16) - v_p(6!) \\ &= 2 + v_p(5) + v_p(19) - v_p(3), \end{aligned} \tag{5.16}$$

$$\begin{aligned} & v_p(S(p^2, p^2-6)) \\ &= \sum_{i=0}^{6} v_p(p^2-i) + v_p(63p^{10} - 1575p^8 + 15435p^6 - 73801p^4 + 171150p^2 - 152696) \\ &\quad - v_p(576) - v_p(7!) \\ &= 2 + v_p(3) + v_p(5) + v_p(6) + v_p(152696) - v_p(576) - v_p(7!) \\ &= 2 + v_p(19087) - 2v_p(3) - v_p(7) \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} v_p(S(p^2, p^2-7)) &= \sum_{i=0}^{7} v_p(p^2-i) + v_p(p^2-6) + v_p(p^2-7) \\ &\quad + v_p(9p^8 - 198p^6 + 1563p^4 - 5182p^2 + 6008) - v_p(144) - v_p(8!) \\ &= 2 + v_p(3) + v_p(5) + 2v_p(6) + 2v_p(7) + v_p(6008) - v_p(144) - v_p(8!) \\ &= 2 + v_p(7) + v_p(751) - v_p(3). \end{aligned} \tag{5.18}$$

Hence the truth of part (vi) follows from (5.15) to (5.18).

This finishes the proof of Proposition 5.2. □

6. Conclusions

Let p be a prime number. Let n and k be positive integers. The computation of the exact p-adic valuation of Stirling numbers is difficult. In Theorem 1.1, we use 2-associated Stirling number $S_2(n,k)$ to represent a formula to calculate the p-adic valuation of Stirling numbers of the second kind $S(n,k)$. The formula of $v_p(S(n,n-k))$ depends on the value of $S_2(i,i-k)$, where $k+2 \leq i \leq 2k$. From this, we arrive at a formula to compute $v_p(S(n,n-k))$, which enables us to show that $v_p((n-k)!S(n,n-k)) < n$ with $0 \leq k \leq \min\{7, n-1\}$ and $p \geq 3$. Let $1 \leq k \leq 7$ and a be a positive integer with $(a,p) = 1$. For $n \geq 3$, by Proposition 5.1 we know that $n-3 \leq v_p(S(ap^n, ap^n-k)) \leq n+1$, and $v_p(S(ap^n, ap^n-k)) = n$ holds if $p > 19087$. Moreover, for any prime number p with $p > 19087$, Propositions 5.1 and 5.2 also imply that $v_p(S(p, p-k)) = 1$ and $v_p(S(p^2, p^2-k)) = 2$.

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Conflict of interest

We declare that we have no conflict of interest.

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