Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(4): 3783–3790.
DOI: 10.3934/math.2020245
Received: 26 February 2020
Accepted: 17 April 2020
Published: 21 April 2020

Research article

On C-ideals and the basis of an ordered semigroup

Ze Gu[1], **Xiang-Yun Xie**[2,*] **and Jian Tang**[3]

[1] School of Mathematics and Statistics, Zhaoqing University, Zhaoqing, Guangdong, P.R. China
[2] School of Mathematics and Computational Science, Wuyi University, Jiangmen, Guangdong, P.R. China
[3] School of Mathematics and Statistics, Fuyang Normal University, Fuyang, Anhui, P.R. China

* **Correspondence:** Email: xyxie@wyu.edu.cn; guze528@sina.com.

Abstract: In this paper, we characterize ordered semigroups containing the greatest ideal and give the conditions of the greatest ideal being a C-ideal in an ordered semigroup. Moreover, we introduce the concept of a basis of an ordered semigroup and study the relationship between the greatest C-ideal and the basis in an ordered semigroup.

Keywords: ordered semigroup; greatest ideal; C-ideal; basis; greatest C-ideal
Mathematics Subject Classification: 06F05, 20M12

1. Introduction and preliminaries

Ideal theory play an important role in studying ordered semigroups. The concepts of ideals, bi-ideals, quasi-ideals, weakly prime ideals and prime ideals in ordered semigroups have been introduced by N. Kehayopulu in [1–3]. Moreover, fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals, weakly prime fuzzy ideals and prime fuzzy ideals in ordered semigroups have been studied in [4–10].

The concept of C-ideals in semigroups has been introduced by I. Fabrici in [11]. The second author Xie of this paper extended the concept to ordered semigroups in [12]. Xie has studied some basic properties of C-ideals in ordered semigroups in [13] and characterized the ordered semigroups in which every proper ideal is a C-ideal in [12]. Mao, Xu and Lian have further studied some properties of C-ideals in ordered semigroups in [14]. Motivated by the previous work, in this paper, we study ordered semigroups containing the greatest ideal and give the conditions of the greatest ideal to be a C-ideal in an ordered semigroup. Moreover, we introduce the concept of a basis of an ordered semigroup and study the relationship between the greatest C-ideal and the basis in an ordered semigroup. If the order relation is trivial in an ordered semigroup S, then S is a semigroup. Consequently, all the results in this paper are true for semigroups.

We call $(S, \cdot, \leq)$ an *ordered semigroup* if $(S, \cdot)$ is a semigroup, $(S, \leq)$ is an ordered set and it satisfies:

$$a \leq b \Rightarrow ax \leq bx,\ xa \leq xb\ \ (\forall a, b, x \in S).\ ([15])$$

We will just use S to denote an ordered semigroup when the operation and order are understood. A nonempty subset I of an ordered semigroup S is called an *ideal* if I satisfies: 1) $IS, SI \subseteq I$; 2) $\forall a \in S, a \leq b \in I \Rightarrow a \in I$. An ideal I of S is called *proper* if $I \neq S$. A proper ideal M of S is called the *greatest ideal* if every proper ideal is contained in M. A proper ideal M of S is called a *maximal ideal* if whenever there exists an ideal N of S such that $M \subsetneq N$, then $N = S$. An ordered semigroup S is called *simple* if S contains no proper ideals. Let H be a nonempty subset of S. Denote

$$(H] := \{x \in S \mid \exists h \in H, x \leq h\};\ \ [H) := \{x \in S \mid \exists h \in H, h \leq x\}.$$

If H has only one element a, then we denote $(H]$ and $[H)$ by $(a]$ and $[a)$ respectively. For any $x \in S$, denote by $I(x)$ the ideal generated by x. Then we have $I(x) = (x \cup Sx \cup xS \cup SxS]$ (see [1, 2]).

Green's relation $\mathcal{J} := \{(x, y) \in S \times S \mid I(x) = I(y)\}$ on an ordered semigroup S was introduced by N. Kehayopulu in [16]. It is easy to see that the relation $\mathcal{J}$ is an equivalence on S. For any $x \in S$, denote by I^x the $\mathcal{J}$-class containing x. We define a relation "$\preccurlyeq$" on the set of all $\mathcal{J}$-classes in S as below:

$$I^x \preccurlyeq I^y \Leftrightarrow I(x) \subseteq I(y)\ \ (\forall x, y \in S).$$

It is routine to verify that "$\preccurlyeq$" is an order relation. A $\mathcal{J}$-class I^x of S is called *maximal* if there is no other $\mathcal{J}$-class I^y such that $I^x \preccurlyeq I^y$. A $\mathcal{J}$-class I^x of S is called the *greatest $\mathcal{J}$-class* if other $\mathcal{J}$-classes are all contained in I^x. From [12], we know that M is a maximal ideal of S if and only if $S - M$ is a maximal $\mathcal{J}$-class.

2. The greatest ideal and C-ideals

Let S be an ordered semigroup. A proper ideal M of S is called a *C-ideal* if $M \subseteq (S(S - M)S]$. We know from [14] the following basic properties of C-ideals.

Lemma 1. *(1)* (Theorem 5, [14]) *If S is not simple, then S contains at least one C-ideal;*

(2) (Theorem 1, [14]) *If S contains two different proper ideals M_1, M_2 such that $M_1 \cup M_2 = S$, then neither of them is a C-ideal of S;*

(3) (Corollary 1, [14]) *If S contains more than one maximal ideal, then none of them is a C-ideal of S;*

(4) (Theorem 4, [14]) *If S contains only one maximal ideal M and M is a C-ideal, then M is the greatest ideal of S;*

(5) (Theorem 2, Theorem 3 [14]) *If M_1 and M_2 are two C-ideals of S, then $M_1 \cup M_2$ and $M_1 \cap M_2$ are C-ideals of S.*

(6) (Theorem 7, [14]) *If S contains the greatest ideal M^* and M^* is a C-ideal, then every proper ideal of S is a C-ideal.*

However, the following result given in [14] is incorrect, we next improve it.

Result 1. (Theorem 6, [14]) *Let S be an ordered semigroup with an identity* 1. *If S contains the greatest ideal, denoted by M^*, then M^* is a C-ideal or there exists $a \in S - (S^3]$ such that $S - [a) \subseteq M^*$.*

Remark 1. *If S has an identity 1, then $S = (S^3]$, i.e. $S - (S^3] = \emptyset$. Consequently, we next improve Result 1 by the following result.*

Theorem 1. *Let S be an ordered semigroup with an identity 1. Then every proper ideal of S is a C-ideal. In particular, if S contains the greatest ideal, denoted by M^*, then M^* is a C-ideal.*

Proof. Let M be a proper ideal of S. Then $1 \notin M$. Suppose that $1 \in M$. Then $S = S \cdot 1 \subseteq SM \subseteq M$, i.e. $S = M$, this is a contradiction. Thus $1 \in S - M$. It follows that $(S(S-M)S] = S$. Therefore $M \subseteq (S(S-M)S]$, i.e. M is a C-ideal.

Since M^* is a proper ideal of S, we have M^* is a C-ideal. □

In the following, we study ordered semigroups containing the greatest ideal and give the conditions of the greatest ideal to be a C-ideal in an ordered semigroup.

Lemma 2. *If $S \neq (S^2]$, then $I(x) = (x]$ for any $x \in S - (S^2]$.*

Proof. Let $x \in S - (S^2]$. Suppose that $I(x) \neq (x]$. Then there exists $a \in S$ or $b \in S$ or $c, d \in S$ such that $ax > x$ or $xb > x$ or $cxd > x$. Since $ax, xb, cxd \in S^2$, we have $x \in (S^2]$. This is a contradiction. □

Theorem 2. *If $S \neq (S^2]$, then S contains the greatest ideal if and only if $S = (a]$ for some $a \in S - (S^2]$.*

Proof. ($\Rightarrow$) Denote by M^* the greatest ideal of S. Let $a \in S - M^*$. Since $(S^2] \subseteq M^*$, $a \in S - (S^2]$. Moreover, $I^a = S - M^*$ is the greatest $\mathcal{J}$-class. Thus $I(x) \subseteq I(a)$ for any $x \in M^*$ and so $M^* \subseteq I(a)$. In addition, $I(a) \cup M^* = S$. It follows that $S = I(a) = (a]$ from Lemma 2.

($\Leftarrow$) Let $S = (a]$ for some $a \in S - (S^2]$. Next we prove that $S - \{a\}$ is an ideal of S. In fact: for any $x, y \in S$ and $b \in S - \{a\}$, we have $xb, by \in S - \{a\}$. Otherwise, $a = xb = by \in S^2$ which contradicts that $a \in S - (S^2]$. Let $c \in S - \{a\}$ and $z \in S$. If $z \leq c$, then $z \in S - \{a\}$. Otherwise, $z = a$ which contradicts that $c < a$. It follows that $S - \{a\}$ is an ideal of S.

Finally, we show that $S - \{a\}$ is the greatest. Let I be a proper ideal of S. Then $a \notin I$ and thus $I \subseteq S - \{a\}$. □

Theorem 3. *Let M^* be the greatest ideal of S.*

(1) If $S = (S^2]$, then M^* is a C-ideal.

(2) If $S \neq (S^2]$, then $M^* = S - \{a\}$ for some $a \in S - (S^2]$.

Proof. (1) Since $(S(S-M^*)S]$ is an ideal of S and M^* is the greatest ideal of S, we have $(S(S-M^*)S] = S$ or $(S(S-M^*)S] \subseteq M^*$. We distinguish three cases.

1) If $(S(S-M^*)S] = S$, then $M^* \subseteq (S(S-M^*)S]$. Hence M^* is a C-ideal.

2) If $(S(S-M^*)S] = M^*$, then M^* is a C-ideal clearly.

3) If $(S(S-M^*)S] \subset M^*$, then $(S^3] = (S(S-M^*)S \cup SM^*S] \subseteq (S(S-M^*)S] \cup (SM^*S] \subset M^* \cup (M^*] = M^* \cup M^* = M^* \subset S$. Since $S = (S^2]$, we have $S = (S^3]$ which contradicts $(S^3] \subset S$.

(2) It can be easily obtained from the proof of Theorem 2. □

Corollary 1. *Let M^* be the greatest ideal of S. If M^* is a C-ideal, then $(S^2] = (S^3]$. In particular, if $S \neq (S^2]$, then M^* is a C-ideal if and only if $M^* = (S^3]$.*

Proof. If $S = (S^2]$, then $(S^2] = (S^3]$ obviously. Next we consider the case of $S \neq (S^2]$. From Theorem 3, we know that $M^* = S - \{a\}$ for some $a \in S - (S^2]$. Since M^* is a C-ideal and $(S^2]$ is a proper ideal, we have $(S^2] \subseteq M^* \subseteq (S(S-M^*)S] = (SaS] \subseteq (S^3] \subseteq (S^2]$. It follows that $M^* = (S^2] = (S^3]$.

Let $S \neq (S^2]$. We need only prove the sufficiency. We know that $S = (a]$ for some $a \in S - (S^2]$ from Theorem 2 and $M^* = S - \{a\}$ from Theorem 3(2). Since $M^* = (S^3]$, we have $M^* = (S^3] = (S(a]S] = (SaS] = (S(S - M^*)S]$. Thus M^* is a C-ideal. □

3. The basis and the greatest C-ideal

In this section, we introduce the concept of a basis of an ordered semigroup and study the relation between the existence of the greatest C-ideal and the existence of a basis in an ordered semigroup.

Definition 1. *A nonempty subset A of an ordered semigroup S is called a basis of S if A satisfies the following conditions:*

1) $(A \cup SA \cup AS \cup SAS] = S$;

2) There is no proper subset B of A such that $(B \cup SB \cup BS \cup SBS] = S$.

Example 1. *We consider the order semigroup $S = \{a, b, c, d\}$ with the multiplication "$\cdot$" and the order relation "$\leq$" below:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (d, d)\}.$$

Let $A = \{c, d\}$. Then $(A \cup SA \cup AS \cup SAS] = S$. However, $d \notin (c \cup Sc \cup cS \cup ScS]$ and $c \notin (d \cup Sd \cup dS \cup SdS]$. Thus A is a basis of S.

Theorem 4. *Let A be a nonempty subset of S. Then A is a basis of S if and only if A satisfies:*

1) For any $x \in S$, there exists $a \in A$ such that $I(x) \subseteq I(a)$;

2) For any $a_1, a_2 \in A$, if $I^{a_1} \preccurlyeq I^{a_2}$, then $a_1 = a_2$.

Proof. ($\Rightarrow$) Obviously, condition 1) can be obtained from Definition 1. Next we prove condition 2).

If $I^{a_1} \preccurlyeq I^{a_2}$, then $I(a_1) \subseteq I(a_2)$. Suppose that $a_1 \neq a_2$. Let $B = A - \{a_1\}$. Then $(A \cup SA \cup AS \cup SAS] \subseteq (B \cup SB \cup BS \cup SBS]$. Therefore $(B \cup SB \cup BS \cup SBS] = S$. This contradicts Definition 1.

($\Leftarrow$) By the condition 1), we have

$$S = \bigcup_{x \in S} I(x) \subseteq \bigcup_{a \in A} I(a) = (A \cup SA \cup AS \cup SAS] \subseteq S.$$

Hence $(A \cup SA \cup AS \cup SAS] = S$. Suppose that there exists $B \subset A$ such that $(B \cup SB \cup BS \cup SBS] = S$. Let $a_1 \in A - B$. Then there exists $b_1 \in B$ or $b_2 \in B$ or $b_3 \in B$ or $b_4, b_5 \in B$, $s \in S$ such that $a_1 \leq b_1$ or $a_1 \leq sb_2$ or $a_1 \leq b_3 s$ or $a_1 \leq b_4 s b_5$. It implies that $I(a_1) \subseteq I(b_1)$ or $I(a_1) \subseteq I(b_2)$ or $I(a_1) \subseteq I(b_3)$ or $I(a_1) \subseteq I(b_4)$, i.e. $I^{a_1} \preccurlyeq I^{b_1}$ or $I^{a_1} \preccurlyeq I^{b_2}$ or $I^{a_1} \preccurlyeq I^{b_3}$ or $I^{a_1} \preccurlyeq I^{b_4}$. This contradicts 2). □

Remark 2. *By Theorem 4, we can see that if A is a basis of S, then every element of A belongs to some maximal $\mathcal{J}$-class and there is only one element in A for every maximal $\mathcal{J}$-class.*

Proposition 1. *Let $\{M_\alpha \mid \alpha \in \Lambda\}$ be a set of all maximal ideals of S, $\hat{M} = \bigcap_{\alpha\in\Lambda} M_\alpha$ and $\bar{M}_\alpha = S - M_\alpha$. Then we have*

1) $S = (\bigcup_{\alpha\in\Lambda} \bar{M}_\alpha) \cup \hat{M}$;

2) If $\alpha \neq \beta$, then $\bar{M}_\alpha \cap \bar{M}_\beta = \emptyset$;

3) If $\alpha \neq \beta$, then $\bar{M}_\alpha \subseteq M_\beta$;

4) If $\alpha \neq \beta$, then $\bar{M}_\alpha\bar{M}_\beta \subseteq \hat{M}$, i.e. $\hat{M} \neq \emptyset$;

5) Let I be an ideal of S and $I \cap \bar{M}_\alpha \neq \emptyset$, then $\bar{M}_\alpha \subseteq I$.

Proof. 1) Since

$$\hat{M} = \bigcap_{\alpha\in\Lambda} M_\alpha = \bigcap_{\alpha\in\Lambda}(S - \bar{M}_\alpha) = S - \bigcup_{\alpha\in\Lambda} \bar{M}_\alpha,$$

we have

$$S = (\bigcup_{\alpha\in\Lambda} \bar{M}_\alpha) \cup \hat{M}.$$

2) We know that $\bar{M}_\alpha$ is a maximal $\mathcal{J}$-class for any $\alpha \in \Lambda$. Thus $\bar{M}_\alpha \cap \bar{M}_\beta = \emptyset$ when $\alpha \neq \beta$.

3) It can be obtained from 2) obviously.

4) Let $\alpha \neq \beta, \mu_\alpha \in \bar{M}_\alpha, \mu_\beta \in \bar{M}_\beta$ and $\mu = \mu_\alpha\mu_\beta \notin \hat{M}$. By 1), we have $S = (\bigcup_{\alpha\in\Lambda} \bar{M}_\alpha) \cup \hat{M}$. Thus there exists $\mu_\gamma \in \bar{M}_\gamma$ such that $\mu = \mu_\gamma$.

i) If $\gamma \neq \alpha$, then $\bar{M}_\alpha \subseteq M_\gamma$. Thus $\bar{M}_\alpha\bar{M}_\beta \subseteq M_\gamma$. Hence $\mu_\gamma \in M_\gamma$ which contradicts $\mu_\gamma \in \bar{M}_\gamma$.

ii) If $\gamma = \alpha$, then $\bar{M}_\beta \subseteq M_\gamma$. It follows that $\bar{M}_\alpha\bar{M}_\beta \subseteq M_\gamma$ Thus $\mu_\gamma \in M_\gamma$ which is also a contradiction.

By *i*) and *ii*), we get $\mu = \mu_\alpha\mu_\beta \in \hat{M}$. Hence $\bar{M}_\alpha\bar{M}_\beta \subseteq \hat{M}$, i.e. $\hat{M} \neq \emptyset$.

5) If I is an ideal of S and $I \cap \bar{M}_\alpha \neq \emptyset$, then $M_\alpha \cup I = S$. Thus, $\bar{M}_\alpha \subseteq I$. □

Remark 3. *1) If S contains the greatest C-ideal, denote it by M^g, then $M^g \subseteq \hat{M}$. Indeed: If there exists $\alpha \in \Lambda$ such that $M^g \nsubseteq M_\alpha$, then $M^g \cup M_\alpha = S$. By Lemma 1(2), we can see that M^g is not a C-ideal. This is a contradiction.*

2) The greatest C-ideal M^g does not necessarily exist in all ordered semigroups. See the following example.

Example 2. *Let $S = \{a, b, c, d, e\}$ with the multiplication "$\cdot$" and the order "$\leq$" below:*

$\cdot$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	c	c
d	d	d	d	d	d
e	d	d	c	d	c

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, b), (d, c), (d, d), (e, c), (e, e)\}.$$

It is easy to check that $(S, \cdot, \leq)$ is an ordered semigroup and $\{a\}, \{d\}, \{e\}$ are C-ideals of S. However, there is not the greatest C-ideal.

In the following, we give the relationship between the basis and the greatest C-ideal M^g of S.

Theorem 5. *Let S contain a basis A. Then S contains the greatest C-ideal M^g. Moreover, $M^g = (S^3] \cap \hat{M}$ where $\hat{M} = \bigcap_{\alpha\in\Lambda} M_\alpha$ and $\{M_\alpha \mid \alpha \in \Lambda\}$ is the family of all maximal ideals of S.*

Proof. For any $a \in A$, I^a is a maximal $\mathcal{J}$-class. Thus $S - I^a$ is a maximal ideal. Hence $\hat{M} \neq \emptyset$. Since $\hat{M}$ and $(S^3]$ are ideals of S, $\hat{M} \cap (S^3] \neq \emptyset$. Denote $N = \hat{M} \cap (S^3]$. Then for any $x \in N$, there exists $c \in S$ such that $x \in (ScS]$. If $c \notin A$, then there exists $b \in A$ such that $I(c) \subsetneq I(b)$. Therefore $c \in (b \cup Sb \cup bS \cup SbS]$. Obviously, $c \neq b$. Next we distinguish two cases.

1) If $c < b$, then $(ScS] \subseteq (SbS]$. Thus $x \in (SbS]$.

2) If $c \in (Sb \cup bS \cup SbS]$, then $(ScS] \subseteq (S(Sb \cup bS \cup SbS]S] \subseteq (S^2bS \cup SbS^2 \cup S^2bS^2] \subseteq (SbS]$. Hence $x \in (SbS]$.

By 1) and 2), we have $x \in (SbS] \subseteq (SAS] \subseteq (S(S - \hat{M})S] \subseteq (S(S - N)S]$ (Because $S - \hat{M} = S - \bigcap_{\alpha\in\Lambda} M_\alpha = \bigcup_{\alpha\in\Lambda} \bar{M}_\alpha$ and every element of A is in some maximal $\mathcal{J}$-class). Consequently, N is a C-ideal.

Finally, we prove that N is the greatest. Let T be a C-ideal of S. Then $T \subseteq (S(S - T)S] \subseteq (S^3]$. By Lemma 1(4), we have every C-ideal is contained in all maximal ideals. Thus $T \subseteq \hat{M}$. Then $T \subseteq (S^3] \cap \hat{M}$. It follows that $N = (S^3] \cap \hat{M}$ is the greatest C-ideal M^g. □

Proposition 2. *Let S contain the greatest C-ideal M^g and $M^g \subsetneq (S^3]$. Then every $\mathcal{J}$-class of $(S^3] - M^g$ is maximal and $I(a) = (SaS]$ for any $a \in (S^3] - M^g$.*

Proof. Since $(S^3]$ and M^g are ideals of S, $(S^3] - M^g$ is a union of some $\mathcal{J}$-classes of S. Let M^γ be any one of them. Then $M^\gamma \subseteq (S^3]$. It implies that there exist $x, y, b \in S$ such that $a \leq xby$ for any $a \in M^\gamma$. Thus $a \in (SbS]$. Next we show $b \in M^\gamma$. If there exists $\delta \neq \gamma$ such that $b \in M^\delta$, then $a \in (SbS]$ which implies that $I(a) \subseteq I(b)$ and $b \notin I(a)$. Otherwise, $b \in I(a)$. Hence $I(b) \subseteq I(a)$ and so $I(a) = I(b)$ which contradicts $\delta \neq \gamma$. Therefore $b \in S - I(a)$. It follows that $a \in (S(S - I(a))S]$. Then $I(a) \subseteq (S(S - I(a))S]$, i.e. $I(a)$ is a C-ideal. Therefore $I(a) \cup M^g$ is a C-ideal properly containing M^g, which is impossible. Thus $b \in M^\gamma$. This implies $I(a) \subseteq (SbS] \subseteq I(b) = I(a)$, hence $I(a) = (SbS] = I(b)$. Next we show that $I(a) = (SaS]$.

1) If $a = b$, then $I(a) = (SaS]$.

2) If $a \neq b$, then $b \in (a \cup Sa \cup aS \cup SaS]$.

i) If $b < a$, then $I(b) = (SbS] \subseteq (SaS] \subseteq I(a) = I(b)$. Thus $I(a) = (SaS]$.

ii) If $b \in (Sa \cup aS \cup SaS]$, then $I(b) = (SbS] \subseteq (S(Sa \cup aS \cup SaS]S] \subseteq (S^2aS \cup SaS^2 \cup S^2aS^2] \subseteq (SaS] \subseteq I(a) = I(b)$. Hence $I(a) = (SaS]$.

Finally, we show that M^γ is maximal. Suppose that $I(a) = (SaS] \subsetneq I(c)$ for $a \in M^\gamma \subseteq (S^3] - M^g$ and some $c \in S$. Then $a \in I(c) = (c \cup Sc \cup cS \cup ScS]$. Obviously, $a \neq c$. We show that $I(a) \subseteq (ScS]$.

1) If $a < c$, then $I(a) = (SaS] \subseteq (ScS]$.

2) If $a \in (Sc \cup cS \cup ScS]$, then $I(a) = (SaS] \subseteq (S(Sc \cup cS \cup ScS]S] \subseteq (S^2cS \cup ScS^2 \cup S^2cS^2] \subseteq (ScS]$.

Since $c \notin I(a)$, we have $I(a) \subseteq (S(S - I(a))S]$, i.e. $I(a)$ is a C-ideal. Therefore $I(a) \cup M^g$ is a C-ideal properly containing M^g, this is a contradiction. It follows that M^γ is a maximal $\mathcal{J}$-class. □

Theorem 6. *Let S contain the greatest C-ideal M^g. If $S \neq (S^2]$ and any two elements of $S - (S^2]$ are incomparable, then S contains a basis.*

Proof. Firstly, we have $M^g \subseteq (S(S - M^g)S] \subseteq (S^3] \subseteq (S^2] \subseteq S$. We denote by M^α a $\mathcal{J}$-class of $S - (S^2]$, by M^β a $\mathcal{J}$-class of $(S^2] - (S^3]$ and by M^γ a $\mathcal{J}$-class of $(S^3] - M^g$.

1) From Lemma 2, we have

$$I(x) = I(y) \Leftrightarrow (x] = (y] \Leftrightarrow x = y \ \ (\forall x, y \in S - (S^2]).$$

Thus there is only one element in M^α. Since any two elements of $S - (S^2]$ are incomparable, M^α is maximal.

2) For any $x \in M^\beta \subseteq (S^2] - (S^3]$, there exist $u, v \in S$ such that $x \le uv$. Here $u, v \in S - (S^2]$. Otherwise $u, v \in (S^2]$, then $x \in (S^3]$ which is a contradiction. Therefore $x \in I(u) = (u]$.

3) From Proposition 2, we get that M^γ is maximal. Since $M^g \subseteq (S(S-M^g)S]$, there exists $z \in S-M^g$ such that $y \in I(z)$ for any $y \in M^g$. It follows that $y \in M^\alpha$ or M^β or M^γ. Next we construct a set A.

i) If $z \in M^\alpha$ or $z \in M^\gamma$, then we choose z into A.

ii) If $z \in M^\beta$, then there exists $u \in M^\alpha$, such that $z \in I(u)$. Hence we can choose u into A.

Now, we have

$$M^g \subseteq (A \cup SA \cup AS \cup SAS];$$
$$(S^2] - (S^3] \subseteq (A \cup SA \cup AS \cup SAS];$$
$$S - (S^2] \subseteq (A \cup SA \cup AS \cup SAS];$$
$$(S^3] - M^g \subseteq (A \cup SA \cup AS \cup SAS].$$

Therefore $S \subseteq (A \cup SA \cup AS \cup SAS]$, i.e. $(A \cup SA \cup AS \cup SAS] = S$.

To prove that A is a basis of S, it remains to show that there is no proper subset $B \subsetneq A$ with the property $(B \cup SB \cup BS \cup SBS] = S$. This is evident, because A is constructed by means of elements of maximal $\mathcal{J}$-classes of S, and from each maximal $\mathcal{J}$-class just one element is chosen into A. □

4. Conclusions

Ideal theory play an important role in studying ordered semigroups. In this paper, we first study ordered semigroups containing the greatest ideal and give the conditions of the greatest ideal to be a C-ideal in an ordered semigroup. Furthermore, we introduce the concept of a basis of an ordered semigroup and establish the relationship between the greatest C-ideal and the basis in an ordered semigroup.

Acknowledgments

We thank the referees whose comments led to significant improvements to this paper and the editor for his/her warm work.

This research was supported by the National Natural Science Foundation of China (No. 11701504, 11801081); the Natural Science Foundation of Guangdong Province (No. 2014A030313625, No. 2018A030313063); the Project for Characteristic Innovation of 2018 Guangdong University (No. 2018KTSCX234); the Teaching Team Project in Guangdong Province (Guangdong teach highletter [2018] no. 179); the Demonstration Project of Grass-roots Teaching and Research Section in Anhui Province (No. 2018jyssf053); the University Natural Science Project of Anhui Province (No. KJ2018A0329) and the Innovation Education Project of Graduate Students of Guangdong Province (No. 2016SFKS-40).

Conflict of interest

The authors declare no conflict of interest.

References

1. N. Kehayopulu, *On weakly prime ideals in ordered semigroups*, Math. Japon., **35** (1990), 1050–1056.
2. N. Kehayopulu, *On prime, weakly prime ideals in ordered semigroups*, Semigroup Forum, **44** (1992), 341–346.
3. N. Kehayopulu, *On completely regular poe-semigroups*, Math. Japon., **37** (1992), 123–130.
4. N. Kehayopulu, M. Tsingelis, *Fuzzy sets in ordered groupoids*, Semigroup Forum, **65** (2002), 128–132.
5. N. Kehayopulu, M. Tsingelis, *Fuzzy bi-ideals in ordered semigroups*, Inform. Sci., **171** (2005), 13–28.
6. N. Kehayopulu, M. Tsingelis, *Regular ordered semigroups in terms of fuzzy subsets*, Inform. Sci., **176** (2006), 3675–3693.
7. X. Y. Xie, J. Tang, *Fuzzy radicals and prime fuzzy ideals of ordered semigroups*, Inform. Sci., **178** (2008), 4357–4374.
8. X. Y. Xie, J. Tang, *Regular ordered semigroups and intra-regular ordered semigroups in tems of fuzzy subsets*, Iranian J. Fuzzy Syst., **7** (2010), 121–140.
9. M. Shabir, A. Khan, *Characterizations of ordered semigroups by the properties of their fuzzy ideals*, Comput. Math. Appl., **59** (2010), 539–549.
10. M. Shabir, A. Khan, *Fuzzy quasi-ideals of ordered semigroups*, Bull. Malays. Math. Sci. Soc, **34** (2011), 87–102.
11. I. Fabrici, *Semigroups containing covered two-sided ideals*, Math. Slovaca, **34** (1984), 355–363.
12. M. F. Wu, X. Y. Xie, *On C-ideals of ordered semigroups*, J. Wuyi University (in Chinese), **9** (1995), 43–46.
13. X. Y. Xie, M. F. Wu, *On po-semigroups containing no maximal ideals*, Southeast Asian Bull Math., **20** (1996), 31–36.
14. H. Y. Mao, X. Z. Xu, X. P. Lian, *On C-ideals of ordered semigroups*, J. Shandong University (in Chinese), **45** (2010), 14–16.
15. L. Fuchs, Partially Ordered Algebraic Systems, New York: Pregamon Press, 1963.
16. N. Kehayopulu, *Note on Green's relations in ordered semigroups*, Math. Japon., **36** (1991), 211–214.

AIMS Press