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Research article

Implicit fractional differential equation with anti-periodic boundary condition involving Caputo-Katugampola type

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Abstract: This paper deals with a nonlinear implicit fractional differential equation with the anti-periodic boundary condition involving the Caputo-Katugampola type. The existence and uniqueness results are established by applying the fixed point theorems of Krasnoselskii and Banach. Further, by using generalized Gronwall inequality the Ulam-Hyers stability results are proved. To demonstrate the effectiveness of the main results, appropriate examples are granted.

Keywords: fractional differential equations; Katugampola fractional operator; Ulam-Hyers stability; fixed point theorems; fractional Gronwall inequality
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1. Introduction

The topic of fractional-order of differential equations has recently evolved as an interesting field of research. In fact, fractional derivatives types supply a luxurious tool for the description of memory and hereditary properties of various materials and processes. More investigators have found that fractional-order differential equations play important roles in many research fields, such as chemical technology, physics, biotechnology, population dynamics, and economics. On the advanced development of the fractional differential equations have been caught much attention recently due to exact description of nonlinear phenomena, for example, an understanding the behavior of a flow and heat transfer at the nanoscale has been a great interest in recent years, one can find more details in the series of papers published [1–6]. In recent years, many classes of differential equations involving the Caputo (Riemann-Liouville, Hilfer, and Hadamard) fractional derivative have been investigated and developed by using different tools from the nonlinear analysis. For more details, see the monographs of Kilbas et al. [7], Malinowska et al. [8], Podlubny [9], and some papers, for instance, [10–12] and the references cited therein.

Recently, in [13] the author introduced a new fractional integral, which generalizes the Riemann-Liouville and Hadamard integrals into a single form. For more properties such as expansion formulas, variational calculus applications, control theoretical applications, convexity, and integral inequalities and Hermite-Hadamard type inequalities of this new operator and similar operators, can be found in [14–17]. The corresponding fractional derivatives were introduced in [8, 18, 19] which so-called Katugampola fractional operators.

The existence and uniqueness results of fractional differential equations involving Caputo-Katugampola derivative are given in [20], the author used the Peano theorem to obtain the existence and uniqueness of solution for the following Cauchy type problem

$$ {}^{c}D_{0^+}^{\alpha;\rho} x(t) = g(t, x(t)), \quad t \in [0, T], \tag{1.1} $$

$$ x^{(k)}(0) = x_0^{(k)}, \;\; k = 0, 1, ..., m-1, \; m = [\alpha]. \tag{1.2} $$

In the same context, R. Almeida in [21], proved the uniqueness of solution of the problem (1.1)–(1.2) involving ${}^{c}D_{a^+}^{\alpha;\rho}$ via Gronwall inequality type. On the other hand, Oliveira and de Oliveira in [22], considered the initial value problem for a nonlinear fractional differential equation including Hilfer-Katugampola derivative of the form

$$ {}^{\rho}D_{a^+}^{\alpha,\beta} x(t) = g(t, x(t)), \quad t \in J = [a, b], \tag{1.3} $$

$$ {}^{\rho}I_{a^+}^{1-\gamma} x(a) = c, \;\; \gamma = \alpha + \beta - \alpha\beta. \tag{1.4} $$

They used the generalized Banach fixed point theorem to investigate the existence and uniqueness results on the problem (1.3)–(1.4).

The recent development of implicit fractional differential equations and the theoretical analysis can be seen in [23–26]. Some anti-periodic boundary value problems for fractional differential equations were also discussed in [27–30]. In order to investigate the different kinds of stability in the Ulam sense for fractional differential equations, we mention the works [31–34].

To the best of our knowledge, the implicit fractional differential equations with anti-periodic boundary conditions and Caputo-Katugampola type have not yet been studied widely till the present day. So, in this paper, we investigate a new class of Caputo-Katugampola type implicit fractional differential equation, that is

$$ {}^{c}D_{a^+}^{\alpha;\rho} x(t) = g(t, x(t), {}^{c}D_{a^+}^{\alpha;\rho} x(t)), \; t \in J = [a, T]. \tag{1.5} $$

$$ x(a) + x(T) = 0, \tag{1.6} $$

where $0 < \alpha < 1$, ${}^{c}D_{a^+}^{\alpha;\rho}$ is the fractional derivatives of order α in the Caputo-Katugampola sense, and $g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is an appropriate function.

The purpose of this paper is to study the existence, uniqueness and Ulam-Hyers stability of solutions of the given problem (1.5)–(1.6). Our study is based on fixed point theorems due to Banach and Krasnoselskii [35], and generalized Gronwall inequality [36].

This is the recent and new work on the boundary value problem for implicit fractional differential equations with an anti-periodic condition involving Caputo-Katugampola fractional derivative. The proposed problem is more generalized, and some it in the literature are the special cases of it. Moreover, our analysis can also be applied to the addressed problems by selecting the with the convenient parameter of ρ, i.e., The Caputo-Katugampola fractional derivative ${}^{c}D_{a^+}^{\alpha;\rho}$ is an interpolator of the following fractional derivatives: standard Caputo ($\rho \to 1$, $a \to 0$) [37] , Caputo-Hadamard ($\rho \to 0$) [38], Liouville ($\rho \to 1$, $a \to 0$) [7], and Weyl ($\rho \to 1$, $a \to -\infty$) [7].

The paper is systematized as follows: In the section 2, we survey briefly the properties of Katugampola fractional integral and Caputo-Katugampola fractional derivative, and we also introduce the fundamental tools related to our analysis and proving some axiom lemmas which play a key role in the sequel. Section 3 and 4 are devoted to the existence, uniqueness and stability results of the problem (1.5)–(1.6) by applying the Krasnoselskii/Banach fixed point theorem, and generalized Gronwall inequality. The last section promotes our outcomes to problem (1.5)–(1.6)) by giving illustrative examples to justify the provided results.

2. Preliminaries

We shall start this section with recall some essential lemmas, basic definitions, lemmas and preliminary facts related to our results throughout the paper. Let $J = [a, T]$ $(-\infty < a < T < \infty)$ be a finite interval of $\mathbb{R}$. Denote $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ endowed with the norm given by

$$||z||_C = \sup_{t\in J} |z(t)| : t \in J\},$$

for $z \in C(J, \mathbb{R})$. $C^n(J, \mathbb{R})$ $(n \in \mathbb{N}_0)$ denotes the set of mappings having n times continuously differentiable on J.

For $a < T$, $c \in \mathbb{R}$ and $1 \leq p < \infty$, define the function space

$$X_c^p(a,T) = \left\{ z : J \to \mathbb{R} : ||z||_{X_c^p} = \left(\int_a^T |t^c z(t)|^p \, \frac{dt}{t} \right)^{\frac{1}{p}} < \infty \right\}.$$

for $p = \infty$,

$$||z||_{X_c^p} = ess \sup_{a\leq t\leq T} [|t^c z(t)|] \, .$$

Definition 2.1. *[13] Let $t > a$ be two reals, $\alpha > 0$, $\rho > 0$, $c \in \mathbb{R}$ and $z \in X_c^p(a, T)$. The left-sided Katugampola fractional integral of order α with dependence on a parameter ρ is defined by*

$$I_{a^+}^{\alpha;\rho} z(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \tau^{\rho-1} \left(t^\rho - \tau^\rho\right)^{\alpha-1} z(\tau) d\tau, \tag{2.1}$$

where, $\Gamma(.)$ is a gamma function.

Definition 2.2. *[18] Let $n - 1 < \alpha < n$, $(n = [\alpha] + 1)$, $\rho > 0$, $c \in \mathbb{R}$ and $z \in X_c^p(a, T)$. The left-sided Katugampola fractional derivative of order α with dependence on a parameter ρ is defined as*

$$D_{a^+}^{\alpha;\rho} z(t) = \left(t^{1-\rho} \frac{d}{dt} \right)^n I_{a^+}^{n-\alpha;\rho} z(t) = \frac{\gamma^n \, \rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^t \tau^{\rho-1} \left(t^\rho - \tau^\rho\right)^{n-\alpha-1} z(\tau) d\tau, \; t > a, \tag{2.2}$$

where $\gamma = \left(t^{1-\rho}\frac{d}{dt}\right)$. *In particular, if* $0 < \alpha < 1, \rho > 0$, *and* $z \in C^1(J, \mathbb{R})$, *we have*

$$D_{a^+}^{\alpha;\rho} z(t) = \left(t^{1-\rho}\frac{d}{dt}\right) I_{a^+}^{1-\alpha;\rho} z(t) = \frac{\gamma\rho^{\alpha}}{\Gamma(1-\alpha)} \int_a^t \tau^{\rho-1} \left(t^{\rho} - \tau^{\rho}\right)^{-\alpha} z(\tau)d\tau,\ t > a.$$

Definition 2.3. *[18] Let* $\alpha \geq 0$, $n = [\alpha] + 1$. *If* $z \in C^n(J, \mathbb{R})$. *The left sided Caputo-Katugampola fractional derivative of order* α *with a parameter* $\rho > 0$ *is defined by*

$$^{C}D_{a^+}^{\alpha;\rho} z(t) = D_{a^+}^{\alpha;\rho}\left[z(t) - \sum_{k=0}^{n-1} \frac{z_{\rho}^{(k)}(a)}{k!}\rho^{-k}\left(t^{\rho} - a^{\rho}\right)^k\right], \tag{2.3}$$

where $z_{\rho}^{(k)}(t) = \left(t^{1-\rho}\frac{d}{dt}\right)^k z(t)$. *In case* $0 < \alpha < 1$, *and* $z \in C^1(J, \mathbb{R})$, *we have*

$$^{C}D_{a^+}^{\alpha;\rho} z(t) = D_{a^+}^{\alpha;\rho}\left[z(t) - z(a)\right]. \tag{2.4}$$

From (2.4) and (2.2), we obtain

$$^{C}D_{a^+}^{\alpha;\rho} z(t) = \frac{\gamma\rho^{\alpha}}{\Gamma(1-\alpha)} \int_a^t \tau^{\rho-1}\left(t^{\rho} - \tau^{\rho}\right)^{-\alpha}\left[z(\tau) - z(a)\right] d\tau,\ t > a,\ \gamma = \left(t^{1-\rho}\frac{d}{dt}\right)$$

Obviously, if $\alpha \notin \mathbb{N}_0$, *and* $z \in C^1(J, \mathbb{R})$, *then the Caputo-Katugampola fractional derivative exists a.e, moreover, we have*

$$\begin{aligned} ^{C}D_{a^+}^{\alpha;\rho} z(t) &= \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} \int_a^t \tau^{\rho-1}\left(t^{\rho} - \tau^{\rho}\right)^{-\alpha} z_{\rho}^{(1)}(\tau)d\tau,\ t > a, \\ &= I_{a^+}^{1-\alpha;\rho} z_{\rho}^{(1)}(t). \end{aligned}$$

Also, if $\alpha \in \mathbb{N}$, *then* ${}^{C}D_{a^+}^{\alpha;\rho} z(t) = z_{\rho}^{(n)}(t)$. *Particularly,* ${}^{C}D_{a^+}^{0;\rho} z(t) = z_{\rho}^{(0)}(t) = z(t)$.

Lemma 2.4. *[13]* $I_{a^+}^{\alpha;\rho}$ *is bounded on the function space* $X_c^p(a, T)$.

Lemma 2.5. *[13] Let* $\alpha > 0, \beta > 0, z \in X_c^p(a, T)$ $(1 \leq p \leq \infty), \rho, c \in \mathbb{R}, \rho \geq c$. *Then we have*

$$I_{a^+}^{\alpha;\rho} I_{a^+}^{\beta;\rho} z(t) = I_{a^+}^{\alpha+\beta;\rho} z(t), \qquad {}^{c}D_{a^+}^{\alpha;\rho}\, {}^{\rho}I_{a^+}^{\alpha} z(t) = z(t).$$

Lemma 2.6. *[13,18] Let* $t > a$, $\alpha, \delta \in (0, \infty)$, *and* $I_{a^+}^{\alpha;\rho}$, $D_{a^+}^{\alpha;\rho}$ *and* ${}^{C}D_{a^+}^{\alpha;\rho}$ *are according to (2.1), (2.2) and (2.3) respectively. Then we have*

$$I_{a^+}^{\alpha;\rho}\left(t^{\rho} - a^{\rho}\right)^{\delta-1} = \frac{\rho^{-\alpha}\,\Gamma(\delta)}{\Gamma(\delta+\alpha)}\left(t^{\rho} - a^{\rho}\right)^{\alpha+\delta-1},$$

$$^{C}D_{a^+}^{\alpha;\rho}\left(t^{\rho} - a^{\rho}\right)^{\delta-1} = \frac{\rho^{+\alpha}\,\Gamma(\delta)}{\Gamma(\delta-\alpha)}\left(t^{\rho} - a^{\rho}\right)^{\delta-\alpha-1},$$

and

$$^{C}D_{a^+}^{\alpha;\rho}\left(t^{\rho} - a^{\rho}\right)^{k} = 0,\ \ \alpha \geq 0,\ k = 0, 1, ..., n-1.$$

Particularly, ${}^{C}D_{a^+}^{\alpha;\rho}(1) = 0$.

Lemma 2.7. *[39] Let $\alpha, \rho > 0$ and $x \in C(J,\mathbb{R}) \cap C^1(J,\mathbb{R})$. Then*

1. *The Caputo-Katugampola fractional differential equation*
$$ {}^cD_{a^+}^{\alpha;\rho} x(t) = 0 $$
has a solution
$$ x(t) = c_0 + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right) + c_2\left(\frac{t^\rho - a^\rho}{\rho}\right)^2 + ... + c_{n-1}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{n-1}, $$
where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$ and $n = [\alpha] + 1$.
2. *If $x, {}^C D_{a^+}^{\alpha;\rho} x \in C(J,\mathbb{R}) \cap C^1(J,\mathbb{R})$. Then*
$$ I_{a^+}^{\alpha;\rho}\, {}^C D_{a^+}^{\alpha;\rho} x(t) = x(t) + c_0 + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right) + c_2\left(\frac{t^\rho - a^\rho}{\rho}\right)^2 + ... + c_{n-1}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{n-1}, \tag{2.5} $$
where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$ and $n = [\alpha] + 1$.

Lemma 2.8. *[36] Let $\alpha > 0$, v, w be two integrable functions and z a continuous function, with domain $[a, T]$. Assume that v and w are nonnegative; and let z is nonnegative and nondecreasing. If*
$$ v(t) \le w(t) + z(t)\rho^{1-\alpha}\int_a^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} v(\tau)d\tau, \quad t \in [a, T], $$
then
$$ v(t) \le w(t) + \int_a^t \left[\sum_{k=1}^{\infty} \frac{\rho^{1-k\alpha}(z(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \tau^{\rho-1}(t^\rho - \tau^\rho)^{k\alpha-1} w(\tau)\right] d\tau, \quad t \in [a, T]. $$

Remark 2.9. *In particular, if $w(t)$ be a nondecreasing function on J. Then we have*
$$ v(t) \le w(t) E_\alpha\left[g(t)\Gamma(\alpha)\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha\right], \ t \in [a, T]. $$
where $E_\alpha(\cdot)$ is the Mittag-Leffler function defined by
$$ E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \ \ x \in \mathbb{C}, \ Re(\alpha) > 0. $$

Theorem 2.10. *[35] (Banach fixed point theorem) Let (X, d) be a nonempty complete metric space with $Q : X \to X$ is a contraction mapping. Then map Q has a fixed point.*

Theorem 2.11. *[35] (Krasnoselskii's fixed point theorem) Let X be a Banach space, let Ω be a bounded closed convex subset of X and let Q_1, Q_2 be mapping from Ω into X such that $Q_1x + Q_2y \in \Omega$ for every pair $x, y \in \Omega$. If Q_1 is contraction and Q_2 is completely continuous, then there exists $z \in \Omega$ such that $Q_1z + Q_2z = z$.*

3. Existence and uniqueness theorems

In this section, our purpose is to discuss the existence and uniqueness of solutions to the fractional boundary value problem (1.5)–(1.6). The following lemma plays a pivotal role in the forthcoming analysis.

Lemma 3.1. *[39] Let* $0 < \alpha < 1, \rho > 0$ *and* $w \in C(J, \mathbb{R})$*. Then the linear anti-periodic boundary value problem*

$$ {}^cD_{a^+}^{\alpha;\rho} x(t) = w(t), \quad t \in J, \tag{3.1} $$

$$ x(a) + x(T) = 0, \tag{3.2} $$

has a unique solution defined by

$$ x(t) = -\frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-1} w(\tau) d\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} w(\tau) d\tau. \tag{3.3} $$

Lemma 3.2. *Assume that* $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous. A function* $x(t)$ *solves the problem (1.5)–(1.6) if and only if it is a fixed-point of the operator* $Q : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ *defined by*

$$ \begin{aligned} Qx(t) \quad = \quad & -\frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-1} g(\tau, x(\tau), {}^c D_{a^+}^{\alpha;\rho} x(\tau)) d\tau \\ & + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} g(\tau, x(\tau), {}^c D_{a^+}^{\alpha;\rho} x(\tau)) d\tau. \end{aligned} \tag{3.4} $$

Our first result is based on Banach's fixed point theorem to obtain the existence of a unique solution of problem (1.5)–(1.6).

Theorem 3.3. *Assume that* $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a continuous satisfies the following condition:*

(H$_1$) *There exists a constant* $0 < L < 1$ *such that:*

$$ |g(t, x_1, x_2) - g(t, y_1, y_2)| \leq L \left[|x_1 - y_1| + |x_2 - y_2| \right], \quad \forall t \in J, \ x_i, y_i \in \mathbb{R}, (i = 1, 2). $$

If

$$ \mathcal{N} = \frac{3}{2}\frac{L\rho^{-\alpha}}{1-L}\frac{(T^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)} < 1, \tag{3.5} $$

then the problem (1.5)–(1.6) has a unique solution on J.

Proof. Now, we first show that the operator $Q : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by (3.4) is well-defined, i.e., we show that $Q\mathcal{S}_r \subseteq \mathcal{S}_r$ where

$$ \mathcal{S}_r = \{x \in C(J, \mathbb{R}), \|x\| \leq r\}, \tag{3.6} $$

with choose $r \geq \frac{\mathcal{M}}{1-\mathcal{N}}$, where $\mathcal{N} < 1$ and

$$ \mathcal{M} = \frac{3}{2}\frac{\mu\rho^{-\alpha}}{1-L}\frac{(T^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)}. $$

and $\sup_{t\in J}|g(t,0,0)| := \mu < \infty$. Set $G_x(t) := g(t, x(t), {}^c D_{a^+}^{\alpha;\rho} x(t))$. For any $x \in \mathcal{S}_r$, we obtain by our hypotheses that

$$\begin{aligned}
|Qx(t)| &\leq \sup_{t\in J}|Qx(t)| \\
&\leq \sup_{t\in J}\left\{\frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}|G_x(\tau)|\,d\tau\right. \\
&\quad \left.+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}|G_x(\tau)|\,d\tau\right\}.
\end{aligned}$$

From (H_1), we have

$$\begin{aligned}
|G_x(\tau)| &= \left|g(\tau, x(\tau), {}^c D_{a^+}^{\alpha;\rho} x(\tau))\right| \\
&\leq \left|g(\tau, x(\tau), {}^c D_{a^+}^{\alpha;\rho} x(\tau)) - g(\tau,0,0)\right| + |g(\tau,0,0)| \\
&\leq L|x(\tau)| + L\left|{}^c D_{a^+}^{\alpha;\rho} x(\tau)\right| + \mu \\
&= Lr + L|G_x(\tau)| + \mu
\end{aligned}$$

which gives

$$|G_x(\tau)| \leq \frac{(Lr+\mu)}{1-L}. \tag{3.7}$$

Therefore,

$$\begin{aligned}
|Qx(t)| &\leq \sup_{t\in J}\left\{\frac{1}{2}\frac{(Lr+\mu)}{1-L}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}d\tau\right. \\
&\quad \left.+\frac{(Lr+\mu)}{1-L}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}|G_x(\tau)|\,d\tau\right\} \\
&\leq \frac{3}{2}\frac{(Lr+\mu)\rho^{-\alpha}}{1-L}\frac{(T^\rho-a^\rho)^\alpha}{\Gamma(\alpha+1)} \\
&= \frac{3}{2}\frac{L\rho^{-\alpha}}{1-L}\frac{(T^\rho-a^\rho)^\alpha}{\Gamma(\alpha+1)}r + \frac{3}{2}\frac{\mu\rho^{-\alpha}}{1-L}\frac{(T^\rho-a^\rho)^\alpha}{\Gamma(\alpha+1)} \\
&= \mathcal{N}r + \mathcal{M} < r,
\end{aligned}$$

$$\|Qx\| < r,$$

which implies that $Qx \in \mathcal{S}_r$. Moreover, by (3.4), and lammas 2.5, 2.6, we obtain

$${}^C D_{a^+}^{\alpha;\rho} Qx(t) = {}^C D_{a^+}^{\alpha;\rho} I_{a^+}^{\alpha;\rho} G_x(t) = G_x(t).$$

Since $G_x(\cdot)$ is continuous on J, the operator ${}^c D_{a^+}^{\alpha;\rho} Qx(t)$ is continuous on J, that is $Q\,\mathcal{S}_r \subseteq \mathcal{S}_r$.

Next, we apply the Banach fixed point theorem to prove that Q has a fixed point. Indeed, it enough to show that Q is contraction map. Let $x_1, x_2 \in C(J,\mathbb{R})$ and for $t \in J$. Then, we have

$$\begin{aligned}
|Qx_1(t) - Qx_2(t)| &\leq \frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}\left|G_{x_1}(\tau)-G_{x_2}(\tau)\right|d\tau \\
&\quad +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\left|G_{x_1}(\tau)-G_{x_2}(\tau)\right|d\tau
\end{aligned}$$

by (H_1), we get

$$\begin{aligned}\left|G_{x_1}(\tau)-G_{x_2}(\tau)\right| &= \left|g(\tau,x_1(\tau),{}^cD_{a^+}^{\alpha;\rho}x_1(\tau))-g(\tau,x_1(\tau),{}^cD_{a^+}^{\alpha;\rho}x_2(\tau))\right|\\ &\leq L\left|x_1-x_2\right|+L\left|{}^cD_{a^+}^{\alpha;\rho}x_1(\tau)-{}^cD_{a^+}^{\alpha;\rho}x_2(\tau)\right|\\ &= L\left|x_1-x_2\right|+L\left|G_{x_1}(\tau)-G_{x_2}(\tau)\right|,\end{aligned}$$

which implies

$$\left|G_{x_1}(\tau)-G_{x_2}(\tau)\right|\leq\frac{L}{1-L}\left|x_1-x_2\right|. \tag{3.8}$$

Then

$$\|Qx_1-Qx_2\|\leq\frac{3}{2}\frac{L}{1-L}\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}(T^\rho-a^\rho)^\alpha\|x_1-x_2\|.$$

Consequently, $\|Qx_1-Qx_2\|\leq\mathcal{N}\|x_1-x_2\|$. Since $\mathcal{N}<1$, the operator Q is contraction mapping. As a consequence of theorem 2.10, then the problem (1.5)–(1.6) has a unique solution. This complete the proof. □

Our second existence result for the problem (1.5)–(1.6) is based on the Krasnoselskii′s fixed point theorem.

Theorem 3.4. *Assume that (H_1) holds. If*

$$\Lambda:=\frac{3}{2}\frac{L}{1-L}\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}(T^\rho-a^\rho)^\alpha<\frac{1}{2},$$

then the problem (1.5)–(1.6) has at least one solution on J.

Proof. Consider the operator $Q:C(J,\mathbb{R})\longrightarrow C(J,\mathbb{R})$ defined by (3.4). Define the ball $\mathcal{S}_{r_0}:=\{x\in C(J,\mathbb{R}):\|x\|\leq r_0\}$, with $r_0\geq 2\mu\Lambda$, where μ is defined as in Theorem 3.3. Furthere, we define the operators Q_1 and Q_2 on $\mathcal{S}_{r_0}$ by

$$Q_1x(t)=-\frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}G_x(\tau)d\tau,$$

and

$$Q_2x(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}G_x(\tau)d\tau.$$

Taking into account that Q_1 and Q_2 are defined on $\mathcal{S}_{r_0}$, and for any $x\in C(J,\mathbb{R})$,

$$Qx(t)=Q_1x(t)+Q_2x(t),\quad t\in J.$$

The proof will be divided into several claims:

Claim 1: $Q_1x_1+Q_2x_2\in\mathcal{S}_{r_0}$ for every $x_1,x_2\in\mathcal{S}_{r_0}$.

For $x_1\in\mathcal{S}_{r_0}$ and using the same arguments in (3.7), we get

$$\left|G_{x_1}(\tau)\right|\leq\frac{(Lr_0+\mu)}{1-L}.$$

Similarly, for $x_2 \in \mathcal{S}_{r_0}$, we obain

$$\left|G_{x_2}(\tau)\right| \leq \frac{(Lr_0+\mu)}{1-L}.$$

Now, for $x_1, x_2 \in \mathcal{S}_{r_0}$ and $t \in J$, we have

$$\begin{aligned}
|Q_1x_1(t)+Q_2x_2(t)| &\leq |Q_1x_1(t)|+|Q_2x_2(t)| \\
&\leq \frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}\left|G_{x_1}(\tau)\right|d\tau \\
&\quad +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\left|G_{x_2}(\tau)\right|d\tau \\
&\leq \frac{3}{2}\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}(T^\rho-a^\rho)^\alpha\frac{(Lr_0+\mu)}{1-L} \\
&\leq \left(\frac{3}{2}\frac{L}{1-L}\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}(T^\rho-a^\rho)^\alpha\right)r_0+\frac{3}{2}\frac{\mu}{1-L}\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}(T^\rho-a^\rho)^\alpha
\end{aligned}$$

which gives

$$\|Q_1x_1+Q_2x_2\| \leq r_0. \tag{3.9}$$

This proves that $Q_1x_1+Q_2x_2 \in \mathcal{S}_{r_0}$ for every $x_1, x_2 \in \mathcal{S}_{r_0}$.

Claim 2 Q_1 is a contration mapping on $\mathcal{S}_{r_0}$.

Since Q is contraction mapping as in Theorem 3.3, then Q_1 is a contraction map too.

Claim 3. The operator Q_2 is completely continuous on $\mathcal{S}_{r_0}$.

First, from the continuity of $G_x(\cdot)$, we conclude that the operator Q_2 is continuous.

Next, It is easy to verify that

$$\|Q_2x\| \leq \frac{(Lr_0+\mu)}{1-L}\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}(T^\rho-a^\rho)^\alpha < r_0,$$

due to definitions of Λ and r_0. This proves that Q_2 is uniformly bounded on $\mathcal{S}_{r_0}$

Finally, we prove that Q_2 maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$, i.e., $(Q\mathcal{S}_{r_0})$ is equicontinuous. We estimate the derivative of $Q_2x(t)$

$$\begin{aligned}
\left|(Q_2x)'(t)\right| &= \left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha-1)}\int_a^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-2}G_x(\tau)d\tau\right| \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha-1)}\int_a^t \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-2}|G_x(\tau)|\,d\tau \\
&\leq \frac{(Lr_0+\mu)}{1-L}\frac{\rho^{-\alpha}}{\Gamma(\alpha)}(T^\rho-a^\rho)^{\alpha-1} := K
\end{aligned}$$

Now, Let $t_1, t_2 \in J$, with $t_1 < t_2$ and for any $x \in \mathcal{S}_{r_0}$. Then we have

$$|Q_2x(t_1)-Q_2x(t_2)| = \int_{t_1}^{t_2}\left|(Q_2x)'(\tau)\right|d\tau \leq K(t_2-t_1).$$

As $t_1 \longrightarrow t_2$ the right-hand side of the above inequality is not dependent on x and tends to zero. Consequently,

$$|Q_2x(t_1)-Q_2x(t_2)| \to 0,\ \forall\ |t_2-t_1| \to 0,\ x \in \mathcal{S}_{r_0}.$$

This proves that Q_2 is equicontinuous on $\mathcal{S}_{r_0}$. According to Arzela-Ascoli Theorem, it follows that Q_2 is relatively compact on $\mathcal{S}_{r_0}$. Hence all the hypotheses of Theorem 2.11 are satisfied. Therefore, we conclude that the problem (1.5)–(1.6) has at least one solution on J. □

4. Ulam-Hyers stability

In this section, we discuss the Ulam-Hyers and generalized Ulam-Hyers stability of Caputo-Katugampola-type for the problem (1.5)–(1.6). The following observations are taken from [33, 37].

Definition 4.1. *The problem (1.5)–(1.6) is Ulam-Hyers stable, if there exists a real number* $K_f > 0$, *such that for each* $\varepsilon > 0$ *and for each solution* $\widetilde{x} \in C(J, \mathbb{R})$ *of the inequality*

$$\left| {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(t) - g(t, \widetilde{x}(t), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(t)) \right| \leq \varepsilon, \quad t \in J, \tag{4.1}$$

there exists a solution $x \in C(J, \mathbb{R})$ *for the problem (1.5)–(1.6) such that*

$$|\widetilde{x}(t) - x(t)| \leq K_f \varepsilon, \quad t \in J.$$

Definition 4.2. *The problem (1.5)–(1.6) is generalized Ulam-Hyers stable if there exists* $\Psi \in C([0, \infty), [0, \infty))$ *with* $\Psi(0) = 0$, *such that for each solution* $\widetilde{x} \in C(J, \mathbb{R})$ *of the inequality*

$$\left| {}^{c}D_{a^+}^{\alpha;\rho}\, \widetilde{x}(t) - g(t, \widetilde{x}(t), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(t)) \right| \leq \varepsilon, \quad t \in J, \tag{4.2}$$

there exists a solution $x \in C(J, \mathbb{R})$ *for the problem (1.5)–(1.6) such that*

$$|\widetilde{x}(t) - x(t)| \leq \Psi(\varepsilon), \ t \in J.$$

Remark 4.3. *Let* $\alpha, \rho > 0$. *A function* $\widetilde{x} \in C(J, \mathbb{R})$ *is a solution of the inequality (4.1) if and only if there exist a function* $h_{\widetilde{x}} \in C(J, \mathbb{R})$ *such that*

1. $|h_{\widetilde{x}}(t)| \leq \varepsilon$ *for all* $t \in J$,
2. $D_{a^+}^{\alpha;\rho}\, \widetilde{x}(t) = g(t, \widetilde{x}(t), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(t)) + h_{\widetilde{x}}(t)$, $t \in J$.

Lemma 4.4. *Let* $\widetilde{x} \in C(J, \mathbb{R})$ *is a solution of the inequality (4.1). Then* $\widetilde{x}$ *is a solution of the following integral inequality:*

$$\begin{aligned} &\left| \widetilde{x}(t) - Z_{\widetilde{x}} - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \tau^{\rho-1} \left(t^\rho - a^\rho\right)^{\alpha-1} g(\tau, \widetilde{x}(\tau), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(\tau)) d\tau \right| \\ \leq\ & \frac{3}{2} \frac{\varepsilon \rho^{-\alpha}}{\Gamma(\alpha+1)} \left(T^\rho - a^\rho\right)^\alpha, \end{aligned}$$

where

$$Z_{\widetilde{x}} = \frac{1}{2} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^T \tau^{\rho-1} \left(T^\rho - a^\rho\right)^{\alpha-1} g(\tau, \widetilde{x}(\tau), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(\tau)) d\tau. \tag{4.3}$$

Proof. In view of Remark 4.3, and Theorem 3.3, we obtain

$$\begin{aligned} \widetilde{x}(t) &= -\frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T \tau^{\rho-1}\left(T^\rho - a^\rho\right)^{\alpha-1}\left[g(\tau,\widetilde{x}(\tau),{}^cD_{a^+}^{\alpha;\rho}\widetilde{x}(\tau)) + h_{\widetilde{x}}(\tau))\right]d\tau \\ &\quad +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}\left(t^\rho - a^\rho\right)^{\alpha-1}\left[g(\tau,\widetilde{x}(\tau),{}^cD_{a^+}^{\alpha;\rho}\widetilde{x}(\tau)) + h_{\widetilde{x}}(\tau)\right]d\tau. \end{aligned} \tag{4.4}$$

It follows that

$$\begin{aligned} &\left|\widetilde{x}(t) - Z_{\widetilde{x}} - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}\left(t^\rho - a^\rho\right)^{\alpha-1} g(\tau,\widetilde{x}(\tau),{}^cD_{a^+}^{\alpha;\rho}\widetilde{x}(\tau))d\tau\right| \\ \leq\ & \frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T \tau^{\rho-1}\left(T^\rho - a^\rho\right)^{\alpha-1}|h_{\widetilde{x}}(t)|\,d\tau \\ & +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}\left(t^\rho - a^\rho\right)^{\alpha-1}|h_{\widetilde{x}}(t)|\,d\tau \\ \leq\ & \frac{\varepsilon}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T \tau^{\rho-1}\left(T^\rho - a^\rho\right)^{\alpha-1}d\tau \\ & +\varepsilon\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}\left(t^\rho - a^\rho\right)^{\alpha-1}d\tau \\ \leq\ & \frac{3}{2}\frac{\varepsilon\rho^{-\alpha}}{\Gamma(\alpha+1)}\left(T^\rho - a^\rho\right)^{\alpha}. \end{aligned}$$

□

Theorem 4.5. *Assume that the hypotheses of Theorem 3.3 are satisfied. Then the problem (1.5)–(1.6) is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$, and $\widetilde{x} \in C(J,\mathbb{R})$ be a function which satisfies the inequality (4.1), and let $x \in C(J,\mathbb{R})$ be the unique solution of the following Caputo-Katugampola fractional differential equation

$$ {}^cD_{a^+}^{\alpha;\rho}\, x(t) = g(t, x(t), {}^cD_{a^+}^{\alpha;\rho}x(t)), \quad t \in J, \tag{4.5}$$

with

$$x(a) = \widetilde{x}(a),\ \ x(T) = \widetilde{x}(T), \tag{4.6}$$

where $0 < \alpha < 1$. Using Lemma 3.1, It is easily seen that $x(\cdot)$ satisfies the integral equation

$$x(t) = Z_x + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} g(\tau, x(\tau), {}^cD_{a^+}^{\alpha;\rho}x(\tau))d\tau,$$

where

$$Z_x = -\frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-1} g(\tau, x(\tau), {}^cD_{a^+}^{\alpha;\rho}x(\tau))d\tau.$$

Applying Lemma 4.4, we obtain

$$\left|\widetilde{x}(t) - Z_{\widetilde{x}} - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} g(\tau, x(\tau), {}^cD_{a^+}^{\alpha;\rho}x(\tau))d\tau\right| \leq V\varepsilon, \tag{4.7}$$

where

$$V := \frac{3}{2}\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\left(T^{\rho}-a^{\rho}\right)^{\alpha}.$$

From (4.6) we can easily get that $|Z_{\widetilde{x}} - Z_x| \to 0$. Indeed, from ($H_1$) and (4.6), we obtain that

$$\begin{aligned} |Z_{\widetilde{x}} - Z_x| &= \Big| \frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^T \tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\alpha-1} g(\tau, \widetilde{x}(\tau), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(\tau))d\tau \\ &\quad - \frac{1}{2}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^T \tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\alpha-1} g(\tau, x(\tau), {}^{c}D_{a^+}^{\alpha;\rho}x(\tau))d\tau \Big| \\ &\le \frac{1}{2} I_{a^+}^{\alpha;\rho} \left| g(T, \widetilde{x}(T), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(T)) - g(T, x(T), {}^{c}D_{a^+}^{\alpha;\rho}x(T)) \right|. \end{aligned}$$

Since,

$$\begin{aligned} & \left| g(T, \widetilde{x}(T), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(T)) - g(T, x(T), {}^{c}D_{a^+}^{\alpha;\rho}x(T)) \right| \\ \le\ & L\,|\widetilde{x}(T) - x(T)| + L \left| {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(T) - {}^{c}D_{a^+}^{\alpha;\rho}x(T) \right| \\ \le\ & \frac{L}{1-L}\,|\widetilde{x}(T) - x(T)| \end{aligned} \tag{4.8}$$

which implies

$$|Z_{\widetilde{x}} - Z_x| \le \frac{L}{2(1-L)} I_{a^+}^{\alpha;\rho}\ |\widetilde{x}(T) - x(T)| \to 0.$$

Hence,

$$x(t) = Z_{\widetilde{x}} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\alpha-1} g(\tau, x(\tau), {}^{c}D_{a^+}^{\alpha;\rho}x(\tau))d\tau.$$

According to (4.7), (H_1) and (4.8), we obtain

$$\begin{aligned} |\widetilde{x}(t) - x(t)| &\le \left| \widetilde{x}(t) - Z_{\widetilde{x}} - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\alpha-1} g(\tau, \widetilde{x}(\tau), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(\tau))d\tau \right| \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\alpha-1} \left| g(\tau, \widetilde{x}(\tau), {}^{c}D_{a^+}^{\alpha;\rho}\widetilde{x}(\tau)) - g(\tau, x(\tau), {}^{c}D_{a^+}^{\alpha;\rho}x(\tau)) \right| d\tau \\ &\le V\varepsilon + \frac{L}{1-L}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\alpha-1} |\widetilde{x}(\tau) - x(\tau)|\, d\tau. \end{aligned}$$

Applying Lemma 2.8, and Remark 2.9, it follows that

$$\begin{aligned} |\widetilde{x}(t) - x(t)| &\le V\varepsilon + \int_a^t \left[\sum_{k=1}^{\infty} \frac{\rho^{1-k\alpha}\left(\frac{L}{1-L}\rho^{1-\alpha}\right)^k}{\Gamma(k\alpha)} \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{k\alpha-1} V\varepsilon \right] d\tau \\ &\le \frac{3}{2}\frac{\varepsilon\rho^{-\alpha}}{\Gamma(\alpha+1)} (T^{\rho}-a^{\rho})^{\alpha} E_{\alpha}\left(\frac{L}{1-L}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha} \right) \\ &\le \frac{3}{2}\frac{\varepsilon\rho^{-\alpha}}{\Gamma(\alpha+1)} (T^{\rho}-a^{\rho})^{\alpha} E_{\alpha}\left(\frac{L}{1-L}\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)^{\alpha} \right). \end{aligned}$$

For $K_f = \frac{3}{2}\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}\left(T^{\rho}-a^{\rho}\right)^{\alpha} E_{\alpha}\left(\frac{L}{1-L}\left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\right)$, we get

$$|\widetilde{x}(t) - x(t)| \leq K_f \varepsilon. \tag{4.9}$$

Therefore the problem (1.5)–(1.6) is Ulam-Hyers stable. □

Corollary 4.6. *Under assumptions of Theorem 4.5, Assume that* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\Psi(0) = 0$. *Then tne problem (3.1)–(3.2) is generalized Ulam-Hyers stable.*

Proof. One can repeat the same processes in Theorem 4.5 with putting $K_f\varepsilon = \Psi(\varepsilon)$, and $\Psi(0) = 0$, we conclude that

$$|\widetilde{x}(t) - x(t)| \leq \Psi(\varepsilon).$$

□

5. Examples

Example 5.1. *Consider the following problem of implicit fractional differential equations involving Caputo Katugampola type and anti-period condition:*

$$ {}^{CK}D_{0^+}^{\frac{1}{2};\frac{1}{2}} x(t) = \begin{cases} \left[\frac{1}{3}e^{\sqrt{t}+1} + \frac{2+|x(t)|+\left|D_{0^+}^{\frac{1}{2};\frac{1}{2}} x(t)\right|}{8e^{2-t}\left(1+|x(t)|+\left|D_{0^+}^{\frac{1}{2};\frac{1}{2}} x(t)\right|\right)}\right], & t \in [0,1], \\ x(0) + x(1) = 0, \end{cases} . \tag{5.1}$$

Set:

$$g(t,u,v) = \left[\frac{1}{3}e^{\sqrt{t}+1} + \frac{2+u+v}{8e^{2-t}(1+u+v)}\right], \quad t \in [0,1], u, v \in \mathbb{R}^+,$$

with $\alpha = \frac{1}{2}$ *and* $\rho = \frac{1}{2}$. *Clearly, the function* $g \in C([0,1])$. *For each* $u, v, u^*, v^* \in \mathbb{R}^+$ *and* $t \in [0,1]$

$$\begin{aligned} |g(t,u,v) - g(t,u^*,v^*)| &= \left|\frac{2+u+v}{8e^{2-t}(1+u+v)} - \frac{2+u^*+v^*}{8e^{2-t}(1+u^*+v^*)}\right| \\ &\leq \frac{1}{8e^{2-t}}\left(|u-u^*| + |v-v^*|\right) \\ &\leq \frac{1}{8e}\left(|u-u^*| + |v-v^*|\right). \end{aligned}$$

Hence, the condition (H_1) *is satisfied with* $L = \frac{1}{8e}$. *It is easy to verify that* $\mathcal{N} = \frac{3}{4(1-\frac{1}{8e})e\sqrt{2\pi}} < 1$. *Since all the assumptions of Theorem 3.3 are fulfilled, therefore problem (5.1) has a unique solution.*

Example 5.2. *Consider the following problem of implicit fractional differential equations involving Caputo Katugampola type and anti-period condition:*

$$\begin{cases} {}^{CK}D_{0^+}^{\frac{1}{3};\frac{3}{2}} x(t) = \frac{|x(t)|+\cos\left|{}^{CK}D_{0^+}^{\frac{3}{2};1} x(t)\right|}{30(t+2)(1+|x(t)|)}, \ t \in [0,1] \\ x(0) = -x(1), \end{cases} \tag{5.2}$$

Set:

$$g(t,u,v) = \frac{u+\cos v}{30(t+2)(1+u)}, \quad t \in [0,1], u, v \in \mathbb{R}^+,$$

with $\alpha = \frac{1}{3}, \rho = \frac{3}{2}$ *and* $T = 1$. *Now, for each* $u, v, u^*, v^* \in \mathbb{R}^+$ *and* $t \in [0,1]$

$$\begin{aligned} |g(t,u,v) - g(t,u^*,v^*)| &= \left| \frac{u+\cos v}{30(t+2)(1+u)} - \frac{u^*+\cos v^*}{30(t+2)(1+u^*)} \right| \\ &\leq \frac{1}{30}\left(|u-u^*| + |v-v^*|\right). \end{aligned}$$

Hence, the condition (H_1) *is satisfied with* $L = \frac{1}{30}$. *It is easy to check that* $\mathcal{N} \approx 0.05 < 1$. *It follows from Theorem 3.3 that problem (5.2) has a unique solution.*

We see that all the required conditions of Theorem 4.5 are satisfied. Hence, the proposed problem (5.1) is Ulam-Hyers, generalized Ulam-Hyers stable.

According to Theorem 4.5, for $\varepsilon > 0$, *any solution* $\widetilde{x} \in C([0,1],\mathbb{R})$ *satisfies the inequality*

$$\left| {}^{CK}D_{0^+}^{\frac{1}{2};\frac{1}{2}}\widetilde{x}(t) - \left[\frac{1}{3}e^{\sqrt{t}+1} + \frac{2 + |\widetilde{x}(t)| + \left|D_{0^+}^{\frac{1}{2};\frac{1}{2}}\widetilde{x}(t)\right|}{8e^{2-t}\left(1 + |\widetilde{x}(t)| + \left|D_{0^+}^{\frac{1}{2};\frac{1}{2}}\widetilde{x}(t)\right|\right)} \right] \right| \leq \varepsilon, \quad t \in [0,1],$$

there exists a solution $x \in C([0,1],\mathbb{R})$ *for the problem (5.1) such that*

$$|\widetilde{x}(t) - x(t)| \leq K_f \varepsilon, \quad t \in [0,1],$$

where $K_f = \sqrt{\frac{6}{\pi}} E_{\frac{1}{2}}\left(\frac{\sqrt{2}}{8e-1}\right)$. *Moreover, if we set* $K_f\varepsilon = \Psi(\varepsilon)$, *and* $\Psi(0) = 0$, *then*

$$|\widetilde{x}(t) - x(t)| \leq \Psi(\varepsilon), \quad t \in [0,1].$$

6. Conclusions

In this paper we studied a class of a nonlinear implicit fractional differential equation with the anti-periodic boundary condition involving the Caputo-Katugampola fractional derivative. The existence and uniqueness and Ulam-Hyers stability results are established by applying some fixed point theorems and generalized Gronwall inequality. In future work, it is worth investigating the existence and Ulam-Hyers-Rassias stability of solutions for the proposed problem (1.5)–(1.6) involving generalized fractional derivative with respect to another function.

As a result of our work, We trust the reported results here will have a positive impact on the development of further applications in engineering and applied sciences.

Conflict of interest

The authors declare that they have no competing interests.

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