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Research article

Random attractors of the stochastic extended Brusselator system with a multiplicative noise

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Abstract: In this paper, we are devoted to study asymptotic dynamics of the stochastic extended Brusselator system with a multiplicative noise. The stochastic extended Brusselator system is composed of three pairs of symmetrical coupling components. We firstly study the pullback absorbing property for the stochastic extended Brusselator system with a multiplicative noise. But coupling terms bring great difficulty on this problem, we use the scaling method and estimate groups to overcome this difficulty. Then, we apply the bootstrap pullback estimations to prove the pullback asymptotic compactness for the stochastic extended Brusselator system with a multiplicative noise. Finally, we show the existence of random attractors. In the study of the existence of random attractors for stochastic dynamics, we use the exponential transformation of the Ornstein-Uhlenbeck process to replace the exponential transformation of Brownian motion, which changes the structure of the original Brusselator equations and produces the non-autonomous terms. Based on this, we have to estimate groups to overcome the difficulties of coupling structure and make more complex estimates.

Keywords: random attractors; stochastic extended Brusselator system; multiplicative noise; pullback asymptotic compactness
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1. Introduction

In 1968, Prigogine and Lefever [1] firstly proposed the original Brusselator equations, which is a great significance reaction-diffusion system. It derives from the simulation of autocatalytic chemical or biochemical reactions, the formation of biological and cellular patterns, such as the chlorite-iodide-malonic acid reaction [2]. Up to 1993, Pearson [3] and Lee [4] discovered a large number of self-replicating pattern formation consistent with cubic-autocatalytic reaction-diffusion systems by experimental methods and numerical simulation methods, and showed that the Brusselator equations or Gray-Scott system [5, 6] exhibit abundant spatial patterns.

Due to its wide application in biology and chemistry such as morphogenesis and trimolecular autocatalytic reactions, etc. Many researchers [7–12] have deeply studied the Brusselator equations. Guo and Han [7] studied attractor and proved spatial chaos by using the technique of unstable manifolds for the Brusselator system in R^N. You [8] obtained the existence of a global attractor and proved the Hausdorff dimension and the fractal dimension are finite for the Brusselator equations, and established global attractor of a coupled two-cell Brusselator model in [9]. Bie [10] showed that if nonlinear term f has sublinear growth then no stationary patterns occur, while if f has superlinear growth, stationary patterns may exist for a general two-cell Brusselator model. Recently, Parshed et al. [11] firstly studied global existence of classical solutions, via construction of an appropriate Lyapunov functional for a four compartment Brusselator type system. Then, they proved global existence of weak solutions and obtained the existence of a global attractor.

In this article, we study the existence of a random attractor for the following stochastic extended Brusselator system:

$$\begin{cases} du = [d_1\Delta u + a - (b+k)u + u^2 v + D_1(w-u) + N\varphi]dt + \rho u \circ dW(t), & (1.1)\\ dv = [d_2\Delta v + bu - u^2 v + D_2(z-v)]dt + \rho v \circ dW(t), & (1.2)\\ d\varphi = [d_3\Delta\varphi + ku - (\lambda+N)\varphi + D_3(\psi-\varphi)]dt + \rho\varphi \circ dW(t), & (1.3)\\ dw = [d_1\Delta w + a - (b+k)w + w^2 z + D_1(u-w) + N\psi]dt + \rho w \circ dW(t), & (1.4)\\ dz = [d_2\Delta z + bw - w^2 z + D_2(v-z)]dt + \rho z \circ dW(t), & (1.5)\\ d\psi = [d_3\Delta\psi + kw - (\lambda+N)\psi + D_3(\varphi-\psi)]dt + \rho\psi \circ dW(t), & (1.6) \end{cases}$$

on $\mathbb{R} \times \mathcal{O}$ with the initial conditions

$$u(t_0,x) = u_0(x), \quad v(t_0,x) = v_0(x), \quad \varphi(t_0,x) = \varphi_0(x), \tag{1.7}$$

$$w(t_0,x) = w_0(x), \quad z(t_0,x) = z_0(x), \quad \psi(t_0,x) = \psi_0(x), \quad x \in \mathcal{O}, \tag{1.8}$$

and boundary conditions

$$u(t,x) = v(t,x) = \varphi(t,x) = w(t,x) = z(t,x) = \psi(t,x) = 0, \quad t > t_0, \quad x \in \partial\mathcal{O}, \tag{1.9}$$

where $\mathcal{O}$ is a bounded domain in $\mathbb{R}^n (n \leq 3)$. Let d_1, d_2, d_3, D_1, D_2, D_3, a, b, k, λ and N are nonnegative numbers. Here $W(t)_{t\in\mathbb{R}}$ is a two-sided real-valued Wiener process defined on the probability space, and the symbol $\circ$ represents the Stratonovich's integration.

In our study of the existence of random attractors for stochastic dynamics, we use the exponential transformation of the Ornstein-Uhlenbeck process to replace the exponential transformation of Brownian motion, which changes the structure of the original Brusselator equations and produces the non-autonomous terms, cf. (2.13)–(2.18). Based on this, we have to overcome the difficulties of coupling structure and make more complex estimates.

It's noticing that six coupling components with partial reversibility constitute the extended Brusselator system, which to some extend reflects the relevant network dynamics, see, e.g., [13, 14]. If $\rho = 0$, system (1.1)–(1.6) reduces to the extended Brusselator system without random terms, which has been established the global dissipative dynamics by You and Zhou in [15]. Furthermore, Tu and You [16] proved random attractor of stochastic Brusselator system with a multiplicative noise, this paper has included the results of [16] when $w, z, \varphi, \psi = 0$.

The structure of this article is as follows. In section 2, we will introduce an important theorem about the existence of random attractors and some basic facts. In section 3, we will study pullback absorbing property of the stochastic extended Brusselator equations. In section 4, we apply the uniform Gronwall inequality to prove the pullback asymptotic compactness. Then, we will get the main results of this paper in section 5.

2. Preliminaries

In this section, we recall an essential theorem for the existence of random attractors. Please note that here we will not introduce some basic concepts associated with random attractors and stochastic dynamical systems. The reader can refer to [16–25] for these knowledge.

Let $(X, \|\cdot\|_X)$ indicates a real separable Banach space with Borel σ-algebra $\mathcal{B}(X)$, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Assume that $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system(MDS), and f is a continuous RDS on X over MDS.

Definition 2.1. A random set in X is a set-valued function $B(\omega) : \Omega \to 2^X$ whose graph $\{(\omega, x) : x \in B(\omega)\} \subset \Omega \times X$ is an element of the product σ-algebra $\mathcal{F} \times B(X)$. If one has a random variable $r(\omega) \geq 0$ such that $\|B(\omega)\| := \sup_{x\in B(\omega)}\|x\| \leq r(\omega)$ for every $\omega \in \Omega$, then $B(\omega)$ is a bounded random set in X. If the set $B(\omega)$ is compact (or precompact) in X for all $\omega \in \Omega$, then the random set $B(\omega)$ is called compact (or precompact). The bounded random set B is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ means that for all $\omega \in \Omega$ and every positive constant ν,

$$\lim_{t\to\infty} e^{-\nu t}\|B(\theta_{-t}\omega)\| = 0.$$

Besides, we say that a random variable $R : (\Omega, \mathcal{F}, \mathbb{P}) \to (0, \infty)$ is tempered if for any $\omega \in \Omega$

$$\lim_{t\to\pm\infty} \frac{1}{t} \log R(\theta_{-t}\omega) = 0.$$

A collection $\mathcal{D}$ of random subsets of X is called inclusion-closed, if $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\hat{D}$=$\{\hat{D}(\omega) \subset D(\omega) : \omega \in \Omega\}$ means that $\hat{D} \in \mathcal{D}$. In this way, we say that the collection $\mathcal{D}$ is a universe. Let $\mathcal{D}$ indicates the universe of all the tempered random sets in X. Noticing that $\mathcal{D}$ contains each bounded non-random sets.

Proposition 2.1. (see [17, 18]) Let $\mathcal{D}$ be a collection of random subsets of a Banach space X and f is a continuous RDS on X over a MDS. Assume that there is a closed pullback absorbing set $\{K(\omega)\}_{\omega\in\Omega}$ and f is pullback asymptotically compact associated with $\mathcal{D}$, thus the RDS f has a unique random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ whose basin is $\mathcal{D}$ and

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} f(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}. \tag{2.1}$$

Define the product Hilbert spaces

$$H = [L^2(O)]^6, \quad E = [H^1(O)]^6, \quad \Pi = [H_0^1(O) \cap H^2(O)]^6.$$

We denote the norm and the inner product by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. $\|\cdot\|_{L^p}(p\neq 2)$ represent the norm of $L^p(O)$ or the product space $\mathbb{L}^p(O)=\left[L^p(O)\right]^6$. According to the Poincáre inequality and the homogeneous Dirichlet boundary condition (1.9), one has a nonnegative constant γ such that

$$\|\nabla\zeta\|^2\geq\gamma\|\zeta\|^2,\quad \forall\zeta\in H_0^1(O)\text{ or }E, \tag{2.2}$$

where $\|\nabla\zeta\|$ denotes the equivalent norm of the space E or the space $H_0^1(O)$.

The linear sectorial operator

$$A=\begin{pmatrix} d_1\Delta & 0 & 0 & 0 & 0 & 0\\ 0 & d_2\Delta & 0 & 0 & 0 & 0\\ 0 & 0 & d_3\Delta & 0 & 0 & 0\\ 0 & 0 & 0 & d_1\Delta & 0 & 0\\ 0 & 0 & 0 & 0 & d_2\Delta & 0\\ 0 & 0 & 0 & 0 & 0 & d_3\Delta \end{pmatrix}:\Pi\to H \tag{2.3}$$

denotes the generator of an analytic C_0-semigroup on the Hilbert space H, see, e.g., [26].

Applying the Sobolev embedding theorem, $H_0^1(O)\hookrightarrow L^6(O)$ is a continuous embedding for $n\leq 3$, therefore one has a nonnegative constant δ satisfies the following Sobolev imbedding inequality

$$\|\varphi\|_{L^6(O)}\leq\delta\|\varphi\|_E=\delta\|\nabla\varphi\|,\quad \forall\varphi\in E. \tag{2.4}$$

Let

$$H(u,v,\varphi,w,z,\phi)=\begin{pmatrix} a-(b+k)u+u^2v+D_1(w-u)+N\varphi\\ bu-u^2v+D_2(z-v)\\ ku-(\lambda+N)\varphi+D_3(\psi-\varphi)\\ a-(b+k)w+w^2z+D_1(u-w)+N\psi\\ bw-w^2z+D_2(v-z)\\ kw-(\lambda+N)\psi+D_3(\varphi-\psi) \end{pmatrix}:E\to H \tag{2.5}$$

is a locally Lipschitz continuous mapping on E. Then, the system (1.1)–(1.9) can be expressed as

$$\frac{df}{dt}=Af+H(f)+\rho f\circ\frac{dW(t)}{dt}, \tag{2.6}$$

$$f(t_0,x)=f_0(x)=\left(u_0(x),v_0(x),\varphi_0(x),w_0(x),z_0(x),\phi_0(x)\right). \tag{2.7}$$

Suppose that $W(t)_{t\in\mathbb{R}}$ is a one-dimensional two-sided real-valued Wiener process defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, where

$$\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\},$$

the Borel σ-algebra $\mathcal{F}$ on Ω is produced by the compact-open topology, and $\mathbb{P}$ is consistent with Wiener measure. The time shift defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is given by

$$\theta_t\omega(\cdot)=\omega(t+\cdot)-\omega(t),\qquad t\in\mathbb{R}.$$

Therefore, $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ constitute a metric dynamical system f, see Arnold [19].

In the rest of this section, we consider the following Ornstein-Uhlenbeck process

$$z(\theta_t\omega) = -\int_{-\infty}^{0} e^s(\theta_t\omega)(s)ds = -\int_{-\infty}^{0} e^s\omega(t+s)ds + \omega(t), \tag{2.8}$$

and $z(\theta_t\omega)$ satisfies the following linear stochastic differential equation

$$dz + zdt = dW(t). \tag{2.9}$$

Proposition 2.2. (see [20]) As defined above $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is the metric dynamical system and the Ornstein-Uhlenbeck process $\{z(\theta_t\omega)\}_{t\in\mathbb{R}}$. Then one has a θ_t-invariant set $\tilde{\Omega} \in \Omega$ of full P-measure such that the following statements are satisfied

(i) The Ornstein-Uhlenbeck process $\{z(\theta_t\omega)\}_{t\in\mathbb{R}}$ has the asymptotically sublinear growth property, i.e.,

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0, \tag{2.10}$$

(ii) $z(\theta_t\omega)$ is continuous with respect to t, and

$$\lim_{t\to\pm\infty} \frac{1}{t-t_0}\int_{t_0}^{t} z(\theta_s\omega)ds = 0, \quad \forall t_0 \in \mathbb{R}. \tag{2.11}$$

Noticing that we consider $\omega \in \tilde{\Omega}$ only and will always write Ω for $\tilde{\Omega}$.

In order to study the random dynamics of the stochastic extended Brusselator system, we usually transform stochastic Eqs. (1.1)–(1.6) to a system of pathwise PDEs.

Let

$$\begin{aligned} &U = e^{-\rho z(\theta_t\omega)}u, \quad V = e^{-\rho z(\theta_t\omega)}v, \quad \Phi = e^{-\rho z(\theta_t\omega)}\varphi, \\ &W = e^{-\rho z(\theta_t\omega)}w, \quad Z = e^{-\rho z(\theta_t\omega)}z, \quad \Psi = e^{-\rho z(\theta_t\omega)}\psi, \end{aligned} \tag{2.12}$$

here $z(\theta_t\omega)$ is the Ornstein-Uhlenbeck process as above in (2.8). Therefore, we have

$$\begin{aligned} &dU = -\rho e^{-\rho z(\theta_t\omega)}u \circ dz + e^{-\rho z(\theta_t\omega)}du, \quad dV = -\rho e^{-\rho z(\theta_t\omega)}v \circ dz + e^{-\rho z(\theta_t\omega)}dv, \\ &d\Phi = -\rho e^{-\rho z(\theta_t\omega)}\varphi \circ dz + e^{-\rho z(\theta_t\omega)}d\varphi, \quad dW = -\rho e^{-\rho z(\theta_t\omega)}w \circ dz + e^{-\rho z(\theta_t\omega)}dw, \\ &dZ = -\rho e^{-\rho z(\theta_t\omega)}z \circ dz + e^{-\rho z(\theta_t\omega)}dz, \quad d\Psi = -\rho e^{-\rho z(\theta_t\omega)}\psi \circ dz + e^{-\rho z(\theta_t\omega)}d\psi. \end{aligned}$$

By using (2.9), we convert Eqs. (1.1)–(1.6) to a random PDE system

$$\begin{cases} \dfrac{dU}{dt} = d_1\Delta U + ae^{-\rho z(\theta_t\omega)} - (b+k)U + e^{2\rho z(\theta_t\omega)}U^2V + D_1(W-U) + N\Phi + \rho z(\theta_t\omega)U, & (2.13) \\ \dfrac{dV}{dt} = d_2\Delta V + bU - e^{2\rho z(\theta_t\omega)}U^2V + D_2(Z-V) + \rho z(\theta_t\omega)V, & (2.14) \\ \dfrac{d\Phi}{dt} = d_3\Delta\Phi + kU - (\lambda+N)\Phi + D_3(\Psi-\Phi) + \rho z(\theta_t\omega)\Phi, & (2.15) \\ \dfrac{dW}{dt} = d_1\Delta W + ae^{-\rho z(\theta_t\omega)} - (b+k)W + e^{2\rho z(\theta_t\omega)}W^2Z + D_1(U-W) + N\Psi + \rho z(\theta_t\omega)W, & (2.16) \\ \dfrac{dZ}{dt} = d_2\Delta Z + bW - e^{2\rho z(\theta_t\omega)}W^2Z + D_2(V-Z) + \rho z(\theta_t\omega)Z, & (2.17) \\ \dfrac{d\Psi}{dt} = d_3\Delta\Psi + kW - (\lambda+N)\Psi + D_3(\Phi-\Psi) + \rho z(\theta_t\omega)\Psi, & (2.18) \end{cases}$$

for all $\omega \in \Omega$, $x \in \mathcal{O}$ and $t > t_0$, with boundary conditions

$$U(t,\omega,x) = V(t,\omega,x) = \Phi(t,\omega,x) = 0, \quad t > t_0 \in \mathbb{R},$$
$$W(t,\omega,x) = Z(t,\omega,x) = \Psi(t,\omega,x) = 0, \quad x \in \partial\mathcal{O},\ \omega \in \Omega,$$

and the initial conditions

$$\begin{aligned} &U(t_0,\omega,x) = U_0(\omega,x) = e^{-\rho z(\theta_{t_0}\omega)} u_0(x), \quad V(t_0,\omega,x) = V_0(\omega,x) = e^{-\rho z(\theta_{t_0}\omega)} v_0(x),\\ &\Phi(t_0,\omega,x) = \Phi_0(\omega,x) = e^{-\rho z(\theta_{t_0}\omega)} \varphi_0(x), \quad W(t_0,\omega,x) = W_0(\omega,x) = e^{-\rho z(\theta_{t_0}\omega)} w_0(x),\\ &Z(t_0,\omega,x) = Z_0(\omega,x) = e^{-\rho z(\theta_{t_0}\omega)} z_0(x), \quad \Psi(t_0,\omega,x) = \Psi_0(\omega,x) = e^{-\rho z(\theta_{t_0}\omega)} \psi_0(x). \end{aligned}$$

We can express the Eqs. (2.13)–(2.18) as

$$\frac{dg}{dt} = Ag + F(g,\theta_t\omega), \quad \forall \omega \in \Omega, \tag{2.19}$$
$$g(t_0,\omega,x) = g_0 = (U_0(\omega,x), V_0(\omega,x), \Phi_0(\omega,x), W_0(\omega,x), Z_0(\omega,x), \Psi_0(\omega,x))^T, \tag{2.20}$$

where

$$F(g,\theta_t\omega) = \begin{pmatrix} ae^{-\rho z(\theta_t\omega)} - (b+k)U + e^{2\rho z(\theta_t\omega)} U^2 V + D_1(W-U) + N\Phi + \rho z(\theta_t\omega)U \\ bU - e^{2\rho z(\theta_t\omega)} U^2 V + D_2(Z-V) + \rho z(\theta_t\omega)V \\ kU - (\lambda+N)\Phi + D_3(\Psi-\Phi) + \rho z(\theta_t\omega)\Phi \\ ae^{-\rho z(\theta_t\omega)} - (b+k)W + e^{2\rho z(\theta_t\omega)} W^2 Z + D_1(U-W) + N\Psi + \rho z(\theta_t\omega)W \\ bW - e^{2\rho z(\theta_t\omega)} W^2 Z + D_2(V-Z) + \rho z(\theta_t\omega)Z \\ kW - (\lambda+N)\Psi + D_3(\Phi-\Psi) + \rho z(\theta_t\omega)\Psi \end{pmatrix}.$$

Similar to deterministic system, we prove the local existence and uniqueness of the weak solution $g(t,\omega;t_0,g_0),\ t \in [t_0, T(\omega,g_0)]$ for some $T(\omega,g_0) > t_0$ by the Galerkin approximations and compactness argument [27]. According to the parabolic regularity theory in [26], each weak solution will become a strong solution for $t > t_0$ in the existence interval. Integrating with Lemma 2.2 in [15], the weak solution $g(t,\omega;t_0,g_0)$ of the random extended Brusselator evolutionary system (2.19)–(2.20) on the maximal existence time interval, which satisfies

$$g(t,\omega;t_0,g_0) \in C([t_0,T_{\max});H) \cap C^1((t_0,T_{\max});H) \cap L^2((t_0,T_{\max});E).$$

Therefore, we need to study that the global existence and uniqueness of the weak solutions for the extended Brusselator random dynamic system (2.19)–(2.20) in the next section.

Then, we find that the system (2.6)–(2.7) generate a continuous RDS $f : \mathbb{R}^+ \times \Omega \times H \to H$ over MDS, which satisfies

$$f(t-\tau,\theta_\tau\omega,f_0) = S(t,\tau,\omega)f_0 = e^{\rho z(\theta_t\omega)} g(t,\omega;\tau,g_0), \quad \forall t \geq \tau,\ \omega \in \Omega. \tag{2.21}$$

Owing to (2.21), the following pullback relation is established

$$f(t,\theta_{-t}\omega,f_0) = e^{\rho z(\omega)} g(0,\omega;-t,e^{\rho z(\theta_{-t}\omega)} f_0), \quad \text{for } t \geq 0, \tag{2.22}$$

which be called the pullback quasi-trajectory from g_0. We will study the pullback asymptotic behavior by establishing the pullback quasi-trajectory.

3. Pullback absorbing property

In this section, we firstly prove the existence and uniqueness of the weak solution by applying the scaling method and estimate groups. Then, we obtain the pullback absorbing property and some necessary estimates. For convenience, the $U(t,\omega;t_0,U_0)$, $V(t,\omega;t_0,V_0)$ and $\Phi(t,\omega;t_0,\Phi_0)$ et al are shorthand for $U(t,\omega)$, $V(t,\omega)$ and $\Phi(t,\omega)$ or U, V and Φ.

Lemma 3.1. Let $R(\omega) > 0$ be a given tempered random variable and for every initial value $f_0 = (u_0, v_0, \varphi_0, w_0, z_0, \psi_0) \in H$, where $\|f_0\| \le R(\omega)$, then one has a time variable $T(R,\omega) \le -1$ such that for all initial time $t_0 \le T(R,\omega)$, the weak solution $g(t,\omega) = (U(t,\omega), V(t,\omega), \Phi(t,\omega), W(t,\omega), Z(t,\omega), \Psi(t,\omega))$ of the random extended Brusselator equations (2.13)–(2.18) exists on $[t_0, 0]$.

Furthermore, suppose $t_0 \le \min\{T(R,\omega), -4\}$, then one has a random variable $M(t,\omega)$ for terminal time $t \in [-4, 0]$ such that the weak solution g satisfies the following inequality

$$\|g(t,\omega;t_0,e^{-\rho z(\theta_{t_0}\omega)}f_0)\|^2 \le M(t,\omega), \quad \forall t \ge t_0,\ \omega \in \Omega. \tag{3.1}$$

Proof. We take the scalar products (2.14) with $V(t)$ and (2.17) with $Z(t)$, and then add them up, which follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|V\|^2 + \|Z\|^2) + d_2(\|\nabla V\|^2 + \|\nabla Z\|^2)\\
=& - e^{2\rho z(\theta_t\omega)} \int_O \Big[(UV - \frac{1}{2}be^{-2\rho z(\theta_t\omega)})^2 + (WZ - \frac{1}{2}be^{-2\rho z(\theta_t\omega)})^2\Big]dx\\
&- \int_O D_2(V-Z)^2 dx + \frac{1}{2}b^2|O|e^{-2\rho z(\theta_t\omega)} + \rho z(\theta_t\omega)(\|V\|^2 + \|Z\|^2)\\
\le& \frac{1}{2}b^2|O|e^{-2\rho z(\theta_t\omega)} + \rho z(\theta_t\omega)(\|V\|^2 + \|Z\|^2).
\end{aligned} \tag{3.2}$$

Applying Poincáre inequality, we get

$$\begin{aligned}
&\frac{d}{dt}(\|V\|^2 + \|Z\|^2) + 2\gamma d_2(\|V\|^2 + \|Z\|^2) \le \frac{d}{dt}(\|V\|^2 + \|Z\|^2) + 2d_2(\|\nabla V\|^2 + \|\nabla Z\|^2)\\
\le& b^2|O|e^{-2\rho z(\theta_t\omega)} + 2\rho z(\theta_t\omega)(\|V\|^2 + \|Z\|^2),
\end{aligned} \tag{3.3}$$

multiplying both sides of (3.3) by $e^{\int_{t_0}^t (2\rho z(\theta_s\omega) - 2\gamma d_2)ds}$ and integrating over the time interval $[t_0, t]$, where $t_0 < -4 < t < 0$, which yields

$$\begin{aligned}
&\|V(t,\omega;t_0,g_0)\|^2 + \|Z(t,\omega;t_0,g_0)\|^2\\
\le& (\|V_0\|^2 + \|Z_0\|^2)e^{\int_{t_0}^t 2\rho z(\theta_s\omega)ds - 2\gamma d_2(t-t_0)} + b^2|O| \int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega) - 2\gamma d_2)ds - 2\rho z(\theta_\tau\omega)} d\tau.
\end{aligned} \tag{3.4}$$

Then, we take the pullback estimate by asymptotic decay of the Ornstein-Uhlenbeck process, which get rid of the dependence on the initial value and time. This play an important role in this paper.

According to (2.10) and (2.11), for any random variable $R(\omega) > 0$, one has a time variable $T_1(R,\omega) < -4$ such that for each $t_0 \le T_1(R,\omega)$ and $t \in [-4, 0]$, which follows that

$$\left\{\begin{aligned}
&\frac{1}{t-t_0}\int_{t_0}^t 6\rho z(\theta_s\omega)ds - \gamma d' \le -\frac{1}{2}\gamma d', && (3.5)\\
&e^{-\frac{1}{2}\gamma d'(t-t_0)} e^{-\rho z(\theta_{t_0}\omega)} R^2(\omega) \le 1, && (3.6)
\end{aligned}\right.$$

where $d' = \min\{d_1, d_2, d_3\}$. Applying (3.5) and (3.6), one has $T_2(R,\omega) < -4$ such that for all $\tau \le T_2(R,\omega)$, we infer that

$$e^{\int_\tau^t (2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)} = e^{(t-\tau)\left(\frac{\int_\tau^t 2\rho z(\theta_s\omega)ds}{t-\tau}-2\gamma d_2-\frac{2\rho z(\theta_\tau\omega)}{t-\tau}\right)} \le e^{-\frac{1}{2}\gamma d'(t-\tau)}, \tag{3.7}$$

and then

$$\int_{-\infty}^{\tau} e^{-\frac{1}{2}\gamma d'(t-\sigma)}d\sigma \le \int_{-\infty}^{T_2} e^{-\frac{1}{2}\gamma d'(t-\sigma)}d\sigma = \frac{2}{\gamma d'}e^{\frac{1}{2}\gamma d'(T_2-t)}. \tag{3.8}$$

Therefore, we obtain that

$$\int_{-\infty}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)}d\tau \tag{3.9}$$

is convergent. In this way, we get

$$\begin{aligned}
&\|V(t,\omega;t_0,g_0)\|^2 + \|Z(t,\omega;t_0,g_0)\|^2\\
\le&(\|V_0\|^2 + \|Z_0\|^2)e^{\int_{t_0}^t 2\rho z(\theta_s\omega)ds-2\gamma d_2(t-t_0)} + b^2|O|\int_{t_0}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)}d\tau\\
\le&1 + b^2|O|\int_{-\infty}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)}d\tau.
\end{aligned} \tag{3.10}$$

Let $y(t,x,\omega) = U(t,x,\omega) + V(t,x,\omega) + W(t,x,\omega) + Z(t,x,\omega)$ and $\xi(t,x,\omega) = \Phi(t,x,\omega) + \Psi(t,x,\omega)$, then by applying (2.13)–(2.18), we deduce that

$$\frac{dy}{dt} = d_1\Delta y - ky + [(d_2-d_1)\Delta(V+Z) + k(V+Z) + 2ae^{-\rho z(\theta_t\omega)}] + N\xi + \rho z(\theta_t\omega)y, \tag{3.11}$$

$$\frac{d\xi}{dt} = d_3\Delta\xi + ky - k(V+Z) - (\lambda+N)\xi + \rho z(\theta_t\omega)\xi. \tag{3.12}$$

Rescaling $\xi = \mu\eta$ with $\mu = k/N$, we have

$$\frac{dy}{dt} = d_1\Delta y - ky + [(d_2-d_1)\Delta(V+Z) + k(V+Z) + 2ae^{-\rho z(\theta_t\omega)}] + k\eta + \rho z(\theta_t\omega)y, \tag{3.13}$$

$$\mu\frac{d\eta}{dt} = \mu d_3\Delta\eta + ky - k(V+Z) - (\mu\lambda+k)\eta + \mu\rho z(\theta_t\omega)\eta. \tag{3.14}$$

Then, we take the inner product (3.13) with $y(t)$ and apply Poincáre inequality, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y\|^2 + d_1\|\nabla y\|^2\\
\le&\int_O [(d_2-d_1)\Delta(V+Z) + k(V+Z) + 2ae^{-\rho z(\theta_t\omega)}]ydx - k\|y\|^2 + k\|\eta\|\|y\| + \rho z(\theta_t\omega)\|y\|^2\\
\le&|d_1-d_2|\|\nabla(V+Z)\|\|\nabla y\| + k\|V+Z\|\|y\| + 2ae^{-\rho z(\theta_t\omega)}|O|^{\frac{1}{2}}\|y\| - k\|y\|^2 + k\|\eta\|\|y\| + \rho z(\theta_t\omega)\|y\|^2\\
\le&\frac{d_1}{4}\|\nabla y\|^2 + \frac{|d_1-d_2|^2}{d_1}\|\nabla(V+Z)\|^2 + \frac{2k^2}{d_1\gamma}\|V+Z\|^2 + \frac{d_1\gamma}{4}\|y\|^2 + \frac{8}{d_1\gamma}a^2|O|e^{-2\rho z(\theta_t\omega)}\\
&- k\|y\|^2 + k\|\eta\|\|y\| + \rho z(\theta_t\omega)\|y\|^2
\end{aligned}$$

$$\le\frac{d_1}{2}\|\nabla y\|^2+\frac{|d_1-d_2|^2}{d_1}\|\nabla(V+Z)\|^2+\frac{2k^2}{d_1\gamma}\|V+Z\|^2+\frac{8}{d_1\gamma}a^2|O|e^{-2\rho z(\theta_t\omega)}$$
$$-k\|y\|^2+k\|\eta\|\|y\|+\rho z(\theta_t\omega)\|y\|^2, \tag{3.15}$$

so, we obtain that

$$\begin{aligned}&\frac{d}{dt}\|y\|^2+d_1\|\nabla y\|^2\\ \le&\frac{2|d_1-d_2|^2}{d_1}\|\nabla(V+Z)\|^2+\frac{4k^2}{d_1\gamma}\|V+Z\|^2+\frac{16}{d_1\gamma}a^2|O|e^{-2\rho z(\theta_t\omega)}\\ &-2k\|y\|^2+2k\|\eta\|\|y\|+2\rho z(\theta_t\omega)\|y\|^2.\end{aligned} \tag{3.16}$$

Taking the inner product (3.14) with $\eta(t)$, we have

$$\begin{aligned}&\frac{1}{2}\mu\frac{d}{dt}\|\eta\|^2+\mu d_3\|\nabla\eta\|^2\\ \le&k\|y\|\|\eta\|+k\|V+Z\|\|\eta\|-(\mu\lambda+k)\|\eta\|^2+\mu\rho z(\theta_t\omega)\|\eta\|^2\\ \le&k\|y\|\|\eta\|+\frac{\mu\gamma d_3}{2}\|\eta\|^2+\frac{k^2}{2\mu\gamma d_3}\|V+Z\|^2-(\mu\lambda+k)\|\eta\|^2+\mu\rho z(\theta_t\omega)\|\eta\|^2,\end{aligned} \tag{3.17}$$

we deduce that

$$\begin{aligned}&\mu\frac{d}{dt}\|\eta\|^2+\mu d_3\|\nabla\eta\|^2\\ \le&2k\|y\|\|\eta\|+\frac{k^2}{\mu\gamma d_3}\|V+Z\|^2-2(\mu\lambda+k)\|\eta\|^2+2\mu\rho z(\theta_t\omega)\|\eta\|^2.\end{aligned} \tag{3.18}$$

Adding (3.16) and (3.18) up, and noticing that

$$-2k\|y\|^2+4k\|\eta\|\|y\|-2(\mu\lambda+k)\|\eta\|^2\le 0,$$

so that

$$\begin{aligned}&\frac{d}{dt}(\|y\|^2+\|\mu^{-\frac{1}{2}}\xi\|^2)+\min\{d_1,d_3\}(\|\nabla y\|^2+\|\mu^{-\frac{1}{2}}\nabla\xi\|^2)\\ \le&\frac{2|d_1-d_2|^2}{d_1}\|\nabla(V+Z)\|^2+(\frac{4k^2}{d_1\gamma}+\frac{k^2}{\mu\gamma d_3})\|V+Z\|^2+2\rho z(\theta_t\omega)(\|y\|^2+\|\mu^{-\frac{1}{2}}\xi\|^2)\\ &+\frac{16}{d_1\gamma}a^2|O|e^{-2\rho z(\theta_t\omega)}.\end{aligned} \tag{3.19}$$

Let $d=\min\{d_1,d_3\}$, $\alpha=\frac{\max\{1,\mu^{-1}\}}{\min\{1,\mu^{-1}\}}$. Then, multiplying both sides of (3.19) by $e^{\int_{t_0}^t(2\rho z(\theta_s\omega)-\gamma d)ds}$ and integrating over $[t_0,t]$, where $t_0<-4<t<0$. Therefore, one has a time variable $T_3(R,\omega)<-4$ such that for each $t_0\le T_3(R,\omega)$ and $t\in[-4,0]$, we obtain that

$$\begin{aligned}&\|y(t,\omega;t_0,g_0)\|^2+\|\xi(t,\omega;t_0,g_0)\|^2\\ \le&\alpha(\|y_0\|^2+\|\xi_0\|^2)e^{\int_{t_0}^t 2\rho z(\theta_s\omega)ds-\gamma d(t-t_0)}+\alpha\int_{t_0}^t e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}\Big[\frac{2|d_1-d_2|^2}{d_1}\|\nabla V(\tau)+\nabla Z(\tau)\|^2\Big]d\tau\end{aligned}$$

$$+\alpha\int_{t_0}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}\Big[(\frac{4k^2}{d_1\gamma}+\frac{k^2}{\mu\gamma d_3})\|V(\tau)+Z(\tau)\|^2\Big]d\tau$$
$$+\alpha\int_{t_0}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}\Big(\frac{16a^2}{d_1\gamma}|\mathcal{O}|e^{-2\rho z(\theta_\tau\omega)}\Big)d\tau. \tag{3.20}$$

By applying Poinc$\acute{a}$re inequality and above estimates (3.10), it follows that

$$\begin{aligned}&\|y(t,\omega;t_0,g_0)\|^2+\|\xi(t,\omega;t_0,g_0)\|^2\\ \le&\alpha+\alpha\int_{t_0}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}\Big[\beta(\|\nabla V(\tau)\|^2+\|\nabla Z(\tau)\|^2)+\frac{16a^2}{d_1\gamma}|\mathcal{O}|e^{-2\rho z(\theta_\tau\omega)}\Big]d\tau,\end{aligned} \tag{3.21}$$

where $\beta=\frac{4|d_1-d_2|^2}{d_1}+\frac{8k^2}{d_1\gamma^2}+\frac{2k^2}{\mu d_3\gamma^2}$.

Then, we multiply both sides of (3.3) by $e^{\int_{t_0}^t(2\rho z(\theta_s\omega)-\gamma d)ds}$ and integrate over $[t_0,t]$, where $t_0<-4<t<0$, one has a time variable $T_4(R,\omega)<-4$ such that

$$\begin{aligned}&2d_2\int_{t_0}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}(\|\nabla V(\tau,\omega;t_0,g_0)\|^2+\|\nabla Z(\tau,\omega;t_0,g_0)\|^2)d\tau\\ \le&e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}(\|V(t_0)\|^2+\|Z(t_0)\|^2)+\int_{t_0}^{t}(\|V(\tau)\|^2+\|Z(\tau)\|^2)(-2\rho z(\theta_\tau\omega)+\gamma d)e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}d\tau\\ &+\int_{t_0}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}\Big[2\rho z(\theta_\tau\omega)(\|V(\tau)\|^2+\|Z(\tau)\|^2)+b^2|\mathcal{O}|e^{-2\rho z(\theta_\tau\omega)}\Big]d\tau\\ \le&1+b^2|\mathcal{O}|\int_{t_0}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}e^{-2\rho z(\theta_\tau\omega)}d\tau+\gamma d\int_{t_0}^{t}(\|V(\tau)\|^2+\|Z(\tau)\|^2)e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}d\tau.\end{aligned} \tag{3.22}$$

Now, we deal with the last integral term in (3.22). Multiplying both sides of (3.4) by $e^{\int_{t_0}^t(2\rho z(\theta_s\omega)-\gamma d)ds}$, and integrating over $[t_0,t]$, where $t_0<-4<t<0$. Then, one has a time variable $T_5(R,\omega)<T_1(R,\omega)$ such that for every $t_0\le T_5(\omega)$, it follows that

$$\begin{aligned}&\int_{t_0}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}(\|V(\tau,\omega;t_0,g_0)\|^2+\|Z(\tau,\omega;t_0,g_0)\|^2)d\tau\\ \le&\int_{t_0}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}\Big[(\|V_0\|^2+\|Z_0\|^2)e^{\int_{t_0}^\tau 2\rho z(\theta_s\omega)ds-2\gamma d_2(\tau-t_0)}\Big]d\tau\\ &+b^2|\mathcal{O}|\int_{t_0}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}\int_{t_0}^{\tau}e^{\int_\xi^\tau(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\xi d\tau\\ \le&\int_{t_0}^{t}e^{\int_{t_0}^t(2\rho z(\theta_s\omega)-\gamma d')ds}(\|V_0\|^2+\|Z_0\|^2)d\tau+b^2|\mathcal{O}|\int_{t_0}^{t}\int_{\xi}^{t}e^{\int_\tau^t(2\rho z(\theta_s\omega)-\gamma d)ds}e^{\int_\xi^\tau(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\tau d\xi\\ \le&(t-t_0)(\|V_0\|^2+\|Z_0\|^2)e^{\int_{t_0}^t(2\rho z(\theta_s\omega)-\gamma d')ds}+b^2|\mathcal{O}|\int_{t_0}^{t}\int_{\xi}^{t}e^{\int_\xi^t 2\rho z(\theta_s\omega)ds-\int_\xi^t\gamma d'ds-2\rho z(\theta_\xi\omega)}d\tau d\xi\\ =&(t-t_0)(\|V_0\|^2+\|Z_0\|^2)e^{\int_{t_0}^t(2\rho z(\theta_s\omega)-\gamma d')ds}+b^2|\mathcal{O}|\int_{t_0}^{t}(t-\xi)e^{\int_\xi^t 2\rho z(\theta_s\omega)ds-\int_\xi^t\gamma d'ds-2\rho z(\theta_\xi\omega)}d\xi\\ \le&1+b^2|\mathcal{O}|\int_{-\infty}^{t}(t-\xi)e^{\int_\xi^t 2\rho z(\theta_s\omega)ds-\int_\xi^t\gamma d'ds-2\rho z(\theta_\xi\omega)}d\xi,\end{aligned} \tag{3.23}$$

where $d' = \min\{d_1, d_2, d_3\}$. Substituting (3.23) into (3.22), we deduce

$$
\begin{aligned}
C_1(t,\omega) =& \int_{t_0}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega)-\gamma d)ds}(\|\nabla V(\tau,\omega;t_0,g_0)\|^2 + \|\nabla Z(\tau,\omega;t_0,g_0)\|^2)d\tau \\
=& \frac{1}{2d_2} + \frac{b^2|O|}{2d_2}\int_{t_0}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega)-\gamma d)ds} e^{-2\rho z(\theta_\tau\omega)} d\tau \\
&+ \frac{\gamma d}{2d_2}\Big[1 + b^2|O|\int_{-\infty}^{t}(t-\xi)e^{\int_\xi^t 2\rho z(\theta_s\omega)ds - \int_\xi^t \gamma d' ds - 2\rho z(\theta_\xi\omega)}d\xi\Big]
\end{aligned} \tag{3.24}
$$

is tempered by (2.10) and (2.11). Then, substituting (3.24) into (3.21), we obtain that for $t_0 \le \min\{T_3(R,\omega), T_4(R,\omega), T_5(R,\omega)\}$

$$
\begin{aligned}
&\|y(t,\omega;t_0,g_0)\|^2 + \|\xi(t,\omega;t_0,g_0)\|^2 \\
\le& \alpha + \alpha\beta C_1(t,\omega) + \frac{16a^2|O|\alpha}{d_1\gamma}\int_{-\infty}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega)-\gamma d)ds - 2\rho z(\theta_\tau\omega)}d\tau \\
=& C_2(t,\omega).
\end{aligned} \tag{3.25}
$$

Owing to (3.4) and (3.25), we get

$$
\begin{aligned}
&\|U(t,\omega;t_0,g_0) + W(t,\omega;t_0,g_0)\|^2 + \|\Phi(t,\omega;t_0,g_0) + \Psi(t,\omega;t_0,g_0)\|^2 \\
=& \|y(t,\omega;t_0,g_0) - (V(t,\omega;t_0,g_0) + Z(t,\omega;t_0,g_0))\|^2 + \|\Phi(t,\omega;t_0,g_0) + \Psi(t,\omega;t_0,g_0)\|^2 \\
\le& 2\|y(t,\omega;t_0,g_0)\|^2 + 4(\|V(t,\omega;t_0,g_0)\|^2 + \|Z(t,\omega;t_0,g_0)\|^2) + \|\xi(t,\omega;t_0,g_0)\|^2 \\
\le& 2C_2(t,\omega) + 4\Big(1 + b^2|O|\int_{t_0}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega)-2\gamma d_2)ds - 2\rho z(\theta_\tau\omega)}d\tau\Big).
\end{aligned} \tag{3.26}
$$

Next, we deal with the other components. Let $p(t,x,\omega) = U(t,x,\omega) + V(t,x,\omega) - W(t,x,\omega) - Z(t,x,\omega)$ and $q(t,x,\omega) = \Phi(t,x,\omega) - \Psi(t,x,\omega)$, we have

$$
\frac{dp}{dt} = d_1\Delta p - (k+2D_1)p + (d_2-d_1)\Delta(V-Z) + [k + 2(D_1-D_2)](V-Z) + Nq + \rho z(\theta_t\omega)p, \tag{3.27}
$$

$$
\frac{dq}{dt} = d_3\Delta q + kp - k(V-Z) - (\lambda+N)q - 2D_3 q + \rho z(\theta_t\omega)q. \tag{3.28}
$$

Rescaling $q = \mu\varpi$ with $\mu = k/N$, we have

$$
\frac{dp}{dt} = d_1\Delta p - (k+2D_1)p + (d_2-d_1)\Delta(V-Z) + [k + 2(D_1-D_2)](V-Z) + k\varpi + \rho z(\theta_t\omega)p, \tag{3.29}
$$

$$
\mu\frac{d\varpi}{dt} = \mu d_3\Delta\varpi + kp - k(V-Z) - (\mu\lambda + k)\varpi - 2\mu D_3\varpi + \mu\rho z(\theta_t\omega)\varpi. \tag{3.30}
$$

Taking the inner product (3.29) with $p(t)$, then using Hölder inequality and Young inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|p\|^2 + d_1\|\nabla p\|^2 + (k+2D_1)\|p\|^2 \\
\le& |d_1 - d_2|\|\nabla(V-Z)\|\|\nabla p\| + [k + 2(D_1-D_2)]\|V-Z\|\|p\| + k\|\varpi\|\|p\| + \rho z(\theta_t\omega)\|p\|^2
\end{aligned}
$$

$$\leq \frac{d_1}{2}\|\nabla p\|^2 + \frac{|d_1 - d_2|^2}{2d_1}\|\nabla(V - Z)\|^2 + \frac{|k + 2(D_1 - D_2)|^2}{8D_1}\|V - Z\|^2$$
$$+ 2D_1\|p\|^2 + k\|\varpi\|\|p\| + \rho z(\theta_t\omega)\|p\|^2, \tag{3.31}$$

thus, we get

$$\frac{d}{dt}\|p\|^2 + d_1\|\nabla p\|^2$$
$$\leq \frac{|d_1 - d_2|^2}{d_1}\|\nabla(V - Z)\|^2 + \frac{|k + 2(D_1 - D_2)|^2}{4D_1}\|V - Z\|^2 - 2k\|p\|^2 + 2k\|\varpi\|\|p\| + 2\rho z(\theta_t\omega)\|p\|^2. \tag{3.32}$$

Next, taking the inner products of (3.30) with $\varpi(t)$, we have

$$\frac{1}{2}\mu\frac{d}{dt}\|\varpi\|^2 + \mu d_3\|\nabla\varpi\|^2$$
$$\leq k\|p\|\|\varpi\| + k\|V - Z\|\|\varpi\| - (\mu\lambda + k + 2\mu D_3)\|\varpi\|^2 + \mu\rho z(\theta_t\omega)\|\varpi\|^2$$
$$\leq 2\mu D_3\|\varpi\|^2 + \frac{k^2}{8\mu D_3}\|V - Z\|^2 + k\|p\|\|\varpi\| - (\mu\lambda + k + 2\mu D_3)\|\varpi\|^2 + \mu\rho z(\theta_t\omega)\|\varpi\|^2$$
$$\leq \frac{k^2}{4\mu D_3}(\|V\|^2 + \|Z\|^2) + k\|p\|\|\varpi\| - (\mu\lambda + k)\|\varpi\|^2 + \mu\rho z(\theta_t\omega)\|\varpi\|^2, \tag{3.33}$$

thus, we obtain

$$\mu\frac{d}{dt}\|\varpi\|^2 + \mu d_3\|\nabla\varpi\|^2$$
$$\leq \frac{k^2}{2\mu D_3}(\|V\|^2 + \|Z\|^2) + 2k\|p\|\|\varpi\| - 2(\mu\lambda + k)\|\varpi\|^2 + 2\mu\rho z(\theta_t\omega)\|\varpi\|^2. \tag{3.34}$$

Adding (3.32) and (3.34) up, and noticing that

$$-2k\|p\|^2 + 4k\|\varpi\|\|p\| - 2(\mu\lambda + k)\|\varpi\|^2 \leq 0. \tag{3.35}$$

Therefore, applying the Poincáre inequality, we have

$$\frac{d}{dt}(\|p\|^2 + \|\mu^{-\frac{1}{2}}q\|^2) + \min\{d_1, d_3\}(\|\nabla p\|^2 + \|\mu^{-\frac{1}{2}}\nabla q\|^2)$$
$$\leq \frac{2|d_1 - d_2|^2}{d_1}(\|\nabla V\|^2 + \|\nabla Z\|^2) + \Big(\frac{|k + 2(D_1 - D_2)|^2}{2D_1} + \frac{k^2}{2\mu D_3}\Big)(\|V\|^2 + \|Z\|^2)$$
$$+ 2\rho z(\theta_t\omega)(\|p\|^2 + \|\mu^{-\frac{1}{2}}q\|^2)$$
$$\leq \kappa(\|\nabla V\|^2 + \|\nabla Z\|^2) + 2\rho z(\theta_t\omega)(\|p\|^2 + \|\mu^{-\frac{1}{2}}q\|^2), \tag{3.36}$$

where $\kappa = \frac{2|d_1-d_2|^2}{d_1} + \frac{|k+2(D_1-D_2)|^2}{2\gamma D_1} + \frac{k^2}{2\gamma\mu D_3}$.

Multiplying both sides of (3.36) by $e^{\int_{t_0}^{t}(2\rho z(\theta_s\omega)-\gamma d)ds}$ and integrating over $[t_0, t]$, where $t_0 < -4 < t < 0$. Therefore, one has a time variable $T_6(R, \omega) < -4$ such that for each $t_0 \leq T_6(R, \omega)$ and $t \in [-4, 0]$, by using (3.24), we obtain that

$$\|p(t, \omega; t_0, g_0)\|^2 + \|q(t, \omega; t_0, g_0)\|^2$$

$$\leq \alpha(\|p_0\|^2 + \|q_0\|^2)e^{\int_{t_0}^{t}(2\rho z(\theta_s\omega)-\gamma d)ds} + \alpha\kappa\int_{t_0}^{t} e^{\int_{\tau}^{t}(2\rho z(\theta_s\omega)-\gamma d)ds}(\|\nabla V\|^2 + \|\nabla Z\|^2)d\tau$$
$$= C_3(t,\omega), \tag{3.37}$$

where $C_3(t,\omega) = \alpha + \alpha\kappa C_1(t,\omega)$. From (3.4), (3.24) and (3.37), we infer that

$$\begin{aligned}
&\|U(t,\omega;t_0,g_0) - W(t,\omega;t_0,g_0)\|^2 + \|\Phi(t,\omega;t_0,g_0) - \Psi(t,\omega;t_0,g_0)\|^2\\
=&\|p(t,\omega) - (V(t,\omega) - Z(t,\omega))\|^2 + \|\Phi(t,\omega) - \Psi(t,\omega)\|^2\\
\leq&2\|p(t,\omega)\|^2 + 4(\|V(t,\omega)\|^2 + \|Z(t,\omega)\|^2) + \|q(t,\omega)\|^2\\
\leq&2C_3(t,\omega) + 4\Big(1 + b^2|O|\int_{t_0}^{t} e^{\int_{\tau}^{t}(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)}d\tau\Big).
\end{aligned} \tag{3.38}$$

By using (3.10), (3.26) and (3.38), we have

$$U(t,\omega) = \frac{1}{2}\Big[\big(U(t,\omega) + W(t,\omega)\big) + \big(U(t,\omega) - W(t,\omega)\big)\Big], \tag{3.39}$$
$$W(t,\omega) = \frac{1}{2}\Big[\big(U(t,\omega) + W(t,\omega)\big) - \big(U(t,\omega) - W(t,\omega)\big)\Big], \tag{3.40}$$
$$\Phi(t,\omega) = \frac{1}{2}\Big[\big(\Phi(t,\omega) + \Psi(t,\omega)\big) + \big(\Phi(t,\omega) - \Psi(t,\omega)\big)\Big], \tag{3.41}$$
$$\Psi(t,\omega) = \frac{1}{2}\Big[\big(\Phi(t,\omega) + \Psi(t,\omega)\big) - \big(\Phi(t,\omega) - \Psi(t,\omega)\big)\Big] \tag{3.42}$$

is uniformly bounded.

Let $T(R,\omega) = \min\{T_1(R,\omega), T_2(R,\omega), T_3(R,\omega), T_4(R,\omega), T_5(R,\omega), T_6(R,\omega)\}$. Then, for $t_0 \leq T(R,\omega)$ and $t \in [-4,0]$, we obtain that

$$\begin{aligned}
\|g(t,\omega;t_0,g_0)\|^2 =&\|U(t,\omega)\|^2 + \|V(t,\omega)\|^2 + \|\Phi(t,\omega)\|^2 + \|W(t,\omega)\|^2 + \|Z(t,\omega)\|^2 + \|\Psi(t,\omega)\|^2\\
\leq&\frac{1}{4}\|(U+W) + (U-W)\|^2 + \frac{1}{4}\|(U+W) - (U-W)\|^2 + \frac{1}{4}\|(\Phi+\Psi) + (\Phi-\Psi)\|^2\\
&+ \frac{1}{4}\|(\Phi+\Psi) - (\Phi-\Psi)\|^2 + \|V\|^2 + \|Z\|^2\\
\leq&\|U+W\|^2 + \|U-W\|^2 + \|\Phi+\Psi\|^2 + \|\Phi-\Psi\|^2 + \|V\|^2 + \|Z\|^2\\
\leq&2C_2(t,\omega) + 2C_3(t,\omega) + 9\Big(1 + b^2|O|\int_{t_0}^{t} e^{\int_{\tau}^{t}(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)}d\tau\Big)\\
=&M(t,\omega).
\end{aligned} \tag{3.43}$$

In this way, we have proved the global existence and uniqueness of the weak solution.

Since $g(t,\omega;\tau,g_0)$ is the weak solution to Eqs. (2.13)–(2.18), then

$$f(t,\omega;\tau,f_0) = S(t,\tau,\omega)f_0 = e^{\rho z(\theta_t\omega)}g(t,\omega;\tau,g_0), \quad t \geq \tau, \tag{3.44}$$

is the solution to Eqs. (1.1)–(1.9), where

$$f_0 = (u_0, v_0, \varphi_0, w_0, z_0, \psi_0), \quad g_0 = e^{-\rho z(\theta_\tau\omega)}f_0. \tag{3.45}$$

Lemma 3.2. For the extended Brusselator random dynamical system f on H over the MDS, one has a $\mathcal{D}$-pullback absorbing set $B_0(\omega)$, where $B_0(\omega)$ is the random ball centered at the origin with the radius $M_0(\omega)$ given by

$$M_0(\omega) =e^{\rho z(\theta_t\omega)}[2C_2(0,\omega) + 2C_3(0,\omega)] + 9e^{\rho z(\theta_t\omega)}\Big(1 + b^2|O| \int_{-\infty}^{0} e^{\int_\tau^0 (2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)} d\tau\Big). \quad (3.46)$$

Proof. According to the Lemma 3.1, we can obtain the consequence of Lemma 3.2. The more details of the proof, see, e.g., [16].

If we want to prove the pullback asymptotic compactness in the next section, we have to establish some necessary estimates for the V-component and Z-component in $L^6(O)$. Based on this, the following Lemma is given.

Lemma 3.3. For every given initial value $f_0 \in E$, for the terminal time $t \in [-4, 0]$, one has a random variable $P(t, \omega) > 0$ such that for any initial time $t_0 \le T_7(\|g_0\|_{L^6}, \omega) \le -4$, we obtain that

$$\|V(t, \omega; t_0, g_0)\|^6_{L^6} + \|Z(t, \omega; t_0, g_0)\|^6_{L^6} \le P(t, \omega), \quad -4 \le t \le 0. \quad (3.47)$$

Proof. We take the scalar product (2.14) with V^3 and (2.17) with Z^3, and add them up, then by using Young inequality, it follows that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|V\|^4 + \|Z\|^4) + 3d_2(\|V(t)\nabla V(t)\|^2 + \|Z(t)\nabla Z(t)\|^2)\\
=& \int_O [bUV^3 + bWZ^3 - e^{2\rho z(\theta_t\omega)}(U^2V^4 + W^2Z^4)]dx\\
&+ D_2 \int_O [(Z - V)V^3 + (V - Z)Z^3]dx + \int_O \rho z(\theta_t\omega)(V^4 + Z^4)dx\\
\le& \int_O \Big[\frac{1}{2}b^2e^{-2\rho z(\theta_t\omega)}(V^2 + Z^2) + \frac{1}{2}e^{2\rho z(\theta_t\omega)}(U^2V^4 + W^2Z^4)\Big]dx\\
&- \int_O e^{2\rho z(\theta_t\omega)}(U^2V^4 + W^2Z^4)dx + \int_O \rho z(\theta_t\omega)(V^4 + Z^4)dx, \quad (3.48)
\end{aligned}$$

where

$$\begin{aligned}
&\int_O [(Z - V)V^3 + (V - Z)Z^3]dx\\
\le& \int_O \Big[- V^4 + (\frac{1}{4}Z^4 + \frac{3}{4}V^4) + (\frac{1}{4}V^4 + \frac{3}{4}Z^4) - Z^4\Big]dx = 0.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\frac{d}{dt}(\|V(t)\|^4_{L^4} + \|Z(t)\|^4_{L^4}) + 3\gamma d_2(\|V(t)\|^4_{L^4} + \|Z(t)\|^4_{L^4})\\
\le&\frac{d}{dt}(\|V(t)\|^4_{L^4} + \|Z(t)\|^4_{L^4}) + 3d_2(\|\nabla V^2(t)\|^2 + \|\nabla Z^2(t)\|^2)\\
\le&2b^2e^{-2\rho z(\theta_t\omega)}(\|V(t)\|^2 + \|Z(t)\|^2) + 4\rho z(\theta_t\omega)(\|V(t)\|^4_{L^4} + \|Z(t)\|^4_{L^4}). \quad (3.49)
\end{aligned}$$

We multiply both sides of (3.49) by $e^{\int_{t_0}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds}$, and integrate over the interval $[t_0, t]$, here $t_0 < t$. Then by applying (3.4), we obtain that

$$\begin{aligned}
&\|V(t,\omega;t_0,g_0)\|_{L^4}^4 + \|Z(t,\omega;t_0,g_0)\|_{L^4}^4\\
\leq&(\|V_0\|_{L^4}^4 + \|Z_0\|_{L^4}^4)e^{\int_{t_0}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds} + 2b^2\int_{t_0}^{t} e^{\int_{\tau}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\tau\omega)}(\|V(\tau,\omega)\|^2 + \|Z(\tau,\omega)\|^2)d\tau\\
\leq&(\|V_0\|_{L^4}^4 + \|Z_0\|_{L^4}^4)e^{\int_{t_0}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds}\\
&+ 2b^2\int_{t_0}^{t} e^{\int_{\tau}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\tau\omega)}(\|V_0\|^2 + \|Z_0\|^2)e^{\int_{t_0}^{\tau}(2\rho z(\theta_s\omega)-2\gamma d_2)ds}d\tau\\
&+ 2b^4|O|\int_{t_0}^{t} e^{\int_{\tau}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\tau\omega)}\int_{t_0}^{\tau} e^{\int_{\xi}^{\tau}(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\xi d\tau\\
=&(\|V_0\|_{L^4}^4 + \|Z_0\|_{L^4}^4)e^{\int_{t_0}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds} + 2b^2(\|V_0\|^2 + \|Z_0\|^2)\int_{t_0}^{t} e^{\int_{t_0}^{t}(4\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)}d\tau\\
&+ 2b^4|O|\int_{t_0}^{t}\int_{\xi}^{t} e^{-2\rho z(\theta_\tau\omega)}e^{\int_{\tau}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds}e^{\int_{\xi}^{\tau}(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\tau d\xi\\
\leq&(\|V_0\|_{L^4}^4 + \|Z_0\|_{L^4}^4)e^{\int_{t_0}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds} + 2b^2(\|V_0\|^2 + \|Z_0\|^2)\int_{t_0}^{t} e^{\int_{t_0}^{t}(4\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)}d\tau\\
&+ 2b^4|O|\int_{t_0}^{t}\int_{\xi}^{t} e^{-2\rho z(\theta_\tau\omega)}e^{\int_{\xi}^{t}(4\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\tau d\xi\\
\leq&(\|V_0\|_{L^4}^4 + \|Z_0\|_{L^4}^4)e^{\int_{t_0}^{t}(4\rho z(\theta_s\omega)-3\gamma d_2)ds} + 2b^2(\|V_0\|^2 + \|Z_0\|^2)e^{\int_{t_0}^{t}(4\rho z(\theta_s\omega)-2\gamma d_2)ds}\int_{t_0}^{t} e^{-2\rho z(\theta_\tau\omega)}d\tau\\
&+ 2b^4|O|\int_{t_0}^{t}\int_{t_0}^{t} e^{-2\rho z(\theta_\tau\omega)}e^{\int_{\xi}^{t}(4\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\tau d\xi. \qquad (3.50)
\end{aligned}$$

Then, taking the inner product of (2.14) with V^5 and (2.17) with Z^5, we get

$$\begin{aligned}
&\frac{1}{6}\frac{d}{dt}(\|V\|^6 + \|Z\|^6) + 5d_2(\|V^2(t)\nabla V(t)\|^2 + \|Z^2(t)\nabla Z(t)\|^2)\\
=&\int_O [bUV^5 - e^{2\rho z(\theta_t\omega)}U^2V^6 + bWZ^5 - e^{2\rho z(\theta_t\omega)}W^2Z^6]dx\\
&+ \int_O [D_2(Z-V)V^5 + D_2(V-Z)Z^5]dx + \int_O \rho z(\theta_t\omega)(V^6 + Z^6)dx, \qquad (3.51)
\end{aligned}$$

by using Young inequality and Hölder inequality, we have

$$\begin{aligned}
&\int_O [bUV^5 - e^{2\rho z(\theta_t\omega)}U^2V^6 + bWZ^5 - e^{2\rho z(\theta_t\omega)}W^2Z^6]dx\\
\leq&\int_O \Big[\frac{1}{2}b^2e^{-2\rho z(\theta_t\omega)}(V^4 + Z^4) - \frac{1}{2}e^{2\rho z(\theta_t\omega)}(U^2V^6 + W^2Z^6)\Big]dx\\
\leq&\int_O \frac{1}{2}b^2e^{-2\rho z(\theta_t\omega)}(V^4 + Z^4)dx, \qquad (3.52)
\end{aligned}$$

and

$$\int_{O}[(Z-V)V^5+(V-Z)Z^5]dx \le \int_{O}\Big[-V^6+(\frac{1}{6}Z^6+\frac{5}{6}V^6)+(\frac{1}{6}V^6+\frac{5}{6}Z^6)-Z^6\Big]dx=0. \quad (3.53)$$

Substituting (3.52) and (3.53) into (3.51), and then applying Poincáre inequality, we obtain that

$$\begin{aligned}&\frac{d}{dt}(\|V(t)\|^6+\|Z(t)\|^6)+3\gamma d_2(\|V(t)\|^6+\|Z(t)\|^6)\\ \le&\frac{d}{dt}(\|V(t)\|^6+\|Z(t)\|^6)+\frac{10}{3}d_2(\|\nabla V^3(t)\|^2+\|\nabla Z^3(t)\|^2)\\ \le&3b^2e^{-2\rho z(\theta_t\omega)}(\|V(t)\|^4+\|Z(t)\|^4)+6\rho z(\theta_t\omega)(\|V(t)\|^6+\|Z(t)\|^6). \quad (3.54)\end{aligned}$$

We multiply both sides of (3.54) by $e^{\int_{t_0}^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds}$, and integrate over $[t_0,t]$, here $t_0<-4<t<0$. Then one has a random variable $T_7(\|g_0\|_{L^6},\omega)\le -4$ such that for any $\omega\in\Omega$, $t_0\le T_7(\|g_0\|_{L^6},\omega)$ and $t\in[-4,0]$, by applying (3.50), we obtain that

$$\begin{aligned}&\|V(t,\omega;t_0,g_0)\|_{L^6}^6+\|Z(t,\omega;t_0,g_0)\|_{L^6}^6\\ \le&(\|V_0\|_{L^6}^6+\|Z_0\|_{L^6}^6)e^{\int_{t_0}^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds}\\ &+3b^2\int_{t_0}^t e^{\int_\eta^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\eta\omega)}(\|V(\eta,\omega)\|_{L^4}^4+\|Z(\eta,\omega)\|_{L^4}^4)d\eta\\ \le&(\|V_0\|_{L^6}^6+\|Z_0\|_{L^6}^6)e^{\int_{t_0}^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds}\\ &+3b^2\int_{t_0}^t e^{\int_\eta^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\eta\omega)}(\|V_0\|_{L^4}^4+\|Z_0\|_{L^4}^4)e^{\int_{t_0}^\eta(4\rho z(\theta_s\omega)-3\gamma d_2)ds}d\eta\\ &+6b^4(\|V_0\|^2+\|Z_0\|^2)\int_{t_0}^t e^{\int_\eta^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\eta\omega)}e^{\int_{t_0}^\eta(4\rho z(\theta_s\omega)-2\gamma d_2)ds}\cdot\int_{t_0}^\eta e^{-2\rho z(\theta_\tau\omega)}d\tau d\eta\\ &+6b^6|O|\int_{t_0}^t e^{\int_\eta^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\eta\omega)}\int_{t_0}^\eta e^{-2\rho z(\theta_\tau\omega)}d\tau\cdot\int_{t_0}^\eta e^{\int_\xi^\eta(4\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\xi d\eta\\ \le&(\|V_0\|_{L^6}^6+\|Z_0\|_{L^6}^6)e^{\int_{t_0}^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds}\\ &+3b^2(\|V_0\|_{L^4}^4+\|Z_0\|_{L^4}^4)\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)}e^{\int_{t_0}^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds}d\eta\\ &+6b^4(\|V_0\|^2+\|Z_0\|^2)\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)}e^{\int_{t_0}^t(6\rho z(\theta_s\omega)-2\gamma d_2)ds}\int_{t_0}^\eta e^{-2\rho z(\theta_\tau\omega)}d\tau d\eta\\ &+6b^6|O|\int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)}d\tau\int_{t_0}^t\int_{t_0}^\eta e^{\int_\eta^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\eta\omega)}\cdot e^{\int_\xi^\eta(4\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\xi d\eta\\ \le&(\|V_0\|_{L^6}^6+\|Z_0\|_{L^6}^6)e^{\int_{t_0}^t 6\rho z(\theta_s\omega)ds-3\gamma d_2(t-t_0)}\\ &+3b^2(\|V_0\|_{L^4}^4+\|Z_0\|_{L^4}^4)e^{\int_{t_0}^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds}\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)}d\eta\\ &+6b^4(\|V_0\|^2+\|Z_0\|^2)e^{\int_{t_0}^t(6\rho z(\theta_s\omega)-2\gamma d_2)ds}\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)}\int_{t_0}^\eta e^{-2\rho z(\theta_\tau\omega)}d\tau d\eta\\ &+6b^6|O|\int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)}d\tau\int_{t_0}^t\int_\xi^t e^{-2\rho z(\theta_\eta\omega)}e^{\int_\xi^t(6\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\eta d\xi\end{aligned}$$

$$
\begin{aligned}
&\le (\|V_0\|_{L^6}^6 + \|Z_0\|_{L^6}^6) e^{\int_{t_0}^t 6\rho z(\theta_s\omega)ds - 3\gamma d_2(t-t_0)} \\
&\quad + 3b^2(\|V_0\|_{L^4}^4 + \|Z_0\|_{L^4}^4) e^{\int_{t_0}^t (6\rho z(\theta_s\omega) - 3\gamma d_2)ds} \int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} d\eta \\
&\quad + 6b^4(\|V_0\|^2 + \|Z_0\|^2) e^{\int_{t_0}^t (6\rho z(\theta_s\omega) - 2\gamma d_2)ds} \int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} d\eta \int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} d\tau \\
&\quad + 6b^6|O| \int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} d\tau \int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} d\eta \int_{t_0}^t e^{\int_\xi^t (6\rho z(\theta_s\omega) - 2\gamma d_2)ds - 2\rho z(\theta_\xi\omega)} d\xi. \qquad (3.55)
\end{aligned}
$$

Therefore, we get

$$
\|V(t,\omega;t_0,g_0)\|_{L^6}^6 + \|Z(t,\omega;t_0,g_0)\|_{L^6}^6 \le P(t,\omega), \qquad (3.56)
$$

where

$$
P(t,\omega) = 3 + 6b^6|O| \int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} d\tau \int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} d\eta \int_{t_0}^t e^{\int_\xi^t (6\rho z(\theta_s\omega) - 2\gamma d_2)ds - 2\rho z(\theta_\xi\omega)} d\xi.
$$

Owing to (3.5)–(3.8), we deduce that $P(t,\omega)$ is convergent. Therefore, we have completed the proof.
Lemma 3.4. Let $R(\omega) > 0$ be a given tempered random variable and for all $t_0 < -4 < t_1 < 0$ and every initial value $f_0 \in H$, where $\|f_0\| \le R(\omega)$. Suppose that $g(t,\omega;t_0,g_0)$ satisfies $\|g(t_1,\omega;t_0,g_0)\| \in E$ with

$$
\|g(t_1,\omega;t_0,g_0)\|_E \le G(\omega),
$$

here $G(\omega) > 0$ is any given random variable. For all $t \in [t_1, 0]$, then one has a random variable $D(t,G,\omega) > 0$ such that

$$
\|V(t,\omega;t_0,g_0)\|_{L^6}^6 + \|Z(t,\omega;t_0,g_0)\|_{L^6}^6 \le D(t,G,\omega), \quad \forall t_0 \le \min\{T(R,\omega), -4\}. \qquad (3.57)
$$

Proof. For any initial time $t_0 \le \min\{T(R,\omega), -4\}$. We integrate (3.49) over $[t_1, t]$ and apply (3.10), which follows that

$$
\begin{aligned}
&\|V(t,\omega;t_0,g_0)\|_{L^4}^4 + \|Z(t,\omega;t_0,g_0)\|_{L^4}^4 \\
\le& (\|V(t_1,\omega;t_0,g_0)\|_{L^4}^4 + \|Z(t_1,\omega;t_0,g_0)\|_{L^4}^4) e^{\int_{t_1}^t (4\rho z(\theta_s\omega) - 3\gamma d_2)ds} \\
&+ 2b^2 \int_{t_1}^t e^{\int_\tau^t (4\rho z(\theta_s\omega) - 3\gamma d_2)ds - 2\rho z(\theta_\tau\omega)} (\|V(\tau,\omega)\|^2 + \|Z(\tau,\omega)\|)^2 d\tau \\
\le& \delta^4 G^4(\omega) e^{\int_{t_1}^t (4\rho z(\theta_s\omega) - 3\gamma d_2)ds} + 2b^2 \int_{t_1}^t e^{\int_\tau^t (4\rho z(\theta_s\omega) - 3\gamma d_2)ds - 2\rho z(\theta_\tau\omega)} d\tau \\
&+ 2b^4|O| \int_{t_1}^t e^{\int_\tau^t (4\rho z(\theta_s\omega) - 3\gamma d_2)ds - 2\rho z(\theta_\tau\omega)} \int_{-\infty}^\tau e^{\int_\xi^\tau (2\rho z(\theta_s\omega) - 2\gamma d_2)ds - 2\rho z(\theta_\xi\omega)} d\xi d\tau \\
\le& \Pi(t,\omega), \qquad (3.58)
\end{aligned}
$$

where δ is the constant of the Sobolev embedding $H_0^1(O) \hookrightarrow L^4(O)$ satisfies

$$
\|\phi\|_{L^4(O)} \le \delta \|\phi\|_E, \quad \forall \phi \in E.
$$

For each initial time $t_0 \leq \min\{T(R,\omega), -4\}$, we integrate (3.54) over $[t_1, t]$, then by using (2.4) and (3.58), we obtain that

$$\begin{aligned}
&\|V(t,\omega;t_0,g_0)\|_{L^6}^6 + \|Z(t,\omega;t_0,g_0)\|_{L^6}^6\\
\leq&(\|V(t_1,\omega;t_0,g_0)\|_{L^6}^6 + \|Z(t_1,\omega;t_0,g_0)\|_{L^6}^6)e^{\int_{t_1}^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds}\\
&+3b^2\int_{t_1}^t e^{\int_\tau^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\tau\omega)}(\|V(\tau,\omega)\|_{L^4}^4 + \|Z(\tau,\omega)\|_{L^4}^4)d\tau\\
\leq&\zeta^6G^6(\omega)e^{\int_{t_1}^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds} + 3b^2\int_{t_1}^t e^{\int_\tau^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\tau\omega)}\Pi(t,\omega)d\tau. \qquad (3.59)
\end{aligned}$$

Therefore,

$$D(t,G,\omega) =\zeta^6G^6(\omega)e^{\int_{t_1}^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds} + 3b^2\int_{t_1}^t e^{\int_\tau^t(6\rho z(\theta_s\omega)-3\gamma d_2)ds-2\rho z(\theta_\tau\omega)}\Pi(t,\omega)d\tau.$$

Then, we have completed the proof of Lemma 3.4.

4. Pullback asymptotic compactness

In this section, we will apply the following uniform Gronwall inequality to study the pullback asymptotically compact of the extended Brusselator random dynamical system f in H, the reader can refer to reference [26] for more details.

Proposition 4.1. Assume $n > 1$ is a given natural number. Let σ, π and χ be nonnegative functions in $L^1([-n,0];\mathbb{R}^+)$. Suppose that σ is absolutely continuous over $[-n,0]$ and it satisfies the following inequality

$$\frac{d\sigma}{dt} \leq \pi\sigma + \chi, \quad \text{for } t \in [-n,0].$$

If

$$\int_t^{t+1}\pi(\tau)d\tau \leq A, \quad \int_t^{t+1}\sigma(\tau)d\tau \leq B, \quad \int_t^{t+1}\chi(\tau)d\tau \leq C, \quad \forall t \in [-n,-1],$$

where A, B and C are some positive constants, then

$$\sigma(t) \leq (B+C)e^A, \quad \text{for } t \in [-n+1,0].$$

Lemma 4.2. Let $R(\omega) > 0$ be a given random variable and for every initial value $f_0 \in H$, where $\|f_0\| \leq R(\omega)$, one has a tempered random variable $K(\omega) > 0$ and a finite time variable $T(R,\omega) > 0$ such that for $t_0 \leq T(R,\omega)$, the weak solution $g(t,\omega;\tau,g_0)$ of the random extended Brusselator Eqs. (2.13)–(2.18) satisfies $g(0,\omega;t_0,g_0) \in E$ and satisfies the following estimate

$$\|g(0,\omega;t_0,g_0)\|_E^2 \leq K(\omega), \quad t_0 \leq T(R,\omega). \qquad (4.1)$$

Proof. We can divide into four steps to get the proof of Lemma 4.2.

Step 1. we establish the estimates of the $H_0^1(\mathcal{O})$-norm for the U-component, V-component, Φ-component, W-component, Z-component and Ψ-component of the solution in $[-4,-1]$.

Step 2. we study the estimates of the U-component and W-component in $[-2,0]$ by applying the uniform Gronwall inequality.

Step 3. we obtain the estimates of the V-component and Z-component in $[-1,0]$ by applying the results of the first and the second steps.

Step 4. we conduct the estimates of the Φ-component and Ψ-component in $[-1,0]$.

Step 1. We study the time-average estimates of the E-norm for the weak solution $g(t,\omega) = (U,V,\Phi,W,Z,\Psi)$. In Lemma 3.4, we have established the estimates of $L^6(\mathcal{O})$-norm of the V-component and Z-component. Noticing that $z(\theta_t\omega)$ is continuous in t, then we obtain that for any given $\omega\in\Omega$, $Z(\omega)=\max_{-4\le\tau\le-1}|z(\theta_\tau\omega)|$ is a positive constant. We integrate (3.2) over $[t,t+1]$, here $-4\le t\le -1$, then by applying (3.10), which follows that

$$
\begin{aligned}
&\int_t^{t+1} 2d_2(\|\nabla V(\tau,\omega;t_0,g_0)\|^2+\|\nabla Z(\tau,\omega;t_0,g_0)\|^2)d\tau\\
\le&\|V(t)\|^2+\|Z(t)\|^2+\int_t^{t+1} b^2|\mathcal{O}|e^{-2\rho z(\theta_\tau\omega)}d\tau\\
&+\int_t^{t+1} 2\rho z(\theta_\tau\omega)\Big(1+b^2|\mathcal{O}|\int_{-\infty}^{\tau} e^{\int_\xi^\tau(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\xi\Big)d\tau\\
\le&1+b^2|\mathcal{O}|\max_{-4\le t\le -1}\int_{-\infty}^{t} e^{\int_\tau^t(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)}d\tau+b^2|\mathcal{O}|\int_{-4}^{0} e^{-2\rho z(\theta_\tau\omega)}d\tau\\
&+\int_{-4}^{0} 2c|z(\theta_\tau\omega)|\Big(1+b^2|\mathcal{O}|\int_{-\infty}^{\tau} e^{\int_\xi^\tau(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\xi\Big)d\tau. \qquad (4.2)
\end{aligned}
$$

Then, for $t_0\le\min\{T(R,\omega),-4\}$ and $-4\le t\le-1$, we obtain that

$$
\int_t^{t+1}(\|\nabla V(\tau,\omega;t_0,g_0)\|^2+\|\nabla Z(\tau,\omega;t_0,g_0)\|^2)d\tau\le\frac{K_1(\omega)}{2d_2}, \qquad (4.3)
$$

where

$$
\begin{aligned}
K_1(\omega)=&1+b^2|\mathcal{O}|\max_{-4\le t\le -1}\int_{-\infty}^{t} e^{\int_\tau^t(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\tau\omega)}d\tau+b^2|\mathcal{O}|\int_{-4}^{0} e^{-2\rho z(\theta_\tau\omega)}d\tau\\
&+\int_{-4}^{0} 2c|z(\theta_\tau\omega)|\Big(1+b^2|\mathcal{O}|\int_{-\infty}^{\tau} e^{\int_\xi^\tau(2\rho z(\theta_s\omega)-2\gamma d_2)ds-2\rho z(\theta_\xi\omega)}d\xi\Big)d\tau.
\end{aligned}
$$

Let $t=-4$, which follows that

$$
\int_{-4}^{-3}(\|\nabla V(\tau,\omega;t_0,g_0)\|^2+\|\nabla Z(\tau,\omega;t_0,g_0)\|^2)d\tau\le\frac{K_1(\omega)}{2d_2}. \qquad (4.4)
$$

According to the Mean Value Theorem, one has a time $t_1\in[-4,-3]$ such that

$$
\|V(t_1,\omega;t_0,g_0)\|_E^2+\|Z(t_1,\omega;t_0,g_0)\|_E^2\le\frac{K_1(\omega)}{2d_2}. \qquad (4.5)
$$

Therefore, by using Lemma 3.4, one has a random variable $D(t, \frac{K_1}{2d_1}, \omega) > 0$ such that

$$\|V(t,\omega;t_0,g_0)\|_{L^6}^6 + \|Z(t,\omega;t_0,g_0)\|_{L^6}^6 \le D(t, \frac{K_1}{2d_1}, \omega), \quad \forall t \in [t_1, 0]. \tag{4.6}$$

Integrating (3.19) over the interval $[t, t+1]$, where $-4 \le t \le -1$. For all $t_0 \le \min\{T(R,\omega), -4\}$, by using (3.25) and (4.5), we obtain that

$$\begin{aligned}
&d\int_t^{t+1} (\|\nabla y(\tau,\omega;t_0,g_0)\|^2 + \|\nabla\xi(\tau,\omega;t_0,g_0)\|^2)d\tau \\
\le&\alpha(\|y(t)\|^2 + \|\xi(t)\|^2) + \alpha\beta\int_t^{t+1} (\|\nabla V(\tau)\|^2 + \|\nabla Z(\tau)\|^2)d\tau \\
&+ \alpha\int_t^{t+1} 2\rho z(\theta_\tau\omega)(\|y(\tau)\|^2 + \|\xi(\tau)\|^2)d\tau + \frac{16a^2\alpha|O|}{d_1\gamma}\int_t^{t+1} e^{-2\rho z(\theta_\tau\omega)}d\tau \\
\le&\alpha\max_{-4\le t\le -1} C_2(t,\omega) + \alpha\beta\frac{K_1(\omega)}{2d_2} + 2\alpha\max_{-4\le t\le -1} C_2(t,\omega)\int_{-4}^0 c|z(\theta_\tau\omega)|d\tau \\
&+ \frac{16a^2\alpha|O|}{d_1\gamma}\int_{-4}^0 e^{-2\rho z(\theta_\tau\omega)}d\tau.
\end{aligned} \tag{4.7}$$

Therefore, we have

$$\int_t^{t+1} (\|\nabla y(\tau,\omega;t_0,g_0)\|^2 + \|\nabla\xi(\tau,\omega;t_0,g_0)\|^2)d\tau \le \frac{K_2(\omega)}{d}, \tag{4.8}$$

where

$$\begin{aligned}
K_2(\omega) =&\alpha\max_{-4\le t\le -1} C_2(t,\omega) + \alpha\beta\frac{K_1(\omega)}{2d_2} + 2\alpha\max_{-4\le t\le -1} C_2(t,\omega)\int_{-4}^0 c|z(\theta_\tau\omega)|d\tau \\
&+ \frac{16a^2\alpha|O|}{d_1\gamma}\int_{-4}^0 e^{-2\rho z(\theta_\tau\omega)}d\tau.
\end{aligned}$$

In this way, for any $t_0 \le \min\{T(R,\omega), -4\}$ and $-4 \le t \le -1$, we deduce that

$$\begin{aligned}
&\int_t^{t+1} (\|\nabla U(\tau,\omega) + \nabla W(\tau,\omega)\|^2 + \|\nabla\Phi(\tau,\omega) + \nabla\Psi(\tau,\omega)\|^2)d\tau \\
=&\int_t^{t+1} (\|\nabla y(\tau,\omega) - \nabla V(\tau,\omega) - \nabla Z(\tau,\omega)\|^2 + \|\nabla\xi(\tau,\omega)\|^2)d\tau \\
\le&\int_t^{t+1} [2\|\nabla y(\tau,\omega)\|^2 + \|\nabla\xi(\tau,\omega)\|^2 + 4(\|\nabla V(\tau,\omega)\|^2 + \|\nabla Z(\tau,\omega)\|^2)]d\tau \\
\le&\frac{2K_2(\omega)}{d} + \frac{2K_1(\omega)}{d_2} = K_3(\omega).
\end{aligned} \tag{4.9}$$

We integrate (3.36) over the interval $[t, t+1]$, where $-4 \le t \le -1$. For all $t_0 \le \min\{T(R,\omega), -4\}$, by using (3.37) and (4.5), we obtain that

$$d\int_t^{t+1} (\|\nabla p(\tau,\omega;t_0,g_0)\|^2 + \|\nabla q(\tau,\omega;t_0,g_0)\|^2)d\tau$$

$$
\begin{aligned}
&\le \alpha(\|p(t)\|^2 + \|q(t)\|^2) + \alpha\kappa \int_t^{t+1} (\|\nabla V(\tau)\|^2 + \|\nabla Z(\tau)\|^2) d\tau \\
&\quad + \alpha \int_t^{t+1} 2\rho z(\theta_\tau\omega)(\|p(\tau)\|^2 + \|q(\tau)\|^2) d\tau \\
&\le \alpha \max_{-4\le t\le -1} C_3(t,\omega) + \alpha\kappa \frac{K_1(\omega)}{2d_2} + 2\alpha \max_{-4\le t\le -1} C_3(t,\omega) \int_{-4}^{0} c|z(\theta_\tau\omega)| d\tau.
\end{aligned} \tag{4.10}
$$

Therefore, we have

$$
\int_t^{t+1} (\|\nabla p(\tau,\omega;t_0,g_0)\|^2 + \|\nabla q(\tau,\omega;t_0,g_0)\|^2) d\tau \le \frac{K_4(\omega)}{d}, \tag{4.11}
$$

where

$$
K_4(\omega) = \alpha \max_{-4\le t\le -1} C_3(t,\omega) + \alpha\kappa \frac{K_1(\omega)}{2d_2} + 2\alpha \max_{-4\le t\le -1} C_3(t,\omega) \int_{-4}^{0} c|z(\theta_\tau\omega)| d\tau.
$$

In this way, for any $t_0 \le \min\{T(R,\omega), -4\}$ and $-4 \le t \le -1$, we deduce that

$$
\begin{aligned}
&\int_t^{t+1} (\|\nabla U(\tau,\omega) - \nabla W(\tau,\omega)\|^2 + \|\nabla\Phi(\tau,\omega) - \nabla\Psi(\tau,\omega)\|^2) d\tau \\
&= \int_t^{t+1} (\|\nabla p(\tau,\omega) - \nabla V(\tau,\omega) + \nabla Z(\tau,\omega)\|^2 + \|\nabla q(\tau,\omega)\|^2) d\tau \\
&\le \int_t^{t+1} \Big(2\|\nabla p(\tau,\omega)\|^2 + \|\nabla q(\tau,\omega)\|^2 + 4(\|\nabla V(\tau,\omega)\|^2 + \|\nabla Z(\tau,\omega)\|^2)\Big) d\tau \\
&\le \frac{2K_4(\omega)}{d} + \frac{2K_1(\omega)}{d_2} = K_5(\omega).
\end{aligned} \tag{4.12}
$$

Thus, for $t_0 \le \min\{T(R,\omega), -4\}$ and $-4 \le t \le -1$, we get by applying (4.9) and (4.12)

$$
\begin{aligned}
&\int_t^{t+1} (\|\nabla U\|^2 + \|\nabla W\|^2 + \|\nabla\Phi\|^2 + \|\nabla\Psi\|^2) d\tau \\
&\le \frac{1}{4} \int_t^{t+1} (\|(\nabla U + \nabla W) + (\nabla U - \nabla W)\|^2 + \|(\nabla U + \nabla W) - (\nabla U - \nabla W)\|^2) d\tau \\
&\quad + \frac{1}{4} \int_t^{t+1} (\|(\nabla\Phi + \nabla\Psi) + (\nabla\Phi - \nabla\Psi)\|^2 + \|(\nabla\Phi + \nabla\Psi) - (\nabla\Phi - \nabla\Psi)\|^2) d\tau \\
&\le \int_t^{t+1} (\|\nabla U + \nabla W\|^2 + \|\nabla U - \nabla W\|^2 + \|\nabla\Phi + \nabla\Psi\|^2 + \|\nabla\Phi - \nabla\Psi\|^2) d\tau \\
&= K_4(\omega) + K_5(\omega) = K_6(\omega).
\end{aligned} \tag{4.13}
$$

Step 2. We take the inner product of (2.13) with $-\Delta U$ and (2.16) with $-\Delta W$, which follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla U\|^2 + \|\nabla W\|^2) + d_1(\|\Delta U\|^2 + \|\Delta W\|^2) + (b+k)(\|\nabla U\|^2 + \|\nabla W\|^2) \\
&= \int_O [-ae^{-\rho z(\theta_t\omega)}(\Delta U + \Delta W) - e^{2\rho z(\theta_t\omega)}(U^2 V \Delta U + W^2 Z \Delta W)] dx
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathcal{O}}[N(\Phi\Delta U+\Psi\Delta W)+D_1(|\nabla U|^2-2\nabla U\cdot\nabla W+|\nabla W|^2)]dx+\rho z(\theta_t\omega)(\|\nabla U\|^2+\|\nabla W\|^2)\\
\leq&\frac{3}{4}d_1(\|\Delta U\|^2+\|\Delta W\|^2)+\frac{a^2|\mathcal{O}|}{d_1}e^{-2\rho z(\theta_t\omega)}+\frac{1}{d_1}e^{4\rho z(\theta_t\omega)}\int_{\mathcal{O}}(U^4V^2+W^4Z^2)dx\\
&+\frac{N^2}{d_1}(\|\Phi\|^2+\|\Psi\|^2)+\rho z(\theta_t\omega)(\|\nabla U\|^2+\|\nabla W\|^2).
\end{aligned}\tag{4.14}
$$

Then, by applying (2.2) and (2.4), we have

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla U\|^2+\|\nabla W\|^2)+2(b+k)(\|\nabla U\|^2+\|\nabla W\|^2)\\
\leq&\frac{2a^2|\mathcal{O}|}{d_1}e^{-2\rho z(\theta_t\omega)}+\frac{2}{d_1}e^{4\rho z(\theta_t\omega)}(\|U\|_{L^6}^4\|V\|_{L^6}^2+\|W\|_{L^6}^4\|Z\|_{L^6}^2)+\frac{2N^2}{d_1}(\|\Phi\|^2+\|\Psi\|^2)\\
&+2\rho z(\theta_t\omega)(\|\nabla U\|^2+\|\nabla W\|^2)\\
\leq&\frac{2a^2|\mathcal{O}|}{d_1}e^{-2\rho z(\theta_t\omega)}+\frac{2N^2}{d_1}(\|\Phi\|^2+\|\Psi\|^2)+2\rho z(\theta_t\omega)(\|\nabla U\|^2+\|\nabla W\|^2)\\
&+\frac{4\delta^4}{d_1}e^{4\rho z(\theta_t\omega)}\Big[(\|U\|^4+\|\nabla U\|^4)\|V\|_{L^6}^2+(\|W\|^4+\|\nabla W\|^4)\|Z\|_{L^6}^2\Big]\\
\leq&\frac{2a^2|\mathcal{O}|}{d_1}e^{-2\rho z(\theta_t\omega)}+\frac{2N^2}{d_1}(\|\Phi\|^2+\|\Psi\|^2)+2\rho z(\theta_t\omega)(\|\nabla U\|^2+\|\nabla W\|^2)\\
&+\frac{4\delta^4}{d_1}e^{4\rho z(\theta_t\omega)}(\|U\|^2+\|W\|^2)^2(\|V\|_{L^6}^2+\|Z\|_{L^6}^2)+\frac{4\delta^4}{d_1}e^{4\rho z(\theta_t\omega)}(\|\nabla U\|^2+\|\nabla W\|^2)^2(\|V\|_{L^6}^2+\|Z\|_{L^6}^2)\\
\leq&\frac{2a^2|\mathcal{O}|}{d_1}e^{-2\rho z(\theta_t\omega)}+\frac{2N^2}{d_1}(\|\Phi\|^2+\|\Psi\|^2)+2\rho z(\theta_t\omega)(\|\nabla U\|^2+\|\nabla W\|^2)\\
&+(\frac{4\delta^4}{d_1\gamma^2}+\frac{4\delta^4}{d_1})e^{4\rho z(\theta_t\omega)}(\|\nabla U\|^2+\|\nabla W\|^2)^2(\|V\|_{L^6}^2+\|Z\|_{L^6}^2).
\end{aligned}\tag{4.15}
$$

We can rewrite (4.15) as the following form

$$
\frac{d\beta_1(t)}{dt}\leq\alpha_1(t)\beta_1(t)+\gamma_1(t),\quad t\in[-3,0],\tag{4.16}
$$

where

$$
\begin{aligned}
\alpha_1(t)&=\Big(\frac{4\delta^4}{d_1\gamma^2}+\frac{4\delta^4}{d_1}\Big)e^{4\rho z(\theta_t\omega)}(\|\nabla U\|^2+\|\nabla W\|^2)(\|V\|_{L^6}^2+\|Z\|_{L^6}^2)+2\rho z(\theta_t\omega),\\
\beta_1(t)&=\|\nabla U\|^2+\|\nabla W\|^2,\\
\gamma_1(t)&=\frac{2a^2|\mathcal{O}|}{d_1}e^{-2\rho z(\theta_t\omega)}+\frac{2N^2}{d_1}(\|\Phi\|^2+\|\Psi\|^2).
\end{aligned}
$$

For $t_0\leq T(R,\omega)$ and $-3\leq t\leq -1$, by applying (4.6) and (4.13), we have

$$
\begin{aligned}
\int_t^{t+1}\alpha_1(\tau)d\tau=&\int_t^{t+1}2\rho z(\theta_\tau\omega)d\tau+\int_t^{t+1}\Big(\frac{4\delta^4}{d_1\gamma^2}+\frac{4\delta^4}{d_1}\Big)e^{4\rho z(\theta_\tau\omega)}(\|\nabla U\|^2+\|\nabla W\|^2)(\|V\|_{L^6}^2+\|Z\|_{L^6}^2)d\tau\\
\leq&2\int_{-3}^{0}c|z(\theta_\tau\omega)|d\tau+\Big(\frac{4\delta^4}{d_1\gamma^2}+\frac{4\delta^4}{d_1}\Big)\max_{-3\leq\tau\leq0}[D^{\frac{1}{3}}(\tau,\frac{K_1}{2d_1},\omega)e^{4\rho z(\theta_\tau\omega)}]K_6(\omega).
\end{aligned}\tag{4.17}
$$

Then, by using (4.13), we obtain

$$\int_t^{t+1}\beta_1(\tau)d\tau=\int_t^{t+1}(\|\nabla U\|^2+\|\nabla W\|^2)d\tau\le K_6(\omega), \tag{4.18}$$

and

$$\begin{aligned}\int_t^{t+1}\gamma_1(\tau)d\tau=&\int_t^{t+1}\Big[\frac{2a^2|O|}{d_1}e^{-2\rho z(\theta_\tau\omega)}+\frac{2N^2}{d_1}(\|\Phi\|^2+\|\Psi\|^2)\Big]d\tau\\ \le&\frac{2a^2|O|}{d_1}\int_{-3}^{0}e^{-2\rho z(\theta_\tau\omega)}d\tau+\frac{2N^2}{d_1\gamma}K_6(\omega).\end{aligned} \tag{4.19}$$

By applying the uniform Gronwall inequality, which follows that

$$\|U(t,\omega;t_0,g_0)\|_E^2+\|W(t,\omega;t_0,g_0)\|_E^2\le K_7(\omega),\quad t\in[-2,0],\ t_0\le T(R,\omega), \tag{4.20}$$

where

$$\begin{aligned}K_7(\omega)=&\Big(K_6(\omega)+\frac{2a^2|O|}{d_1}\int_{-3}^{0}e^{-2\rho z(\theta_\tau\omega)}d\tau+\frac{2N^2}{d_1\gamma}K_6(\omega)\Big)\cdot\\ &\exp\Big[\Big(\frac{4\delta^4}{d_1\gamma^2}+\frac{4\delta^4}{d_1}\Big)\max_{-3\le\tau\le0}[D^{\frac{1}{3}}(\tau,\frac{K_1}{2d_1},\omega)e^{4\rho z(\theta_\tau\omega)}]K_6(\omega)+2\int_{-3}^{0}c|z(\theta_\tau\omega)|d\tau\Big].\end{aligned}$$

Step 3. Taking the scalar product of (2.14) with $-\Delta V$ and (2.17) with $-\Delta Z$, and adding up their results, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla V\|^2+\|\nabla Z\|^2)+d_2(\|\Delta V\|^2+\|\Delta Z\|^2)\\ =&\int_O[-bU\Delta V-bW\Delta Z+e^{2\rho z(\theta_t\omega)}(U^2V\Delta V+W^2Z\Delta Z)]dx\\ &-\int_O D_2(|\nabla Z|^2-2\nabla Z\cdot\nabla V+|\nabla V|^2)dx+\rho z(\theta_t\omega)(\|\nabla V\|^2+\|\nabla Z\|^2)\\ \le&(\frac{d_2}{4}+\frac{d_2}{4})(\|\Delta V\|^2+\|\Delta Z\|^2)+\frac{b^2}{d_2}(\|U\|^2+\|W\|^2)\\ &+\frac{1}{d_2}e^{4\rho z(\theta_t\omega)}\int_O(U^4V^2+W^4Z^2)dx+\rho z(\theta_t\omega)(\|\nabla V\|^2+\|\nabla Z\|^2).\end{aligned} \tag{4.21}$$

Therefore, we obtain that

$$\begin{aligned}&\frac{d}{dt}(\|\nabla V\|^2+\|\nabla Z\|^2)+d_2(\|\Delta V\|^2+\|\Delta Z\|^2)\\ \le&\frac{2b^2}{d_2}(\|U\|^2+\|W\|^2)+\frac{2}{d_2}e^{4\rho z(\theta_t\omega)}(\|U\|_{L^6}^4\|V\|_{L^6}^2+\|W\|_{L^6}^4\|Z\|_{L^6}^2)+2\rho z(\theta_t\omega)(\|\nabla V\|^2+\|\nabla Z\|^2)\\ \le&\frac{4\delta^6}{d_2}e^{4\rho z(\theta_t\omega)}[(\|U\|^4+\|\nabla U\|^4)\|\nabla V\|^2+(\|W\|^4+\|\nabla W\|^4)\|\nabla Z\|^2]+\frac{2b^2}{d_2}(\|U\|^2+\|W\|^2)\\ &+2\rho z(\theta_t\omega)(\|\nabla V\|^2+\|\nabla Z\|^2)\end{aligned}$$

$$\leq \frac{4\delta^6}{d_2} e^{4\rho z(\theta_t\omega)}[(\|U\|^2 + \|W\|^2)^2 + (\|\nabla U\|^2 + \|\nabla W\|^2)^2](\|\nabla V\|^2 + \|\nabla Z\|^2) + \frac{2b^2}{d_2}(\|U\|^2 + \|W\|^2)$$
$$+ 2\rho z(\theta_t\omega)(\|\nabla V\|^2 + \|\nabla Z\|^2). \tag{4.22}$$

Then, we can rewrite (4.22) as the following form

$$\frac{d\beta_2(t)}{dt} \leq \alpha_2(t)\beta_2(t) + \gamma_2(t), \quad t \in [-2, 0], \tag{4.23}$$

where

$$\alpha_2(t) = \frac{4\delta^6}{d_2} e^{4\rho z(\theta_t\omega)}\Big[(\|U\|^2 + \|W\|^2)^2 + (\|\nabla U\|^2 + \|\nabla W\|^2)^2\Big] + 2\rho z(\theta_t\omega),$$
$$\beta_2(t) = \|\nabla V\|^2 + \|\nabla Z\|^2, \quad \gamma_2(t) = \frac{2b^2}{d_2}(\|U\|^2 + \|W\|^2).$$

For $t_0 \leq T(R, \omega)$ and $-2 \leq t \leq -1$, by applying (4.20), we infer that

$$\int_t^{t+1} \alpha_2(\tau)d\tau = \int_t^{t+1} \frac{4\delta^6}{d_2} e^{4\rho z(\theta_\tau\omega)}\Big[(\|U\|^2 + \|W\|^2)^2 + (\|\nabla U\|^2 + \|\nabla W\|^2)^2\Big]d\tau + \int_t^{t+1} 2\rho z(\theta_\tau\omega)d\tau$$
$$\leq \Big(\frac{4\delta^6}{d_2\gamma^2} + \frac{4\delta^6}{d_2}\Big) \max_{-2\leq\tau\leq 0} e^{4\rho z(\theta_\tau\omega)} K_7^2(\omega) + 2\int_{-2}^{0} c|z(\theta_\tau\omega)|d\tau. \tag{4.24}$$

Then, by using (4.3), we obtain

$$\int_t^{t+1} \beta_2(\tau)d\tau = \int_t^{t+1} (\|\nabla V\|^2 + \|\nabla Z\|^2)d\tau \leq \frac{K_1(\omega)}{2d_2}, \tag{4.25}$$

and by applying (4.20), we have

$$\int_t^{t+1} \gamma_2(\tau)d\tau = \int_t^{t+1} \frac{2b^2}{d_2}(\|U\|^2 + \|W\|^2)d\tau \leq \frac{2b^2}{d_2\gamma} K_7(\omega). \tag{4.26}$$

Owing to the uniform Gronwall inequality, which follows that

$$\|V(t, \omega; t_0, g_0)\|_E^2 + \|Z(t, \omega; t_0, g_0)\|_E^2 \leq K_8(\omega), \quad t \in [-1, 0],\ t_0 \leq T(R, \omega), \tag{4.27}$$

where

$$K_8(\omega) = \Big(\frac{K_1(\omega)}{2d_2} + \frac{2b^2}{d_2\gamma} K_7(\omega)\Big) \cdot \exp\Big[\Big(\frac{4\delta^6}{d_2\gamma^2} + \frac{4\delta^6}{d_2}\Big) \max_{-2\leq\tau\leq 0} e^{4\rho z(\theta_\tau\omega)} K_7^2(\omega) + 2\int_{-2}^{0} c|z(\theta_\tau\omega)|d\tau\Big].$$

Step 4. Taking the scalar product of (2.15) with $-\Delta\Phi$ and (2.18) with $-\Delta\Psi$, and adding up their results, we get

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\Phi\|^2 + \|\nabla\Psi\|^2) + d_3(\|\Delta\Phi\|^2 + \|\Delta\Psi\|^2) + (\lambda + N)(\|\nabla\Phi\|^2 + \|\nabla\Psi\|^2)$$
$$= \int_O (-kU\Delta\Phi - kW\Delta\Psi)dx - \int_O D_3(|\nabla\Phi|^2 - 2\nabla\Phi \cdot \nabla\Psi + |\nabla\Psi|^2)dx + \rho z(\theta_t\omega)(\|\nabla\Phi\|^2 + \|\nabla\Psi\|^2)$$

$$\leq \frac{d_3}{2}(\|\Delta\Phi\|^2 + \|\Delta\Psi\|^2) + \frac{k^2}{2d_3}(\|U\|^2 + \|W\|^2) + \rho z(\theta_t\omega)(\|\nabla\Phi\|^2 + \|\nabla\Psi\|^2). \tag{4.28}$$

Then, we have

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\Phi\|^2 + \|\nabla\Psi\|^2) + d_3(\|\Delta\Phi\|^2 + \|\Delta\Psi\|^2)\\ \leq &\frac{k^2}{d_3}(\|U\|^2 + \|W\|^2) + 2\rho z(\theta_t\omega)(\|\nabla\Phi\|^2 + \|\nabla\Psi\|^2).\end{aligned} \tag{4.29}$$

Then, we can rewrite (4.29) as the following form

$$\frac{d\beta_3(t)}{dt} \leq \alpha_3(t)\beta_3(t) + \gamma_3(t), \quad t \in [-2, 0], \tag{4.30}$$

where

$$\alpha_3(t) = 2\rho z(\theta_t\omega), \quad \beta_3(t) = \|\nabla\Phi\|^2 + \|\nabla\Psi\|^2, \quad \gamma_3(t) = \frac{k^2}{d_3}(\|U\|^2 + \|W\|^2).$$

For $t_0 \leq T(R, \omega)$ and $-2 \leq t \leq -1$, we infer that

$$\int_t^{t+1} \alpha_3(\tau)d\tau = \int_t^{t+1} 2\rho z(\theta_\tau\omega)d\tau \leq 2\int_{-2}^0 c|z(\theta_\tau\omega)|d\tau, \tag{4.31}$$

then, by using (4.13), we obtain

$$\int_t^{t+1} \beta_3(\tau)d\tau = \int_t^{t+1} (\|\nabla\Phi\|^2 + \|\nabla\Psi\|^2)d\tau \leq K_6(\omega), \tag{4.32}$$

and applying (4.20), we have

$$\int_t^{t+1} \gamma_3(\tau)d\tau = \int_t^{t+1} \frac{k^2}{d_3}(\|U\|^2 + \|W\|^2)d\tau \leq \frac{k^2}{d_3\gamma}K_7(\omega). \tag{4.33}$$

Owing to the uniform Gronwall inequality, which follows that

$$\|\Phi(t, \omega; t_0, g_0)\|_E^2 + \|\Psi(t, \omega; t_0, g_0)\|_E^2 \leq K_9(\omega), \quad t \in [-1, 0],\ t_0 \leq T(R, \omega), \tag{4.34}$$

where

$$K_9(\omega) = \Big(K_6(\omega) + \frac{k^2}{d_3\gamma}K_7(\omega)\Big) \cdot \exp\{2\int_{-2}^0 c|z(\theta_\tau\omega)|\}.$$

Finally, let $t = 0$ in (4.20), (4.27) and (4.34). Therefore, (4.1) holds with $K(\omega) = K_7(\omega) + K_8(\omega) + K_9(\omega)$. In this way, we have completed the proof of Lemma 4.2.

5. The existence of random attractors

In this section, we obtain the existence of a pullback random attractor for the stochastic extended Brusselator system in H.

Theorem 5.1. The extended Brusselator random dynamical system f has a unique pullback random attractor.

Proof. In Lemma 3.2, we obtained that the RDS f has a bounded pullback absorbing set associated with the universe $\mathcal{D}$. In Lemma 4.2, due to the embedding $E \hookrightarrow H$ is a compact mapping, which means that the RDS f is pullback asymptotically compact in H by applying the results of Proposition 1.8.3 in [28]. Then, by Proposition 2.1, the existence of pullback random attractor $\mathcal{A}(\omega) = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is proved for the RDS f, which is given by

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} f(t, \theta_{-t}\omega, B_0(\theta_{-t}\omega))}. \tag{5.1}$$

Therefore, we have completed the proof of Theorem 5.1.

6. Conclusion

In this paper, we prove the existence of random attractors for stochastic dynamics by using the exponential transformation of the Ornstein-Uhlenbeck process to replace the exponential transformation of Brownian motion, which changes the structure of the original Brusselator equations and produces the non-autonomous terms, cf. (2.13)–(2.18). Based on this, we have to overcome the difficulties of coupling structure and make more complex estimates.

If $\rho = 0$, system (1.1)–(1.6) reduces to the extended Brusselator system without random terms, which has been established the global dissipative dynamics by You and Zhou in [15]. Furthermore, Tu and You [16] proved random attractor of stochastic Brusselator system with a multiplicative noise, this paper has included the results of [16] when $w, z, \varphi, \psi = 0$.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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