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Research article

Completely monotonic degree of a function involving trigamma and tetragamma functions

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Abstract: Let $\psi(x)$ be the digamma function. In the paper, the author reviews backgrounds and motivations to compute complete monotonic degree of the function $\Psi(x) = [\psi'(x)]^2+\psi''(x)$ with respect to $x \in (0,\infty)$, confirms that completely monotonic degree of the function $\Psi(x)$ is 4, finds a relation between strongly completely monotonic functions and completely monotonic degrees, provides a proof for the relation between strongly completely monotonic functions and completely monotonic degrees, proves a property of logarithmically concave functions, and poses two open problems on lower bound for convolution of logarithmically concave functions and on completely monotonic degree of a function involving $\Psi(x)$.

Keywords: completely monotonic degree; completely monotonic function; trigamma function; tetragamma function; strongly completely monotonic function; logarithmically concave function; convolution; open problem
Mathematics Subject Classification: Primary: 33B15; Secondary: 26A12, 26A48, 26A51, 42A85, 44A10, 44A35

1. Preliminaries

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $0 \le (-1)^{k-1} f^{(k-1)}(x) < \infty$ for $x \in I$ and $k \in \mathbb{N}$, where $f^{(0)}(x)$ means $f(x)$ and $\mathbb{N}$ is the set of all positive integers. See [1–3]. Theorem 12b in [3] states that a necessary and sufficient condition for a function f to be completely monotonic on the infinite interval $(0,\infty)$ is that the integral $f(t) = \int_0^\infty e^{-ts}\,\mathrm{d}\,\tau(s)$ converges for $s \in (0,\infty)$, where $\tau(s)$ is nondecreasing on $(0,\infty)$. In other words, a function is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform of a nonnegative measure. This is one of many reasons why many mathematicians have been investigating completely monotonic functions for many decades.

Definition 1.1 ([4–9])**.** *Let $f(x)$ be a completely monotonic function on $(0,\infty)$ and denote $f(\infty) = \lim_{x\to\infty} f(x)$. If for some $r \in \mathbb{R}$ the function $x^r[f(x) - f(\infty)]$ is completely monotonic on $(0,\infty)$ but $x^{r+\varepsilon}[f(x) - f(\infty)]$ is not for any positive number $\varepsilon > 0$, then we say that the number r is completely monotonic degree of $f(x)$ with respect to $x \in (0,\infty)$; if for all $r \in \mathbb{R}$ each and every $x^r[f(x) - f(\infty)]$ is completely monotonic on $(0,\infty)$, then we say that completely monotonic degree of $f(x)$ with respect to $x \in (0,\infty)$ is ∞.*

The notation $\deg_{\text{cm}}^{x}[f(x)]$ has been designed in [4] to denote completely monotonic degree r of $f(x)$ with respect to $x \in (0,\infty)$. It is clear that completely monotonic degree $\deg_{\text{cm}}^{x}[f(x)]$ of any completely monotonic function $f(x)$ with respect to $x \in (0,\infty)$ is at leat 0. It was proved in [6] that completely monotonic degree $\deg_{\text{cm}}^{x}[f(x)]$ equals ∞ if and only if $f(x)$ is nonnegative and identically constant. This definition slightly modifies the corresponding one stated in [4] and related references therein. For simplicity, in what follows, we sometimes just say that $\deg_{\text{cm}}^{x}[f(x)]$ is completely monotonic degree of $f(x)$.

Why do we compute completely monotonic degrees? One can find simple but significant reasons in the second paragraph of [7] or in the papers [10–13] and closely related references therein. Completely monotonic degree is a new notion introduced in very recent years. See [4,6,9,11,12,14–22] and closely related references. This new notion can be used to more accurately measure and differentiate complete monotonicity. For example, the functions $\frac{1}{x^\alpha}$ and $\frac{1}{x^\beta}$ for $\alpha,\beta > 0$ and $\alpha \ne \beta$ are both completely monotonic on $(0,\infty)$, but they are different completely monotonic functions. How to quantitatively measure their differences? How to quantitatively differentiate them from each other? The notion of completely monotonic degrees can be put to good use: The completely monotonic degrees of $\frac{1}{x^\alpha}$ and $\frac{1}{x^\beta}$ with respect to $x \in (0,\infty)$ for $\alpha,\beta > 0$ and $\alpha \ne \beta$ are α and β respectively.

The classical Euler's gamma function $\Gamma(x)$ can be defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}\,t$. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, the derivatives $\psi'(x)$ and $\psi''(x)$ are respectively called the tri- and tetragamma functions. As a whole, the derivatives $\psi^{(k)}(x)$ for $k \ge 0$ are called polygamma functions. For new results on $\Gamma(z)$ and $\psi^{(k)}(x)$ in recent years, please refer to [7, 11, 23–29] and closely related references therein.

Why do we still study the gamma and polygamma functions $\Gamma(z)$ and $\psi^{(k)}(z)$ for $k \ge 0$ nowadays? Because this kind of functions are not elementary and are the most applicable functions in almost all aspects of mathematics and mathematical sciences.

2. Backgrounds and motivations

Let

$$\Psi(x) = [\psi'(x)]^2 + \psi''(x), \quad x \in (0,\infty). \tag{2.1}$$

In [30], it was established that the inequality

$$\Psi(x) > \frac{p(x)}{900x^4(x+1)^{10}} \tag{2.2}$$

holds for $x > 0$, where

$$p(x) = 75x^{10} + 900x^9 + 4840x^8 + 15370x^7 + 31865x^6 + 45050x^5 + 44101x^4 + 29700x^3 + 13290x^2 + 3600x + 450.$$

It is clear that the inequality

$$\Psi(x) > 0 \tag{2.3}$$

for $x > 0$ is a weakened version of the inequality (2.2). This inequality was deduced and recovered in [31, 32]. The inequality (2.3) was also employed in [31–34]. This inequality has been generalized in [33, 35–37]. For more information about the history and background of this topic, please refer to the expository and survey articles [11, 38–41] and plenty of references therein.

In the paper [42], it was proved that, among all functions $[\psi^{(m)}(x)]^2 + \psi^{(n)}(x)$ for $m, n \in \mathbb{N}$, only the function $\Psi(x)$ is nontrivially completely monotonic on $(0, \infty)$.

In [43, 44], the functions

$$\frac{x+12}{12x^4(x+1)} - \Psi(x), \quad \Psi(x) - \frac{x^2+12}{12x^4(x+1)^2}, \quad \Psi(x) - \frac{p(x)}{900x^4(x+1)^{10}}$$

were proved to be completely monotonic on $(0, \infty)$. From this, we obtain

$$\max\left\{\frac{x^2+12}{12x^4(x+1)^2}, \frac{p(x)}{900x^4(x+1)^{10}}\right\} < \Psi(x) < \frac{x+12}{12x^4(x+1)} \tag{2.4}$$

for $x > 0$. In [45], the function

$$h_\lambda(x) = \Psi(x) - \frac{x^2+\lambda x+12}{12x^4(x+1)^2} \tag{2.5}$$

was proved to be completely monotonic on $(0, \infty)$ if and only if $\lambda \le 0$, and so is $-h_\lambda(x)$ if and only if $\lambda \ge 4$; Consequently, the double inequality

$$\frac{x^2+\mu x+12}{12x^4(x+1)^2} < \Psi(x) < \frac{x^2+\nu x+12}{12x^4(x+1)^2} \tag{2.6}$$

holds on $(0, \infty)$ if and only if $\mu \le 0$ and $\nu \ge 4$. The inequality (2.6) refines and sharpens the right-hand side inequality in (2.4).

It was remarked in [40] that a divided difference version of the inequality (2.3) has been implicitly obtained in [46]. The divided difference form of the function $\Psi(x)$ and related functions have been investigated in the papers [47–51] and closely related references therein. There is a much complete list of references in [52].

In [14, 16], among other things, it was deduced that the functions $x^2\Psi(x)$ and $x^3\Psi(x)$ are completely monotonic on $(0, \infty)$. Equivalently,

$$\deg_{\mathrm{cm}}^{x}[\Psi(x)] \ge 2 \quad \text{and} \quad \deg_{\mathrm{cm}}^{x}[\Psi(x)] \ge 3. \tag{2.7}$$

Motivated by these results, we naturally pose the following two questions:

1. is the function $x^4\Psi(x)$ completely monotonic on $(0, \infty)$?
2. is $\alpha \le 4$ the necessary and sufficient condition for the function $x^\alpha\Psi(x)$ to be completely monotonic on $(0, \infty)$?

In other words, is the constant 4 completely monotonic degree of $\Psi(x)$ with respect to $x \in (0, \infty)$?

3. Five lemmas

In order to affirmatively and smoothly answer the above questions, we need five lemmas below.

Lemma 3.1 ([29]). *For $n \in \mathbb{N}$ and $x > 0$,*

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1-e^{-t}} e^{-xt} \,\mathrm{d}\, t. \tag{3.1}$$

Lemma 3.2 ([3,29]). *Let $f_i(t)$ for $i = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$ and suppose that there exist some constants $M_i > 0$ and $c_i \ge 0$ such that $|f_i(t)| \le M_i e^{c_i t}$ for $i = 1, 2$. Then*

$$\int_0^\infty \biggl[\int_0^t f_1(u) f_2(t-u) \,\mathrm{d}\, u\biggr] e^{-st} \,\mathrm{d}\, t = \int_0^\infty f_1(u) e^{-su} \,\mathrm{d}\, u \int_0^\infty f_2(v) e^{-sv} \,\mathrm{d}\, v. \tag{3.2}$$

Lemma 3.3 ([53]). *Let $f(x,t)$ is differentiable in t and continuous for $(x, t) \in \mathbb{R}^2$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \int_{x_0}^t f(x,t) \,\mathrm{d}\, x = f(t,t) + \int_{x_0}^t \frac{\partial f(x,t)}{\partial t} \,\mathrm{d}\, x.$$

Lemma 3.4 ([54–56]). *If f_i for $1 \le i \le n$ are nonnegative Lebesgue square integrable functions on $[0, a)$ for all $a > 0$, then*

$$f_1 * \cdots * f_n(x) \ge \frac{x^{n-1}}{(n-1)!} \exp\Biggl[\frac{n-1}{x^{n-1}} \int_0^x (x-u)^{n-2} \sum_{j=1}^n \ln f_j(u) \,\mathrm{d}\, u\Biggr] \tag{3.3}$$

for all $n \ge 2$ and $x \ge 0$, where $f_i * f_j(x)$ denotes the convolution $\int_0^x f_i(t) f_j(x-t) \,\mathrm{d}\, t$.

Lemma 3.5 ([29]). *As $z \to \infty$ in $|\arg z| < \pi$,*

$$\begin{aligned}
\psi'(z) &\sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} + \frac{1}{42z^7} - \frac{1}{30z^9} + \cdots, \\
\psi''(z) &\sim -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^4} + \frac{1}{6z^6} - \frac{1}{6z^8} + \frac{3}{10z^{10}} - \frac{5}{6z^{12}} + \cdots, \\
\psi^{(3)}(z) &\sim \frac{2}{z^3} + \frac{3}{z^4} + \frac{2}{z^5} - \frac{1}{z^7} + \frac{4}{3z^9} - \frac{3}{z^{11}} + \frac{10}{z^{13}} - \cdots.
\end{aligned}$$

The formulas listed in Lemma 3.5 are special cases of [29].

4. Completely monotonic degree of $\Psi(x)$ with respect to $x \in (0, \infty)$ is 4

Now we are in a position to compute completely monotonic degree of the function $\Psi(x)$.

Theorem 4.1. *Completely monotonic degree of $\Psi(x)$ defined by (2.1) with respect to $x \in (0, \infty)$ is 4. In other words,*

$$\deg_{\mathrm{cm}}^{x}[\Psi(x)] = 4. \tag{4.1}$$

Proof. Using the integral representation (3.1) and the formula (3.2) gives

$$\begin{aligned}\Psi(x) &= \left[\int_0^\infty \frac{t}{1-e^{-t}}e^{-xt}\,\mathrm{d}\,t\right]^2 - \int_0^\infty \frac{t^2}{1-e^{-t}}e^{-xt}\,\mathrm{d}\,t\\ &= \int_0^\infty\left[\int_0^t \frac{s(t-s)}{(1-e^{-s})[1-e^{-(t-s)}]}\,\mathrm{d}\,s - \frac{t^2}{1-e^{-t}}\right]e^{-xt}\,\mathrm{d}\,t\\ &= \int_0^\infty q(t)e^{-xt}\,\mathrm{d}\,t,\end{aligned}$$

where

$$q(t) = \int_0^t \sigma(s)\sigma(t-s)\,\mathrm{d}\,s - t\sigma(t) \quad\text{and}\quad \sigma(s) = \begin{cases}\dfrac{s}{1-e^{-s}}, & s\ne 0\\ 1, & s=0.\end{cases} \tag{4.2}$$

Direct calculations reveal

$$\begin{aligned}\sigma'(s) &= 1 + \frac{1-s}{e^s-1} - \frac{s}{(e^s-1)^2},\\ \sigma''(s) &= \frac{s-2}{e^s-1} + \frac{3s-2}{(e^s-1)^2} + \frac{2s}{(e^s-1)^3},\\ \sigma^{(3)}(s) &= \frac{3-s}{e^s-1} + \frac{9-7s}{(e^s-1)^2} - \frac{6(2s-1)}{(e^s-1)^3} - \frac{6s}{(e^s-1)^4},\\ \sigma^{(4)}(s) &= \frac{s-4}{e^s-1} + \frac{15s-28}{(e^s-1)^2} + \frac{2(25s-24)}{(e^s-1)^3} + \frac{12(5s-2)}{(e^s-1)^4} + \frac{24s}{(e^s-1)^5},\\ \sigma^{(5)}(s) &= \frac{5-s}{e^s-1} + \frac{75-31s}{(e^s-1)^2} - \frac{10(18s-25)}{(e^s-1)^3} - \frac{30(13s-10)}{(e^s-1)^4} - \frac{120(3s-1)}{(e^s-1)^5} - \frac{120s}{(e^s-1)^6},\\ \sigma^{(6)}(s) &= \frac{s-6}{e^s-1} + \frac{3(21s-62)}{(e^s-1)^2} + \frac{2(301s-540)}{(e^s-1)^3} + \frac{60(35s-39)}{(e^s-1)^4}\\ &\quad + \frac{240(14s-9)}{(e^s-1)^5} + \frac{360(7s-2)}{(e^s-1)^6} + \frac{720s}{(e^s-1)^7},\end{aligned}$$

and

$$\begin{gathered}\sigma(0)=1,\quad \sigma'(0)=\frac{1}{2},\quad \sigma''(0)=\frac{1}{6},\quad \sigma^{(3)}(0)=0,\\ \sigma^{(4)}(0) = -\frac{1}{30},\quad \sigma^{(5)}(0)=0,\quad \sigma^{(6)}(0)=\frac{1}{42}.\end{gathered}$$

Further differentiating consecutively brings out

$$\begin{aligned}[\ln\sigma''(s)]' &= -\frac{(s-3)e^{2s}+4se^s+s+3}{[(s-2)e^s+s+2](e^s-1)},\\ [\ln\sigma''(s)]'' &= -\frac{e^{4s}-4(s^2-3s+4)e^{3s}-(4s^2-30)e^{2s}-4(s^2+3s+4)e^s+1}{(e^s-1)^2[(s-2)e^s+s+2]^2}\\ &\triangleq -\frac{h_1(s)}{(e^s-1)^2[(s-2)e^s+s+2]^2},\\ h_1'(s) &= 4[e^{3s}-(3s^2-7s+9)e^{2s}-(2s^2+2s-15)e^s-s^2-5s-7]e^s\end{aligned}$$

$$
\begin{aligned}
&\triangleq 4h_2(s)e^s,\\
h_2'(s) &= 3e^{3s} - (6s^2 - 8s + 11)e^{2s} - (2s^2 + 6s - 13)e^s - 2s - 5,\\
h_2''(s) &= 9e^{3s} - 2(6s^2 - 2s + 7)e^{2s} - (2s^2 + 10s - 7)e^s - 2,\\
h_2^{(3)}(s) &= [27e^{2s} - 8e^s(3s^2 + 2s + 3) - 2s^2 - 14s - 3]e^s\\
&\triangleq h_3(s)e^s,\\
h_3'(s) &= 54e^{2s} - 8(3s^2 + 8s + 5)e^s - 2(2s + 7),\\
h_3''(s) &= 4[27e^{2s} - 2(3s^2 + 14s + 13)e^s - 1],\\
h_3^{(3)}(s) &= 8(27e^s - 3s^2 - 20s - 27)e^s\\
&> 0
\end{aligned}
$$

for $s \in (0, \infty)$, and

$$
h_3''(0) = h_3'(0) = h_3(0) = h_2^{(3)}(0) = h_2''(0) = h_2'(0) = h_2(0) = h_1'(0) = h_1(0) = 0.
$$

This means that

$$
\begin{gathered}
h_3''(s) > 0, \quad h_3'(s) > 0, \quad h_3(s) > 0, \quad h_2^{(3)}(s) > 0,\\
h_2''(s) > 0, \quad h_2'(s) > 0, \quad h_2(s) > 0, \quad h_1'(s) > 0, \quad h_1(s) > 0
\end{gathered}
$$

for $s \in (0, \infty)$. Therefore, the derivative $[\ln \sigma''(s)]''$ is negative, that is, the function $\sigma''(s)$ is logarithmically concave, on $(0, \infty)$. Hence, for any given number $t > 0$,

1. the function $\sigma''(s)\sigma''(t - s)$ is also logarithmically concave with respect to $s \in (0, t)$;
2. the function $\sigma''(s)$ is decreasing and $\sigma(s)$ is not concave on $(0, \infty)$.

By Lemma 3.3 and integration-by-part, straightforward computations yield

$$
\begin{aligned}
q'(t) &= \int_0^t \sigma(s)\sigma'(t - s)\,\mathrm{d}\,s + \sigma(0)\sigma(t) - [t\sigma'(t) + \sigma(t)]\\
&= \int_0^t \sigma(s)\sigma'(t - s)\,\mathrm{d}\,s - t\sigma'(t),\\
q''(t) &= \int_0^t \sigma(s)\sigma''(t - s)\,\mathrm{d}\,s + \sigma(t)\sigma'(0) - [\sigma'(t) + t\sigma''(t)]\\
&= -\int_0^t \sigma(s)\frac{\mathrm{d}\,\sigma'(t - s)}{\mathrm{d}\,s}\,\mathrm{d}\,s + \sigma(t)\sigma'(0) - [\sigma'(t) + t\sigma''(t)]\\
&= \int_0^t \sigma'(s)\sigma'(t - s)\,\mathrm{d}\,s - t\sigma''(t),\\
q^{(3)}(t) &= \int_0^t \sigma'(s)\sigma''(t - s)\,\mathrm{d}\,s + \frac{1}{2}\sigma'(t) - \sigma''(t) - t\sigma^{(3)}(t),\\
q^{(4)}(t) &= \int_0^t \sigma'(s)\sigma^{(3)}(t - s)\,\mathrm{d}\,s + \frac{1}{6}\sigma'(t) + \frac{1}{2}\sigma''(t) - 2\sigma^{(3)}(t) - t\sigma^{(4)}(t)\\
&= -\int_0^t \sigma'(s)\frac{\mathrm{d}\,\sigma''(t - s)}{\mathrm{d}\,s}\,\mathrm{d}\,s + \frac{1}{6}\sigma'(t) + \frac{1}{2}\sigma''(t) - 2\sigma^{(3)}(t) - t\sigma^{(4)}(t)
\end{aligned}
$$

$$= \int_0^t \sigma''(s)\sigma''(t-s)\,\mathrm{d}\,s + \sigma''(t) - 2\sigma^{(3)}(t) - t\sigma^{(4)}(t)$$
$$= 2\int_0^{t/2} \sigma''(s)\sigma''(t-s)\,\mathrm{d}\,s + \sigma''(t) - 2\sigma^{(3)}(t) - t\sigma^{(4)}(t),$$

and

$$q(0) = q'(0) = q''(0) = 0, \quad q^{(3)}(0) = \frac{1}{12}, \quad q^{(4)}(0) = \frac{1}{6}.$$

Applying Lemma 3.4 to $f_1 = f_2 = \sigma''$ and $n = 2$ leads to

$$\int_0^t \sigma''(s)\sigma''(t-s)\,\mathrm{d}\,s \ge t\exp\left[\frac{2}{t}\int_0^t \ln\sigma''(u)\,\mathrm{d}\,u\right].$$

Hence, the validity of the inequality

$$t\exp\left[\frac{2}{t}\int_0^t \ln\sigma''(u)\,\mathrm{d}\,u\right] + \sigma''(t) - 2\sigma^{(3)}(t) - t\sigma^{(4)}(t) > 0 \tag{4.3}$$

implies the positivity of $q^{(4)}(t)$ on $(0,\infty)$.

When $t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t) \le 0$, the inequality (4.3) is clearly valid.

When $t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t) > 0$, the inequality (4.3) can be rearranged as

$$\int_0^t \ln\sigma''(u)\,\mathrm{d}\,u > \frac{t}{2}\ln\frac{t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)}{t}.$$

Let

$$F(t) = \int_0^t \ln\sigma''(u)\,\mathrm{d}\,u - \frac{t}{2}\ln\frac{t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)}{t}.$$

Differentiating twice produces

$$F'(t) = \ln\sigma''(t) - \frac{1}{2}\ln\frac{t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)}{t} - \frac{t^2\sigma^{(5)}(t) + 2t\sigma^{(4)}(t) - (t+2)\sigma^{(3)}(t) + \sigma''(t)}{2[t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)]}$$

and

$$\begin{aligned} F''(t) &= \frac{\sigma^{(3)}(t)}{\sigma''(t)} - \frac{t^2\sigma^{(5)}(t) + 2t\sigma^{(4)}(t) - (t+2)\sigma^{(3)}(t) + \sigma''(t)}{2t[t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)]} \\ &\quad - \frac{1}{2[t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)]^2}\begin{pmatrix} [t^2\sigma^{(6)}(t) + 4t\sigma^{(5)}(t) - t\sigma^{(4)}(t)][t\sigma^{(4)}(t) \\ +2\sigma^{(3)}(t) - \sigma''(t)] - [t^2\sigma^{(5)}(t) \\ +2t\sigma^{(4)}(t) - (t+2)\sigma^{(3)}(t) + \sigma''(t)] \\ \times[t\sigma^{(5)}(t) + 3\sigma^{(4)}(t) - \sigma^{(3)}(t)] \end{pmatrix} \\ &\triangleq \frac{Q(t)}{2t\sigma''(t)[t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)]^2}, \end{aligned}$$

where

$$Q(t) = 2t\sigma^{(3)}(t)[t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)]^2 - \sigma''(t)[t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)][t^2\sigma^{(5)}(t) + 2t\sigma^{(4)}(t)$$

$$
\begin{aligned}
&- (t+2)\sigma^{(3)}(t) + \sigma''(t)] - t\sigma''(t)\Big\{[t^2\sigma^{(6)}(t) + 4t\sigma^{(5)}(t) - t\sigma^{(4)}(t)][t\sigma^{(4)}(t) + 2\sigma^{(3)}(t) - \sigma''(t)]\\
&- [t^2\sigma^{(5)}(t) + 2t\sigma^{(4)}(t) - (t+2)\sigma^{(3)}(t) + \sigma''(t)][t\sigma^{(5)}(t) + 3\sigma^{(4)}(t) - \sigma^{(3)}(t)]\Big\}\\
\triangleq\ & \frac{e^{3t}R(t)}{(e^t-1)^{15}}
\end{aligned}
$$

and

$$
\begin{aligned}
R(t) = {}& e^{9t}(t^5 - 12t^4 + 70t^3 - 160t^2 + 192t - 128) - e^{8t}(16t^7 - 220t^6 + 1219t^5 - 3220t^4\\
&+ 4490t^3 - 3248t^2 + 1152t - 768) - 4e^{7t}(37t^7 - 423t^6 + 1397t^5 - 1409t^4\\
&- 1020t^3 + 2632t^2 - 732t + 456) - 4e^{6t}(225t^7 - 1281t^6 + 1213t^5 + 3127t^4\\
&- 4372t^3 - 2648t^2 + 1020t - 504) - 2e^{5t}(908t^7 - 1514t^6 - 6493t^5 + 8710t^4\\
&+ 12754t^3 - 1216t^2 - 1656t + 336) - 2e^{4t}(908t^7 + 1710t^6 - 5489t^5 - 12370t^4\\
&+ 594t^3 + 4880t^2 + 696t + 336) - 4e^{3t}(225t^7 + 1263t^6 + 1771t^5 - 887t^4 - 3208t^3\\
&- 728t^2 + 12t - 168) - 4e^{2t}(37t^7 + 353t^6 + 1099t^5 + 1337t^4 + 272t^3 - 632t^2\\
&- 108t + 24) - e^{t}(16t^7 + 180t^6 + 827t^5 + 1864t^4 + 2226t^3 + 1312t^2 + 240t + 96)\\
&+ t^5 + 8t^4 + 30t^3 + 48t^2 + 48t + 32.
\end{aligned}
$$

Differentiating and taking the limit $t \to 0$ about 76 times respectively by the same approach as either the proof of the positivity of $\theta(t)$ in [43], or proofs of the absolute monotonicity of the functions f_1, f_2, f_3 and h_1, h_2, h_3, h_4 in [57], or the proof of the positivity of $h_1(s)$ on page 3396 in this paper, we can verify the positivity of $R(t)$ on $(0,\infty)$. In [58], a stronger conclusion than the positivity of $R(t)$ on $(0,\infty)$ was proved in details. This means that $Q(t) > 0$ on $(0,\infty)$ and $F''(t) > 0$. Accordingly, the derivative $F'(t)$ is strictly increasing. Because

$$
\begin{aligned}
F'(8) &= 4 + \frac{3(6e^{32} + 729e^{24} + 2825e^{16} + 1483e^{8} + 77)}{8e^{32} + 270e^{24} + 150e^{16} - 374e^{8} - 54}\\
&\quad + \frac{1}{2}\ln\frac{8(5+3e^8)}{(e^8-1)(27 + 214e^8 + 139e^{16} + 4e^{24})}\\
&= -0.24428\dots
\end{aligned}
$$

and

$$
\begin{aligned}
F'(10) &= 5 + \frac{72e^{40} + 4715e^{30} + 16563e^{20} + 8241e^{10} + 409}{19e^{40} + 440e^{30} + 186e^{20} - 568e^{10} - 77}\\
&\quad + \frac{1}{2}\ln\frac{80(3+2e^{10})^2}{(e^{10}-1)(77 + 645e^{10} + 459e^{20} + 19e^{30})}\\
&= 0.20823\dots,
\end{aligned}
$$

which are numerically calculated with the help of the software Mathematica, the unique zero of $F'(t)$ locates on the open interval $(8,10)$. Consequently, the unique minimum of the function $F(t)$ attains on the interval $(8,10)$. Since

$$
F(t) = F(t_0) + (t-t_0)F'(t_0) + \frac{(t-t_0)^2}{2}F''(\xi) > F(t_0) + (t-t_0)F'(t_0)
$$

for $t, t_0 \in [8, 10]$, where ξ locates between t_0 and t, numerically calculating with the help of the software Mathematica gains

$$
\begin{aligned}
2F(t) &> [F(8) + (t-8)F'(8)] + [F(10) + (t-10)F'(10)] \\
&= F(8) + F(10) - [8F'(8) + 10F'(10)] + [F'(8) + F'(10)]t \\
&> \int_0^8 \ln\sigma''(u)\,\mathrm{d}\,u - 4\ln\frac{e^8(27 + 214e^8 + 139e^{16} + 4e^{24})}{2(e^8-1)^5} + \int_0^{10} \ln\sigma''(u)\,\mathrm{d}\,u \\
&\quad - 5\ln\frac{e^{10}(77 + 645e^{10} + 459e^{20} + 19e^{30})}{5(e^{10}-1)^5} - 0.1281 - 0.0361t \\
&> \int_0^8 \ln\sigma''(u)\,\mathrm{d}\,u + \int_0^{10} \ln\sigma''(u)\,\mathrm{d}\,u + 72.492 - 0.1281 - 0.361 \\
&> \int_0^8 \ln\sigma''(u)\,\mathrm{d}\,u + \int_0^{10} \ln\sigma''(u)\,\mathrm{d}\,u + 72 \\
&> \frac{1}{3}\left[\sum_{k=1}^{24} \ln\sigma''\left(\frac{k}{3}\right) + \sum_{k=1}^{30} \ln\sigma''\left(\frac{k}{3}\right)\right] + 72 \\
&> -29 - 43 + 72 \\
&= 0
\end{aligned}
$$

on the interval $[8, 10]$. In conclusion, the inequality (4.3) is valid and the fourth derivative $q^{(4)}(t)$ is positive on $(0, \infty)$.

Integrating by parts successively results in

$$
\begin{aligned}
x^4\Psi(x) &= x^4\int_0^\infty q(t)e^{-xt}\,\mathrm{d}\,t = -x^3\int_0^\infty q(t)\frac{\mathrm{d}\,e^{-xt}}{\mathrm{d}\,t}\,\mathrm{d}\,t = -x^3\left[q(t)e^{-xt}\Big|_{t=0}^{t=\infty} - \int_0^\infty q'(t)e^{-xt}\,\mathrm{d}\,t\right] \\
&= x^3\int_0^\infty q'(t)e^{-xt}\,\mathrm{d}\,t = x^2\int_0^\infty q''(t)e^{-xt}\,\mathrm{d}\,t = x\int_0^\infty q^{(3)}(t)e^{-xt}\,\mathrm{d}\,t = -\int_0^\infty q^{(3)}(t)\frac{\mathrm{d}\,e^{-xt}}{\mathrm{d}\,t}\,\mathrm{d}\,t \\
&= -\left[q^{(3)}(t)e^{-xt}\Big|_{t=0}^{t=\infty} - \int_0^\infty q^{(4)}(t)\frac{\mathrm{d}\,e^{-xt}}{\mathrm{d}\,t}\,\mathrm{d}\,t\right] = \frac{1}{12} + \int_0^\infty q^{(4)}(t)e^{-xt}\,\mathrm{d}\,t.
\end{aligned}
$$

From the positivity of $q^{(4)}(t)$ on $(0, \infty)$, it follows that the function $x^4\Psi(x)$ is completely monotonic on $(0, \infty)$. In other words,

$$
\deg_{\mathrm{cm}}^{x}[\Psi(x)] \ge 4. \tag{4.4}
$$

Suppose that the function

$$
f_\alpha(x) = x^\alpha\Psi(x)
$$

is completely monotonic on $(0, \infty)$. Then

$$
f'_\alpha(x) = x^{\alpha-1}\{\alpha\Psi(x) + x[2\psi'(x)\psi''(x) + \psi^{(3)}(x)]\} \le 0
$$

on $(0, \infty)$, that is,

$$
\alpha \le -\frac{x[2\psi'(x)\psi''(x) + \psi^{(3)}(x)]}{\Psi}(x) \triangleq \phi(x), \quad x > 0.
$$

From Lemma 3.5, it follows

$$\lim_{x\to\infty}\phi(x) = -\lim_{x\to\infty}\left\{\frac{x}{[\frac{1}{x}+\frac{1}{2x^2}+O(\frac{1}{x^2})]^2+[-\frac{1}{x^2}-\frac{1}{x^3}+O(\frac{1}{x^3})]}\right.$$
$$\left.\times\left[2\left(\frac{1}{x}+\frac{1}{2x^2}+O\left(\frac{1}{x^2}\right)\right)\left(-\frac{1}{x^2}-\frac{1}{x^3}+O\left(\frac{1}{x^3}\right)\right)+\left(\frac{2}{x^3}+\frac{3}{x^4}+O\left(\frac{1}{x^4}\right)\right)\right]\right\}$$
$$= 4.$$

As a result, we have

$$\deg_{\mathrm{cm}}^{x}[\Psi(x)] \le 4. \tag{4.5}$$

Combining (4.4) with (4.5) yields (4.1). The proof of Theorem 4.1 is complete. □

5. Strongly completely monotonic functions and completely monotonic degree

Recall from [59] that a function f is said to be strongly completely monotonic on $(0,\infty)$ if it has derivatives of all orders and $(-1)^n x^{n+1} f^{(n)}(x)$ is nonnegative and decreasing on $(0,\infty)$ for all $n \ge 0$.

Theorem 5.1 ([18])**.** *A function $f(x)$ is strongly completely monotonic on $(0,\infty)$ if and only if the function $xf(x)$ is completely monotonic on $(0,\infty)$.*

This theorem implies that the set of completely monotonic functions whose completely monotonic degrees are not less than 1 with respect to $x \in (0,\infty)$ coincides with the set of strongly completely monotonic functions on $(0,\infty)$.

Because not finding a proof for [18] anywhere, we now provide a proof for Theorem 5.1 as follows.

Proof of Theorem 5.1. If $xf(x)$ is completely monotonic on $(0,\infty)$, then

$$(-1)^k[xf(x)]^{(k)} = (-1)^k[xf^{(k)}(x)+kf^{(k-1)}(x)] = \frac{(-1)^k x^{k+1} f^{(k)}(x) - k[(-1)^{k-1}x^k f^{(k-1)}(x)]}{x^k} \ge 0$$

on $(0,\infty)$ for all integers $k \ge 0$. From this and by induction, we obtain

$$(-1)^k x^{k+1} f^{(k)}(x) \ge k[(-1)^{k-1}x^k f^{(k-1)}(x)] \ge k(k-1)[(-1)^{k-2}x^{k-1} f^{(k-2)}(x)] \ge \cdots$$
$$\ge [k(k-1)\cdots 4\cdot 3]x^3 f''(x) \ge [k(k-1)\cdots 4\cdot 3\cdot 2]x^2 f'(x) \ge k!xf(x) \ge 0$$

on $(0,\infty)$ for all integers $k \ge 0$. So, the function $f(x)$ is strongly completely monotonic on $(0,\infty)$.

Conversely, if $f(x)$ is a strongly completely monotonic function on $(0,\infty)$, then

$$(-1)^k x^{k+1} f^{(k)}(x) \ge 0$$

and

$$[(-1)^k x^{k+1} f^{(k)}(x)]' = \frac{(k+1)[(-1)^k x^{k+1} f^{(k)}(x)] - (-1)^{k+1} x^{k+2} f^{(k+1)}(x)}{x} \le 0$$

hold on $(0,\infty)$ for all integers $k \ge 0$. Hence, it follows that $xf(x) \ge 0$ and $(-1)^{k+1}[xf(x)]^{(k+1)}$ on $(0,\infty)$ for all integers $k \ge 0$. As a result, the function $xf(x)$ is completely monotonic on $(0,\infty)$. The proof of Theorem 5.1 is complete. □

6. A property of logarithmically concave functions

Now we prove a property of logarithmically concave functions.

Theorem 6.1. *If $f(x)$ is differentiable and logarithmically concave (or logarithmically convex, respectively) on $(-\infty, \infty)$, then the product $f(x)f(\lambda - x)$ for any fixed number $\lambda \in \mathbb{R}$ is increasing (or decreasing, respectively) with respect to $x \in (-\infty, \frac{\lambda}{2})$ and decreasing (or increasing, respectively) with respect to $x \in (\frac{\lambda}{2}, \infty)$.*

Proof. Taking the logarithm of $f(x)f(\lambda - x)$ and differentiating give

$$\{\ln[f(x)f(\lambda - x)]\}' = \frac{f'(x)}{f(x)} - \frac{f'(\lambda - x)}{f(\lambda - x)}.$$

In virtue of the logarithmic concavity of $f(x)$, it follows that the function $\frac{f'(x)}{f(x)}$ is decreasing and $\frac{f'(\lambda-x)}{f(\lambda-x)}$ is increasing on $(-\infty, \infty)$. From the obvious fact that $\{\ln[f(x)f(\lambda - x)]\}'|_{x=\lambda/2} = 0$, it is deduced that $\{\ln[f(x)f(\lambda - x)]\}' < 0$ for $x > \frac{\lambda}{2}$ and $\{\ln[f(x)f(\lambda - x)]\}' > 0$ for $x < \frac{\lambda}{2}$. Hence, the function $f(x)f(\lambda - x)$ is decreasing for $x > \frac{\lambda}{2}$ and increasing for $x < \frac{\lambda}{2}$.

For the case of $f(x)$ being logarithmically convex, it can be proved similarly. □

7. Remarks and two open problems

In this section, we list several remarks on our main results and pose two open prblems.

Remark 7.1. The function $\sigma(s)$ defined in (4.2) is a special case of the function

$$g_{a,b}(s) = \begin{cases} \dfrac{s}{b^s - a^s}, & s \neq 0, \\ \dfrac{1}{\ln b - \ln a}, & s = 0, \end{cases}$$

where a, b are positive numbers and $a \neq b$. Some special cases of the function $g_{a,b}(s)$ and their reciprocals have been investigated and applied in many papers such as [6, 8, 60–75]. This subject was also surveyed in [76]. Recently, it was discovered that the derivatives of the function $\frac{\sigma(s)}{s} = \frac{1}{1-e^{-s}}$ have something to do with the Stirling numbers of the first and second kinds in combinatorics and number theory. For detailed and more information, please refer to [77–89].

By Theorem 6.1, it can be deduced that the function $\sigma''(s)\sigma''(t - s)$ is increasing with respect to $s \in (0, \frac{t}{2})$ and decreasing with respect to $s \in (\frac{t}{2}, t)$, where σ is defined in (4.2).

The techniques used in the proof of Theorem 6.1 was ever utilized in the papers [70, 90–92] and closely related references therein.

Remark 7.2. The result obtained in Theorem 4.1 in this paper affirmatively answers those questions asked on page 3393 at the end of Section 2. Therefore, the result in Theorem 4.1 strengthens, improves, and sharpens those results in (2.7). This implies that other results established in [14, 16] can also be further improved, developed, or amended.

Remark 7.3 (First open problem). Motivated by Lemma 3.4, the proof of Theorem 4.1, and Theorem 6.1, we pose the following open problem: when f_i for $1 \leq i \leq n$ are all logarithmically concave on $[0, a)$ for all $a > 0$, can one find a stronger lower bound than the one in (3.3) for the convolution $f_1 * f_2 * \cdots * f_n(x)$?

Remark 7.4 (Second open problem). We conjecture that the completely monotonic degrees with respect to $x \in (0, \infty)$ of the functions $h_\lambda(x)$ and $-h_\mu(x)$ defined by (2.5) are 4 if and only if $\lambda \le 0$ and $\mu \ge 4$. In other words,

$$\deg_{\mathrm{cm}}^{x}[h_\lambda(x)] = \deg_{\mathrm{cm}}^{x}[-h_\mu(x)] = 4$$

if and only if $\lambda \le 0$ and $\mu \ge 4$.

Remark 7.5. This paper is a revised and shortened version of the preprint [93].

8. Conclusions

In ths paper, the author proved that the completely monotonic degree of the function $[\psi'(x)]^2+\psi''(x)$ with respect to $x \in (0, \infty)$ is 4, verified that the set of all strongly completely monotonic functions on $(0, \infty)$ coincides with the set of functions whose completely monotonic degrees are greater than or equal to 1 on $(0, \infty)$, presented a property of logarithmically concave functions, and posed two open problems on a stronger lower bound of the convolution of finite many functions and on completely monotonic degree of a kind of completely monotonic functions on $(0, \infty)$.

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Conflict of interest

The author declares that he have no conflict of interest.

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