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Research article

Nonlocal problems of fractional systems involving left and right fractional derivatives at resonance

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Abstract: In this paper, we study a class of nonlocal boundary value problems of fractional systems which involves left and right fractional derivatives at resonance. By using the coincidence degree theory, the solvability results for the problems are obtained under the resonant conditions. As an application of our results, we also deal with the existence result for the solution of fractional differential equation which involves both left and right fractional derivatives and satisfies certain boundary conditions under the resonant conditions. Finally, some examples are presented to illustrate our main results.

Keywords: fractional system; left and right fractional derivative; nonlocal problem; resonance condition
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1. Introduction

In the recent decades, fractional differential equation has received extensive attention in mathematical theory and application research, see [1–5] and the references therein. A great deal of research results have been obtained in the theory and application of fractional differential equations, see [6–18] and the references therein. Meanwhile, the differential equations with left and right fractional derivatives are also playing an important role in many different applications. For example, in [19, 20], this type of differential equations is used to describe the temperature distribution of building walls while in [21], it is used to simulate the movement of particulate matter in the process of silo emptying. The theoretical research of this kind of problem has also attracted lots of attention, see [22–31].

In this paper, we study the following fractional systems which involve both left and right fractional derivatives

$$\begin{cases} {}_0^C D_t^{\alpha} u_1(t) = f_1(t, u_1(t), u_2(t)),\ t \in (0,1)\,, \\ {}_t^C D_1^{\beta} u_2(t) = f_2(t, u_1(t), u_2(t)),\ t \in (0,1)\,, \end{cases} \tag{1.1}$$

with the nonlocal boundary conditions

$$\begin{cases} u_1(0) = r u_2(1),\ u_2(0) = \int_0^1 \omega(t) u_1(t) \mathrm{d}t, \\ u_1'(0) = 0,\ u_2'(1) = 0, \end{cases} \tag{1.2}$$

where $1 < \alpha, \beta \le 2$, ${}_0^C D_t^{\alpha}$ and ${}_t^C D_1^{\beta}$ represent the left and right Caputo fractional derivative operator, respectively. $f_i \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$ may be nonlinear functions, and $\omega \in C([0,1],[0,+\infty))$ is a given function, r is a real.

The purpose of this paper is to obtain the existence results for solutions of boundary value problem (1.1) and (1.2) under the resonant condition

$$r \int_0^1 \omega(t) \mathrm{d}t = 1. \tag{1.3}$$

As an application of our results, we deal with the existence result of the solution to fractional differential equation under the resonant conditions which involves both left and right fractional derivatives

$${}_t^C D_1^{\beta} \left({}_0^C D_t^{\alpha} u(t) \right) = g(t, u(t), {}_0^C D_t^{\alpha} u(t)),\ t \in (0,1)\,, \tag{1.4}$$

which satisfies certain nonlocal boundary conditions.

2. Preliminaries

In this section, we show some basic definitions for the fractional calculus and related lemmas which are used to establish the main results.

Definition 2.1. (See [1,3]) *Suppose $\gamma > 0$, then the order γ Riemann-Liouville left fractional integral and Caputo left fractional derivative of function $y : [0,1] \to \mathbb{R}$ are defined by*

$${}_0 I_t^{\gamma} y(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} y(s) \mathrm{d}s, \ \text{and} \ \ {}_0^C D_t^{\gamma} y(t) = {}_0 I_t^{n-\gamma} (\frac{\mathrm{d}}{\mathrm{d}t})^n y(t),$$

respectively, provided the right sides exist. And the order γ Riemann-Liouville right fractional integral and Caputo right fractional derivative of y are given by

$${}_t I_1^{\gamma} y(t) = \frac{1}{\Gamma(\gamma)} \int_t^1 (s-t)^{\gamma-1} y(s) \mathrm{d}s, \ \ \text{and} \ {}_t^C D_1^{\gamma} y(t) = (-1)^n {}_t I_1^{n-\gamma} (\frac{\mathrm{d}}{\mathrm{d}t})^n y(t),$$

respectively, provided the right-side integral converges, where n is an integer with $n-1 < \gamma < n$.

Lemma 2.1. (See [1,3]) *For $n-1 < \gamma < n$, n is a positive integer, then the general solution of fractional differential equation ${}_0^C D_t^{\gamma} y(t) = 0$ is given by*

$$y(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

and the general solution of fractional differential equation ${}_t^C D_1^\gamma y(t) = 0$ *is given by*

$$y(t) = d_0 + d_1(1-t) + \cdots + d_{n-1}(1-t)^{n-1},$$

where $c_j,\ d_j \in \mathbb{R},\ j = 0, 1, \cdots, n.$

Definition 2.2. (see [32], P39) *Let* $\mathcal{X}$*,* $\mathcal{Y}$ *be normed vector spaces,* $L : \operatorname{Dom} L \subset \mathcal{X} \to \mathcal{Y}$ *a linear mapping. The mapping* L *will be called a Fredholm mapping of index zero if*
(a) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$*;*
(b) $\operatorname{Im} L$ *is closed in* $\mathcal{Y}$.

Lemma 2.2. (Generalized Krasnosel'skii theorem, see [32], P32) *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be Banach spaces. Let* $L : \operatorname{dom} L \subset \mathcal{X} \to \mathcal{Y}$ *be a Fredholm mapping of index zero,* $N : \mathcal{X} \to \mathcal{Y}$ *be an L-compact mapping in* $\overline{\Omega}$ *with* Ω *open, bounded, symmetric with respect to the origin and containing it. If*

$$(L-N)x \neq \lambda(L-N)(-x)$$

for every $x \in \operatorname{Dom} L \cap \partial\Omega$ *and every* $\lambda \in [0, 1]$*, where* $\partial\Omega$ *is the boundary of* Ω *with respect to* $\mathcal{X}$*, then equation* $Lx = Nx$ *has at least one solution in* Ω*.*

3. The existence of solutions for the systems

In this section, we present the existence results of the solutions of boundary value problem (1.1) and (1.2).

Let

$$\mathcal{X} = \mathcal{Y} = \{\mathbf{u} = (u_1,\ u_2)^T :\ u_i \in C[0, 1],\ i = 1, 2\}$$

be endowed with the norm

$$\|\mathbf{u}\| = \|(u_1,\ u_2)^T\| = \max\{\max_{t\in[0,1]} |u_1(t)|,\ \max_{t\in[0,1]} |u_2(t)|\}.$$

Then $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ are Banach spaces.

Denote vector functions

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},\quad \mathbf{f}(t, \mathbf{u}(t)) = \begin{pmatrix} f_1(t, u_1(t), u_2(t)) \\ f_2(t, u_1(t), u_2(t)) \end{pmatrix},$$

and an operator

$$L = \begin{pmatrix} {}_0^C D_t^\alpha & 0 \\ 0 & {}_t^C D_1^\beta \end{pmatrix}.$$

Let $L :\ \operatorname{Dom} L \subseteq \mathcal{X} \to \mathcal{Y}$ by

$$L\mathbf{u}(t) = L(u_1,\ u_2)^T = \begin{pmatrix} {}_0^C D_t^\alpha & 0 \\ 0 & {}_t^C D_1^\beta \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} {}_0^C D_t^\alpha u_1(t) \\ {}_t^C D_1^\beta u_2(t) \end{pmatrix}, \tag{3.1}$$

where

$$\operatorname{Dom} L = \{\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{X} :\ \begin{pmatrix} {}_0^C D_t^\alpha u_1(t) \\ {}_t^C D_1^\beta u_2(t) \end{pmatrix} \in \mathcal{Y},\ u_1(t),\ u_2(t) \text{ satisfy boundary conditions (1.2)}\}.$$

Define $N: \mathcal{X} \to \mathcal{Y}$ by

$$N\mathbf{u}(t) = \mathbf{f}(t, \mathbf{u}(t)) = \begin{pmatrix} f_1(t, u_1(t), u_2(t)) \\ f_2(t, u_1(t), u_2(t)) \end{pmatrix}.$$

It is clear that boundary value problem (1.1) and (1.2) is equivalent to the following operator equation

$$L\mathbf{u} = N\mathbf{u}.$$

Lemma 3.1. *Let L be defined by* (3.1). *Then L is a Fredholm operator of index zero.*

Proof. Obviously L is a linear operator. Next, we consider the kernel of the linear operator L.

$$\mathrm{Ker}L = \{\mathbf{u} \in \mathrm{Dom}\, L \subseteq \mathcal{X} | \ L\mathbf{u} = \mathbf{0}\},$$

which implies that

$$L\mathbf{u}(t) = L(u_1,\ u_2)^T = \begin{pmatrix} {}_0^C D_t^{\alpha} u_1(t) \\ {}_t^C D_1^{\beta} u_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}, \ \text{ for any } \mathbf{u} \in \mathrm{Ker}L. \tag{3.2}$$

Then

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} c_0 + c_1 t \\ d_0 + d_1(1-t) \end{pmatrix}. \tag{3.3}$$

Take the boundary conditions $u_1'(0) = 0$ and $u_2'(1) = 0$ into account, and we can get $c_1 = d_1 = 0$.

In view of $u_1(0) = ru_2(1)$ and $u_2(0) = \int_0^1 \omega(t)u_1(t)\mathrm{d}t$ and the resonant condition $r\int_0^1 \omega(t)\mathrm{d}t = 1$, we can show that $c_0 = rd_0$. Therefore,

$$\mathrm{Ker}L = \{\mathbf{u} \in \mathrm{Dom}\, L \subseteq \mathcal{X}: \ \mathbf{u}(t) = d\begin{pmatrix} r \\ 1 \end{pmatrix},\ d \in \mathbb{R}\},$$

which implies that $\dim\mathrm{Ker}L = 1$.

Following, we denote

$$\rho(s) := \int_s^1 \omega(\tau)(\tau - s)^{\alpha-1}\mathrm{d}\tau, \qquad \mathbf{\Lambda}(s) := \begin{pmatrix} \frac{1}{\Gamma(\alpha)}\rho(s) \\ -\frac{1}{\Gamma(\beta)}s^{\beta-1} \end{pmatrix}. \tag{3.4}$$

And for $\mathbf{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \mathcal{Y}$, we denote

$$\langle \mathbf{\Lambda},\ \mathbf{y} \rangle := \int_0^1 \mathbf{\Lambda}^{\mathrm{T}}(s)\mathbf{y}(s)\mathrm{d}s = \frac{1}{\Gamma(\alpha)}\int_0^1 \rho(s)y_1(s)\mathrm{d}s - \frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}y_2(s)\mathrm{d}s. \tag{3.5}$$

We prove that

$$\mathrm{Im}L = \Big\{\mathbf{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \mathcal{Y}: \ \langle \mathbf{\Lambda},\ \mathbf{y} \rangle = 0\Big\}. \tag{3.6}$$

Since

$$\mathrm{Im}L = \Big\{\mathbf{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \mathcal{Y}: \ \text{there exists } \mathbf{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \mathrm{Dom}L \text{ such that } L\mathbf{u} = \mathbf{y}\Big\},$$

for any $\mathbf{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \mathrm{Im}L$, there exists $\mathbf{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \mathrm{Dom}L$ such that

$$L\mathbf{u}(t) = L(u_1,\, u_2)^T = \begin{pmatrix} {}_0^C D_t^{\alpha} u_1(t) \\ {}_t^C D_1^{\beta} u_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}. \tag{3.7}$$

Then

$$u_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s) \mathrm{d}s + c_0 + c_1 t,$$

$$u_2(t) = \frac{1}{\Gamma(\beta)} \int_t^1 (s-t)^{\beta-1} y_2(s) \mathrm{d}s + d_0 + d_1(1-t).$$

And

$$u_1'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y_1(s) \mathrm{d}s + c_1,$$

$$u_2'(t) = -\frac{1}{\Gamma(\beta-1)} \int_t^1 (s-t)^{\beta-2} y_2(s) \mathrm{d}s + d_1.$$

By the boundary conditions $u_1'(0) = u_2'(1) = 0$, we get $c_1 = d_1 = 0$. And by $u_1(0) = ru_2(1)$, $u_2(0) = \int_0^1 \omega(t) u_1(t) \mathrm{d}t$, we have

$$c_0 = rd_0 \tag{3.8}$$

and

$$\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} y_2(s) \mathrm{d}s + d_0 = \frac{1}{\Gamma(\alpha)} \int_0^1 \omega(s) (\int_0^s (\tau - s)^{\alpha-1} y_1(\tau) \mathrm{d}\tau + c_0) \mathrm{d}s. \tag{3.9}$$

Then

$$\frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} y_2(s) \mathrm{d}s + d_0 = \frac{1}{\Gamma(\alpha)} \int_0^1 (\int_s^1 (\tau - s)^{\alpha-1} \omega(\tau) \mathrm{d}\tau) y_1(s) \mathrm{d}s + rd_0 \int_0^1 \omega(s) \mathrm{d}s.$$

It follows

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (\int_s^1 (\tau - s)^{\alpha-1} \omega(\tau) \mathrm{d}\tau) y_1(s) \mathrm{d}s - \frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} y_2(s) \mathrm{d}s = 0 \tag{3.10}$$

from the resonant condition $r \int_0^1 \omega(s) \mathrm{d}s = 1$. That is

$$\frac{1}{\Gamma(\alpha)} \int_0^1 \rho(s) y_1(s) \mathrm{d}s - \frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} y_2(s) \mathrm{d}s = 0. \tag{3.11}$$

Then, Eq. (3.11) is equivalent to

$$\langle \mathbf{\Lambda},\, \mathbf{y} \rangle = \int_0^1 \mathbf{\Lambda}^{\mathrm{T}}(s) \mathbf{y}(s) \mathrm{d}s = 0 \tag{3.12}$$

and

$$\mathrm{Im}L \subseteq \Big\{ \mathbf{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \mathcal{Y} :\ \langle \mathbf{\Lambda},\, \mathbf{y} \rangle = 0 \Big\}.$$

On the other hand, for every $\mathbf{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \left\{\mathbf{y} \in \mathcal{Y} : \langle \mathbf{\Lambda}, \mathbf{y} \rangle = 0\right\}$, let

$$\mathbf{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)\mathrm{d}s \\ \frac{1}{\Gamma(\beta)} \int_t^1 (s-t)^{\beta-1} y_2(s)\mathrm{d}s \end{pmatrix},$$

then $u \in \mathrm{Dom}L$ and $L\mathbf{u} = \mathbf{y}$. So

$$\left\{\mathbf{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \mathcal{Y} : \langle \mathbf{\Lambda}, \mathbf{y} \rangle = 0\right\} \subseteq \mathrm{Im}L.$$

Hence (3.6) holds.

Following we prove that $\mathrm{Im}L$ is closed.

Assume that $\mathbf{y}_k = \begin{pmatrix} y_{k,1}(t) \\ y_{k,2}(t) \end{pmatrix} \in \mathrm{Im}L$, and $\lim\limits_{k\to\infty} \mathbf{y}_k = \mathbf{y}_0 = \begin{pmatrix} y_{0,1}(t) \\ y_{0,2}(t) \end{pmatrix}$.

Since $\lim\limits_{k\to\infty} \|\mathbf{y}_k - \mathbf{y}_0\| = 0$, then $\lim\limits_{k\to\infty} |y_{k,i} - y_{0,i}| = 0$ for $i = 1, 2$.

Because $\mathbf{y}_k \in \mathrm{Im}L$ which implies $y_{k,i} \in C[0,1]$ for $k = 1, 2, \cdots$ and $i = 1, 2$, then $y_{0,i} \in C[0,1]$ for $i = 1, 2$, which implies $\mathbf{y}_0 \in \mathcal{Y}$.

By (3.6), we can get that

$$\langle \mathbf{\Lambda}, \mathbf{y}_k \rangle = \frac{1}{\Gamma(\alpha)} \int_0^1 \rho(s) y_{k,1}(s)\mathrm{d}s - \frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} y_{k,2}(s)\mathrm{d}s = 0.$$

Then

$$\langle \mathbf{\Lambda}, \mathbf{y}_0 \rangle = \frac{1}{\Gamma(\alpha)} \int_0^1 \rho(s) y_{0,1}(s)\mathrm{d}s - \frac{1}{\Gamma(\beta)} \int_0^1 s^{\beta-1} y_{0,2}(s)\mathrm{d}s = 0,$$

which implies that $\mathbf{y}_0 \in \mathrm{Im}L$ and $\mathrm{Im}L$ is a closed in $\mathcal{Y}$.

Because $\mathbf{\Lambda}$ is a fixed vector function, then $\langle \mathbf{\Lambda}, \mathbf{y} \rangle \in \mathbb{R}$, for any $\mathbf{y} \in \mathcal{Y}$, which implies $\dim(\mathcal{Y}/\mathrm{Im}L) = 1$.

So

$$\mathrm{codim}(\mathrm{Im}L) = \dim(\mathcal{Y}/\mathrm{Im}L) = 1 = \dim\mathrm{Ker}L.$$

Therefore, we get that the linear operator L is a Fredholm operator with index zero. For the definition of Fredholm operator with index zero, see Definition 2.2.

■

Define $P : \mathcal{X} \to \mathcal{X}$ by

$$P\mathbf{u} = P\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \frac{u_1(0) + r u_2(1)}{2r} \begin{pmatrix} r \\ 1 \end{pmatrix}.$$

Then P is a linear continuous projector operator. We can easily check that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathcal{X} = \mathrm{Ker}P \oplus \mathrm{Ker}L$.

So the operator $L|_{\mathrm{Dom}\,L\cap\mathrm{Ker}P} : \mathrm{Dom}\,L \cap \mathrm{Ker}P \longrightarrow \mathrm{Im}L$ is reversible.

For every $y \in \mathrm{Im}L$, there exists $u \in \mathrm{Dom}\,L \cap \mathrm{Ker}P$ such that

$$L\mathbf{u}(t) = \begin{pmatrix} {}_0^C D_t^\alpha u_1(t) \\ {}_t^C D_1^\beta u_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \mathbf{y} \in \mathcal{Y}.$$

Then

$$u_1(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y_1(s)\mathrm{d}s + c_0 + c_1 t,$$

$$u_2(t) = \frac{1}{\Gamma(\beta)}\int_t^1 (s-t)^{\beta-1}y_2(s)\mathrm{d}s + d_0 + d_1(1-t).$$

Combining the boundary conditions and noticing $u \in \mathrm{Dom}\,L \cap \mathrm{Ker}P$, and we can get that $c_0 = c_1 = d_0 = d_1 = 0$. Then

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y_1(s)\mathrm{d}s \\ \frac{1}{\Gamma(\beta)}\int_t^1 (s-t)^{\beta-1}y_2(s)\mathrm{d}s \end{pmatrix} = L_P^{-1}\mathbf{y}(t), \tag{3.13}$$

where L_P^{-1} is the inverse of $L|_{\mathrm{Dom}\,L\cap\mathrm{Ker}P}$.

Define the operator $Q : \mathcal{Y} \to \mathcal{Y}/\mathrm{Im}L$ by

$$Q\mathbf{y} = Q\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = -\Gamma(\beta+1)\langle \mathbf{\Lambda},\ \mathbf{y}\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.14}$$

Let $\mathbf{y_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

$$\langle \mathbf{\Lambda},\ \mathbf{y_0}\rangle = -\frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}\mathrm{d}s = -\frac{1}{\Gamma(\beta+1)},$$

and $Q^2 = Q$. That is, Q is a linear continuous projector operator.

We can easily see that $\mathrm{Ker}Q = \mathrm{Im}L$ and $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$.

Since $\mathbf{f}$ is continuous, it follows that Lemma 3.2 holds from (3.13) and (3.14).

Lemma 3.2. *$N : \mathcal{X} \to \mathcal{Y}$ is an L-compact operator.*

Denote

(H1) The functions $f_i \in C([0,1]\times\mathbb{R}^2, \mathbb{R})$, there exist constants $\sigma_i \geq 0$ and functions $a_i,\ b_i \in C([0,1],\ [0,\ +\infty))$, $i = 1,2$, such that

$$|f_i(t,x_2,y_2) - f_i(t,x_1,y_1)| \leq a_i(t)|x_1 - x_2|^{\sigma} + b_i(t)|y_1 - y_2|^{\sigma}, \quad i = 1,2,$$

for any $t \in [0,1]$, $x_j,\ y_j \in \mathbb{R}$, $j = 1,2$.

(H2) The functions $f_i \in C([0,1]\times\mathbb{R}^2, \mathbb{R})$ and

$$\limsup_{|x|+|y|\to\infty}\sup_{t\in[0,1]}\frac{|f_i(t,x,y)|}{|x|+|y|} < R_i,\ i = 1,2,$$

where $R_1 = \frac{\Gamma(\alpha+1)}{2}$ and $R_2 = \frac{\Gamma(\beta+1)}{2}$.

For convenience, let

$$m_0 = \max\{\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(a_1(s)+a_2(s))\mathrm{d}s,\ \frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}(b_1(s)+b_2(s))\mathrm{d}s\}, \tag{3.15}$$

and

$$f_0 = \max\{\frac{1}{\Gamma(\alpha+1)}\max_{t\in[0,1]}|f_1(t,0,0)|,\ \frac{1}{\Gamma(\beta+1)}\max_{t\in[0,1]}|f_2(t,0,0)|\}. \tag{3.16}$$

Theorem 3.1. *Suppose (H1) holds and* $0 \leq \sigma < 1$*. Then boundary value problem* (1.1) *and* (1.2) *has at least one solution.*

Proof. If $\mathbf{u} \in \mathrm{Dom}L \cap \mathrm{Ker}P$ satisfies the following equation

$$(L-N)(\mathbf{u}) = \lambda(L-N)(-\mathbf{u}), \quad \lambda \in [0,1], \tag{3.17}$$

then

$$L(\mathbf{u}) = \frac{1}{1+\lambda}(N(\mathbf{u}) - \lambda N(-\mathbf{u})), \quad \lambda \in [0,1] \tag{3.18}$$

and

$$\mathbf{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = L_P^{-1}(\frac{1}{1+\lambda}(N(\mathbf{u}) - \lambda N(-\mathbf{u}))).$$

We can get

$$\begin{aligned}
|u_1(t)| =& |\frac{1}{1+\lambda} \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(f_1(s,u_1(s),u_2(s)) - \lambda f_1(s,-u_1(s),-u_2(s)))\mathrm{d}s| \\
\leq & \frac{1}{\Gamma(\alpha)(1+\lambda)} \int_0^t (t-s)^{\alpha-1}|f_1(s,u_1(s),u_2(s)) - \lambda f_1(s,-u_1(s),-u_2(s))|\mathrm{d}s \\
\leq & \frac{1}{\Gamma(\alpha)(1+\lambda)} \int_0^1 (1-s)^{\alpha-1}|(f_1(s,u_1(s),u_2(s)) - \lambda f_1(s,-u_1(s),-u_2(s)))|\mathrm{d}s \\
\leq & \frac{1}{\Gamma(\alpha)(1+\lambda)} \int_0^1 (1-s)^{\alpha-1}(|f_1(s,u_1(s),u_2(s)) - f_1(s,0,0)| \\
& + \lambda|f_1(s,-u_1(s),-u_2(s)) - f_1(s,0,0)| + (1+\lambda)|f_1(s,0,0)|)\mathrm{d}s \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}[(a_1(s)|u_1(s)|^\sigma + a_2(s)|u_2(s)|^\sigma) + |f_1(s,0,0)|]\mathrm{d}s \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}[(a_1(s) + a_2(s))||\mathbf{u}||^\sigma + |f_1(s,0,0)|]\mathrm{d}s \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(a_1(s) + a_2(s))\mathrm{d}s||\mathbf{u}||^\sigma + f_0
\end{aligned}$$

and

$$\max_{t\in[0,1]} |u_1(t)| \leq m_0||\mathbf{u}||^\sigma + f_0, \tag{3.19}$$

where m_0 and f_0 are given by (3.15) and (3.16).

Similarly, we can show

$$\max_{t\in[0,1]} |u_2(t)| \leq m_0||\mathbf{u}||^\sigma + f_0. \tag{3.20}$$

As a result,

$$||\mathbf{u}|| \leq m_0||\mathbf{u}||^\sigma + f_0. \tag{3.21}$$

Since $0 \leq \sigma < 1$, we take $M \geq \max\{(2m_0)^{\frac{1}{1-\sigma}}, 2f_0\} + 1$ and

$$\Omega = \{\mathbf{u} \in \mathcal{X} \cap \mathrm{Ker}P : ||\mathbf{u}|| < M\}.$$

Then Ω is open, bounded, symmetric with respect to the origin and containing it.

If $\mathbf{u} \in \mathrm{Dom}\, L \cap \partial\Omega$ and satisfies (3.17), by (3.21), we can show that

$$M = \|\mathbf{u}\| \leq m_0 M^{\sigma} + f_0 < \frac{M^{1-\sigma}}{2} \cdot M^{\sigma} + \frac{M}{2} = M,$$

which is a contradiction.

Therefore, we can obtain that

$$(L - N)\mathbf{u} \neq \lambda(L - N)(-\mathbf{u}),\ \mathbf{u} \in \mathrm{Dom}\, L \cap \partial\Omega \text{ and } \lambda \in [0, 1].$$

By Lemma 3.2, N is an L-compact operator.

According to Lemma 2.2, we have the equation $L\mathbf{u} = N\mathbf{u}$ has at least one solution on $\mathrm{Dom}\, L \cap \overline{\Omega}$. Namely, boundary value problem (1.1) and (1.2) has at least one solution.

■

Theorem 3.2. *Suppose (H1) holds. If $\sigma = 1$ and $m_0 < 1$, then boundary value problem* (1.1) *and* (1.2) *has at least one solution.*

Proof. Since $m_0 < 1$, we take $M > \frac{f_0}{1-m_0}$ and

$$\Omega = \{\mathbf{u} \in \mathcal{X} \cap \mathrm{Ker} P : \|\mathbf{u}\| < M\}.$$

Then Ω is open, bounded, symmetric with respect to the origin and containing it.

If $\mathbf{u} \in \mathrm{Dom} L \cap \mathrm{Ker} P$ satisfies equation (3.17), similar to the proof of the theorem 3.1, we can get that

$$\|\mathbf{u}\| \leq m_0 \|\mathbf{u}\| + f_0. \tag{3.22}$$

It follows

$$\|\mathbf{u}\| < m_0 M + (1 - m_0)M = M,\ \mathbf{u} \in \mathrm{Dom}\, L \cap \partial\Omega$$

from (3.22). We can show that

$$(L - N)\mathbf{u} \neq \lambda(L - N)(-\mathbf{u}),\ \mathbf{u} \in \mathrm{Dom}\, L \cap \partial\Omega \text{ and } \lambda \in [0, 1].$$

By Lemma 3.2, $N : \mathcal{X} \to \mathcal{Y}$ is an L-compact operator.

According to Lemma 2.2, we have the equation $L\mathbf{u} = N\mathbf{u}$ has at least one solution on $\mathrm{Dom}\, L \cap \overline{\Omega}$. Namely, boundary value problem (1.1) and (1.2) has at least one solution.

■

Theorem 3.3. *Suppose (H2) holds, then boundary value problem* (1.1) *and* (1.2) *has at least one solution.*

Proof. For $i = 1, 2$, let $\epsilon_i = \frac{1}{2}(R_i - \limsup_{|x|+|y|\to\infty} \sup_{t\in[0,1]} \frac{|f_i(t,x,y)|}{|x|+|y|}) > 0$. By (H2), there exist constants M_i such that

$$|f_i(t, x, y)| \leq (R_i - \epsilon_i)(|x| + |y|),\ \text{for } |x| + |y| > M_i,\ \ i = 1, 2.$$

Since f_i are continuous, there exist constants R_{0i} such that

$$R_{0i} = \max\{f_i(t, x, y) :\ t \in [0, 1] \text{ and } |x| + |y| \leq M_i\},\ \ i = 1, 2.$$

We have

$$|f_i(t,x,y)| \leq R_{0i} + R_i(|x|+|y|), \text{ for } x,y \in \mathbb{R}. \tag{3.23}$$

Let $M > \max\{\frac{R_{01}}{\Gamma(\alpha+1)-2(R_1-\epsilon_1)}, \frac{R_{02}}{\Gamma(\beta+1)-2(R_2-\epsilon_1)}\}$ and

$$\Omega = \{\mathbf{u} \in \mathcal{X} \cap \mathrm{Ker}P : \|\mathbf{u}\| < M\}.$$

Then Ω is open, bounded, symmetric with respect to the origin and containing it.

If $\mathbf{u} \in \mathrm{Dom}L \cap \mathrm{Ker}P$ satisfies Eq. (3.17), we can get that

$$\begin{aligned}
|u_1(t)| =& |\frac{1}{1+\lambda} \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(f_1(s,u_1(s),u_2(s)) - \lambda f_1(s,-u_1(s),-u_2(s)))\mathrm{d}s| \\
\leq& \frac{1}{\Gamma(\alpha)(1+\lambda)} \int_0^t (t-s)^{\alpha-1}|f_1(s,u_1(s),u_2(s)) - \lambda f_1(s,-u_1(s),-u_2(s))|\mathrm{d}s \\
\leq& \frac{1}{\Gamma(\alpha)(1+\lambda)} \int_0^1 (1-s)^{\alpha-1}\big(|(f_1(s,u_1(s),u_2(s))| + \lambda|f_1(s,-u_1(s),-u_2(s)))|\big)\mathrm{d}s \\
\leq& \frac{1}{\Gamma(\alpha)(1+\lambda)} \int_0^1 (1-s)^{\alpha-1}(1+\lambda)\Big(R_{01} + (R_1-\epsilon_1)(|u_1(s)|+|u_2(s)|)\Big)\mathrm{d}s \\
\leq& \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(R_{01} + 2(R_1-\epsilon_1)\|\mathbf{u}\|)\mathrm{d}s \\
=& \frac{1}{\Gamma(\alpha+1)} \cdot (R_{01} + 2(R_1-\epsilon_1)\|\mathbf{u}\|).
\end{aligned}$$

Hence, if $\mathbf{u} \in \mathrm{Dom}\,L \cap \partial\Omega$ and $\lambda \in [0,1]$, then

$$\begin{aligned}
\max_{t\in[0,1]} |u_1(t)| \leq& \frac{1}{\Gamma(\alpha+1)} \cdot (R_{01} + 2(R_1-\epsilon_1)\|\mathbf{u}\|) \\
<& \frac{1}{\Gamma(\alpha+1)} \cdot \big(\Gamma(\alpha+1) - 2(R_1-\epsilon_1))M + 2(R_1-\epsilon_1)M\big) \\
=& M.
\end{aligned} \tag{3.24}$$

Similarly, we can show

$$\max_{t\in[0,1]} |u_2(t)| < M, \ \mathbf{u} \in \mathrm{Dom}\,L \cap \partial\Omega \text{ and } \lambda \in [0,1]. \tag{3.25}$$

It follows

$$M = \|\mathbf{u}\| < M, \ \mathbf{u} \in \mathrm{Dom}\,L \cap \partial\Omega \text{ and } \lambda \in [0,1]$$

from (3.24) and (3.25), which is a contradiction. We show that

$$(L-N)\mathbf{u} \neq \lambda(L-N)(-\mathbf{u}), \ \mathbf{u} \in \mathrm{Dom}\,L \cap \partial\Omega \text{ and } \lambda \in [0,1].$$

By Lemma 3.2, N is an L-compact operator.

According to Lemma 2.2, we have that the equation $L\mathbf{u} = N\mathbf{u}$ has at least one solution on $\mathrm{Dom}\,L \cap \overline{\Omega}$. That is, boundary value problem (1.1) and (1.2) has at least one solution.

■

4. The existence of solutions for the fractional equations

As an application of Theorem 3.2, in this section, we consider the existence of solutions for the following fractional differential equation which involves the left and right derivatives

$$ {}_t^C D_1^{\beta}\left({}_0^C D_t^{\alpha} u(t)\right) = g(t, u(t), {}_0^C D_t^{\alpha} u(t)),\ t \in (0,1) \tag{4.1}$$

with the following nonlocal boundary conditions

$$\begin{cases} u(0) = r\,{}_0^C D_t^{\alpha} u(1),\ {}_0^C D_t^{\alpha} u(0) = \int_0^1 \omega(t)u(t)\mathrm{d}t, \\ u'(0) = 0,\ ({}_0^C D_t^{\alpha} u(t))'|_{t=1} = 0, \end{cases} \tag{4.2}$$

under the resonant condition $r\int_0^1 \omega(s)ds = 1$.

Theorem 4.1. *Assume $r\int_0^1 \omega(s)ds = 1$, there exist $p,\ q \in C([0,1],\ [0,+\infty))$ such that*

$$|g(t, x_2, y_2) - g(t, x_1, y_1)| \le p(t)|x_2 - x_1| + q(t)|y_2 - y_1|,$$

for any $t \in [0,1]$, $x_j,\ y_j \in \mathbb{R}$, $j = 1,2$. If

$$m_0 := \max\left\{\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}p(s)\mathrm{d}s,\ \frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}(1+q(s))\mathrm{d}s\right\} < 1,$$

then boundary value problem (4.1) *and* (4.2) *has at least one solution.*

Proof. Let $u_1(t) = u(t)$, $u_2(t) = {}_0^C D_t^{\alpha} u(t)$, $f_1(t,x,y) = y$ and $f_2(t,x,y) = g(t,x,y)$. Then Eq. (4.1) is equivalent to the following the system

$$\begin{cases} {}_0^C D_t^{\alpha} u_1(t) = u_2(t) := f_1(t, u_1(t), u_2(t)), \\ {}_t^C D_1^{\beta} u_2(t) = g(t, u(t),\ {}_0^C D_t^{\alpha} u(t)) := f_2(t, u_1(t), u_2(t)), \end{cases} \tag{4.3}$$

and boundary conditions (4.2) is equivalent to (1.2).

We can easily check that all conditions in Theorem 3.2 are satisfied for the Eq. (4.1) with the boundary conditions (4.2).

By Theorem 3.2, we can get that the conclusion of Theorem 4.1 holds. ∎

5. Examples

In this section, we give out some examples to illustrate our main results.

Example 5.1 We consider the following fractional integral boundary value problems of the nonlinear fractional differential system

$$\begin{cases} {}_0^C D_t^{\frac{3}{2}} u_1(t) = \frac{1}{3}\arctan\left(2(1-t)^{\frac{1}{2}}u_1(t) + 3t^{\frac{1}{4}}u_2(t)\right) + e^t,\ t \in (0,1), \\ {}_t^C D_1^{\frac{5}{4}} u_2(t) = \frac{1}{4}\arctan\left(3(1-t)^{\frac{1}{2}}u_1(t) + 2t^{\frac{1}{4}}u_2(t)\right) + e^{-t},\ t \in (0,1), \\ u_1(0) = \frac{3\sqrt{\pi}}{4}u_2(1),\ u_2(0) = {}_0I_t^{\frac{3}{2}} u_1(1), \\ u_1'(0) = 0,\ u_2'(1) = 0, \end{cases} \tag{5.1}$$

where $\alpha = \frac{3}{2}, \beta = \frac{5}{4}, r = \frac{3\sqrt{\pi}}{4}$. And the condition $u_2(0) = {}_0I_t^{\frac{3}{2}} u_1(1)$ is equivalent to

$$u_2(0) = \frac{2}{\sqrt{\pi}} \int_0^1 (1-t)^{\frac{1}{2}} u_1(t)\mathrm{d}t = \int_0^1 \omega(t) u_1(t)\mathrm{d}t,$$

where $\omega(t) = \frac{2}{\sqrt{\pi}}(1-t)^{\frac{1}{2}}$. And

$$r\int_0^1 \omega(t)\mathrm{d}t = r({}_0I_t^{\frac{3}{2}}(1)) = \frac{r}{\Gamma(\frac{3}{2})} \int_0^1 (1-t)^{\frac{1}{2}}\mathrm{d}t = 1. \tag{5.2}$$

So boundary value problem (5.1) is a resonance problem.

Let $f_1(t,x,y) = \frac{1}{3}\arctan(2(1-t)^{\frac{1}{2}}x + 3t^{\frac{1}{4}}y) + e^t$ and $f_2(t,x,y) = \frac{1}{4}\arctan(3(1-t)^{\frac{1}{2}}x + 2t^{\frac{1}{4}}y) + e^{-t}$, $a_1(t) = 2(1-t)^{\frac{1}{2}}$, $a_2(t) = 3(1-t)^{\frac{1}{2}}$, $b_1(t) = 3t^{\frac{1}{4}}$, $b_2(t) = 2t^{\frac{1}{4}}$, then we can easily check f_1 and f_2 satisfy the condition (H1) with $\sigma = 1$. And we can get that

$$\begin{aligned} m_0 =& \max\{\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(a_1(s)+a_2(s))\mathrm{d}s,\ \frac{1}{\Gamma(\beta)}\int_0^1 s^{\beta-1}(b_1(s)+b_2(s))\mathrm{d}s\} \\ =& \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(a_1(s)+a_2(s))\mathrm{d}s \\ \approx& 0.940316 < 1, \end{aligned}$$

and

$$f_0 = \max\{\max_{t\in[0,1]} \frac{1}{\Gamma(\alpha+1)}|f_1(t,0,0)|,\ \max_{t\in[0,1]} \frac{1}{\Gamma(\beta+1)}|f_2(t,0,0)|\} \approx 2.04484.$$

As a result, all conditions in Theorem 3.2 hold. By Theorem 3.2, boundary value problem (5.1) has at least one solution.

Example 5.2 We consider the following boundary value problem

$$\begin{cases} {}_0^CD_t^{\frac{3}{2}} u_1(t) = \frac{2}{3}(1-t)^{\frac{1}{2}}(u_1(t))^{\frac{3}{2}} + \frac{4}{5}t^{\frac{1}{4}}(u_2(t))^{\frac{5}{4}} + e^t,\ t\in(0,1), \\ {}_t^CD_1^{\frac{5}{4}} u_2(t) = \frac{4}{5}t^{\frac{1}{4}}(u_1(t))^{\frac{5}{4}} + \frac{2}{3}(1-t)^{\frac{1}{2}}(u_2(t))^{\frac{3}{2}} + e^{-t},\ t\in(0,1), \\ u_1(0) = \frac{3\sqrt{\pi}}{4}u_2(1),\ u_2(0) = {}_0I_t^{\frac{3}{2}}u_1(1), \\ u_1'(0) = 0,\ u_2'(1) = 0. \end{cases} \tag{5.3}$$

Let $f_1(t,x,y) = \frac{2}{3}(1-t)^{\frac{1}{2}}x^{\frac{3}{2}} + \frac{4}{5}t^{\frac{1}{4}}y^{\frac{5}{4}} + e^t$ and $f_2(t,x,y) = \frac{4}{5}t^{\frac{1}{4}}x^{\frac{5}{4}} + \frac{2}{3}(1-t)^{\frac{1}{2}}y^{\frac{3}{2}} + e^{-t}$, $a_1(t) = 2(1-t)^{\frac{1}{2}}$, $a_2(t) = 2(1-t)^{\frac{1}{2}}$, $b_1(t) = 3t^{\frac{1}{4}}$, $b_2(t) = 3t^{\frac{1}{4}}$. Other parameters are same as the ones in Example 5.1. Then we can easily check f_i, $i = 1,2$, satisfy the conditions of (H1), where $\sigma = \frac{1}{2} < 1$.

Then the conditions in Theorem 3.1 hold. By Theorem 3.1, boundary value problem (5.3) has at least one solution.

Example 5.3 We consider the integral boundary value problem of fractional differential equation as following

$$\begin{cases} {}_t^CD_1^{\beta}\left({}_0^CD_t^{\alpha}u(t)\right) = \frac{1}{4}\arctan(3(1-t)^{\frac{1}{2}}u(t) + 2t^{\frac{1}{4}}{}_0^CD_t^{\alpha}u(t)) + e^{-t},\ t\in(0,1), \\ u(0) = \frac{3\sqrt{\pi}}{4}{}_0^CD_t^{\alpha}u(1),\ {}_0^CD_t^{\alpha}u(0) = {}_0I_t^{\frac{3}{2}}u(1), \\ u'(0) = 0,\ ({}_0^CD_t^{\alpha}u(t))'|_{t=1} = 0. \end{cases} \tag{5.4}$$

Similar to Example 5.1, we can check that the conditions in Theorem 4.1 hold. As a result, boundary value problem (5.4) has at least one solution by Theorem 4.1.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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