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Research article

Involution on prime rings with endomorphisms

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Abstract: Let $\mathcal{R}$ be a prime ring with involution $'*'$ and $\psi : \mathcal{R} \to \mathcal{R}$ be an endomorphism on $\mathcal{R}$. In this article, we study the action of involution $'*'$, and the effect of endomorphism ψ satisfying $[\psi(x), \psi(x^*)] - [x, x^*] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. In particular, we prove that any centralizing involution on prime rings with involution of characteristic different from two is of the first kind or $\mathcal{R}$ satisfies s_4, the standard polynomial identity in four variables. Further, we establish that if a prime ring $\mathcal{R}$ with involution of characteristic different from two admits a non-trivial endomorphism ψ such that $[\psi(x), \psi(x^*)] - [x, x^*] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4 and $[\psi(x), x] = 0$ for all $x \in \mathcal{R}$.

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1. Introduction

This research has been motivated by the recent work's of authors [1, 13]. In what follows, $\mathcal{R}$ is an associative ring with $\mathcal{Z}(\mathcal{R})$ the center of $\mathcal{R}$, C the extended centroid of a prime ring $\mathcal{R}$(see [16] for details). A ring $\mathcal{R}$ is said to be n-torsion free, if for $x \in \mathcal{R}$, $nx = 0$ implies $x = 0$, where $n \geq 2$ is an integer. A ring $\mathcal{R}$ is called a prime ring if $a\mathcal{R}b = (0)$ (where $a, b \in \mathcal{R}$) implies $a = 0$ or $b = 0$. As usual, $[x, y]$ and $x \circ y$ will denote the commutator $xy - yx$ and the anti-commutator $xy + yx$, respectively. We shall use the following basic identities related to commutators and anti-commutators:$[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = [x, y]z + y[x, z]$ for all $x, y, z \in \mathcal{R}$. Moreover, $xo(yz) = (xoy)z - y[x, z] = y(x \circ z) + [x, y]z$ and $(xy)oz = (xoz)y + x[y, z] = x(y \circ z) - [x, z]y$ for all $x, y, z \in \mathcal{R}$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation on $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. A derivation d is said to be inner if there exists $a \in \mathcal{R}$ such that $d(x) = [a, x]$ for all $x \in \mathcal{R}$. Following [16], an additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^* x^*$ and (ii) $(x^*)^* = x$ hold for all $x \in \mathcal{R}$.

A ring equipped with involution is called ring with involution or $*$-ring. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of $\mathcal{R}$ will be denoted by $H(\mathcal{R})$ and $S(\mathcal{R})$, respectively. The involution is said to be of the first kind if $\mathcal{H}(\mathcal{R}) \subseteq \mathcal{Z}(\mathcal{R})$, otherwise it is said to be of the second kind. We say that involution $'*'$ is a centralizing (resp. commuting) if $[x, x^*] \in \mathcal{Z}(\mathcal{R})$ (resp. $[x, x^*] = 0$) for all $x \in \mathcal{R}$. We refer the reader to [16] for justification and amplification for the above mentioned notations and key definitions. We say that a map $f : \mathcal{R} \to \mathcal{R}$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in \mathcal{R}$. The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory and ring theory (see [6, 32] for references). Following [9], let $\mathcal{S}$ be a subset of R, a map $f : \mathcal{S} \to \mathcal{R}$ is said to be strong commutativity preserving (SCP) on $\mathcal{S}$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{S}$. The standard polynomial identity s_4 in four variables is defined as $s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$, where $(-1)^\sigma$ is $+1$ or -1 according to σ being an even or odd permutation in symmetric group S_4.

In the beginning of nineties, after the development of the theory of centralizing and commuting maps, Brešar [8] started the characterization of centralizing, commuting and commutativity preserving maps. Moreover, he established some classical theorems (see for example [6,7] and references their in). Commutativity preserving maps first introduced by Watkins [33] and extended to strong commutativity preserving maps by Bell and Mason [5]. Inspired by their works, Bell and Daif [4] demonstrated the commutativity of prime and semiprime rings admitting derivations and endomorphisms, which satisfy the strong commutativity preserving maps on right ideals (see also [30,31] for details). In 1994, Brešar and Miers [9] characterized an additive map $f : \mathcal{R} \to \mathcal{R}$ satisfying SCP on semiprime ring $\mathcal{R}$ and showed that f is of the form $f(x) = \lambda x + \mu(x)$ for all $x \in \mathcal{R}$, $\lambda \in C$, $\lambda^2 = 1$ and $\mu : \mathcal{R} \to C$ is an additive map, where C is the extended centroid of $\mathcal{R}$. Later, Lin and Liu [24] extended this result to Lie ideals of prime rings. Chasing to this, several techniques have been developed to investigate the behavior of SCP maps using restrictions on polynomials, invoking derivations, generalized derivations etcetera. The related results concerning these maps on prime and semiprime rings can be found in [13, 14, 17, 19–23, 25, 31], where further references can be obtained.

The study of additive maps in rings with involution was initiated by Brešar et al. (see [10] for more details) to describe the centralizing maps on the skew-symmetric elements in prime rings. In the same vein, Lin and Liu [25] characterized SCP maps on skew symmetric elements of prime rings with involution. Later Liau et al. [20] improved above mentioned result for non additive SCP maps. Interestingly, in the year 2015, the authors together with Dar [3] studied the SCP maps in different way for prime rings with involution. For instance, they established the following: Let $\mathcal{R}$ be a prime ring with involution of the second kind such that $char(\mathcal{R}) \neq 2$. Let δ be a nonzero derivation of $\mathcal{R}$ such that $[\delta(x), \delta(x^*)] - [x, x^*] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative. Recently, Dar and Khan [12] extended this result for generalized derivations (see also [2, 11, 26–29, 34] for some recent results in rings with involution). In [13], Deng and Ashraf studied SCP maps in the case of endomorphisms without involution. To be precise, they proved the following result:

Theorem 1.1 Let $\mathcal{R}$ ba a prime ring with characteristic different from two. Suppose that $\psi : \mathcal{R} \to \mathcal{R}$ is a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(y)] - [x, y] \in \mathcal{Z}(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.

In view of Theorem 1.1 just mentioned above, it is natural to ask a question. What happens if we take x^* in place of y in this result ? The main purpose of this paper is to answer the above mentioned question. Precisely, we prove that if a prime ring $\mathcal{R}$ with involution of characteristic different from two admits a non-trivial endomorphism ψ such that $[\psi(x), \psi(x^*)] - [x, x^*] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4 and $[\psi(x), x] = 0$ for all $x \in \mathcal{R}$. In order to give the complete answer to a question just mentioned above, first we need to prove that any centralizing (resp. commuting) involution on s_4-torsion free prime ring is of the first kind. Finally, we conclude the paper with some open problems.

2. The results

Inspired from above mentioned result, we prove the following theorems in the setting of rings with involution.

Theorem 2.1 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(x^*)] - [x, x^*] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4 and $[\psi(x), x] = 0$ for all $x \in \mathcal{R}$.

Theorem 2.2 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(x^*)] - (x \circ x^*) \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4.

Corollary 2.3 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(y)] - (x \circ y) \in \mathcal{Z}(\mathcal{R})$ for all $x, y \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4.

The proof of main theorems is based on the key Proposition 2.4 below. In proof of Proposition 2.4, we will use the theory of commuting and centralizing maps, studied and given by Brešar (viz.; [6–8] and references therein). We refer the reader to [7] for introductory account on the theory of commuting and centralizing maps. For full treatment on this theory we refer the reader to [6]. We begin with our first result which is motivated by the result proved in [1, 11], respectively. The proof of our first proposition relies heavily on the classical theorem and techniques due to Brešar [7], although we have attempted to make our exposition as self-contained as possible. In particular, in Proposition 2.4 below, we adapt Brešar's argument [7, Theorem 3.2] and characterizes centralizing involution of prime rings. We begin with the following propositions which are necessary for establishment of the proofs of our main results of the present paper.

Proposition 2.4 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If $[x, x^*] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4.

Proof. By the assumption, we have

$$[x, x^*] \in \mathcal{Z}(\mathcal{R}) \tag{2.1}$$

for all $x \in \mathcal{R}$. In view of [7, Theorem 3.2], there exists $\lambda \in C$ and an additive map $\xi : \mathcal{R} \to C$ such that

$$x^* = \lambda x + \xi(x) \tag{2.2}$$

for all $x \in \mathcal{R}$. Taking $x^* = x$ in (2.2), we get

$$x = \lambda x^* + \xi(x^*) \tag{2.3}$$

for all $x \in \mathcal{R}$. Application of (2.2) yields

$$x = \lambda(\lambda x + \xi(x)) + \xi(x^*) = \lambda^2 x + \lambda\xi(x) + \xi(x^*) \tag{2.4}$$

for all $x \in \mathcal{R}$. By commuting above relation with y, it follows

$$[x, y] = \lambda^2[x, y] \tag{2.5}$$

for all $x, y \in \mathcal{R}$. Now commuting (2.2) with y, we obtain

$$[x^*, y] = \lambda[x, y] \tag{2.6}$$

for all $x, y \in \mathcal{R}$. Left multiplication of (2.6) with $\lambda \in C$ and application of (2.5) yields

$$\lambda[x^*, y] = \lambda^2[x, y] = [x, y] \tag{2.7}$$

for all $x, y \in \mathcal{R}$. Replace x by yx in (2.7), we obtain

$$\lambda[x^*, y]y^* = y[x, y] \tag{2.8}$$

for all $x, y \in \mathcal{R}$. Right multiplication of (2.7) with y^* yields

$$\lambda[x^*, y]y^* = [x, y]y^* \tag{2.9}$$

for all $x, y \in \mathcal{R}$. In view of (2.8) and (2.9), we have

$$[x, y]y^* = y[x, y] \tag{2.10}$$

for all $x, y \in \mathcal{R}$. Again replacing x by y^*x in (2.7) and proceeding as above, we arrive at

$$[x, y]y = y^*[x, y] \tag{2.11}$$

for all $x, y \in \mathcal{R}$. Combination of (2.10) with (2.11) gives

$$[x, y](y^* + y) = (y^* + y)[x, y] \tag{2.12}$$

Replacement of y by $h + k$, where $h \in \mathcal{H}(\mathcal{R})$ and $k \in \mathcal{S}(\mathcal{R})$, in above relation yields

$$[[x, h], h] + [[x, k], h]] = 0 \tag{2.13}$$

for all $x \in \mathcal{R}$, $h \in \mathcal{H}(\mathcal{R})$ and $k \in \mathcal{S}(\mathcal{R})$. Replacing h by $-h$ in above expression and combining with (2.13), we get

$$[[x, h], h]] = 0 \tag{2.14}$$

for all $x \in \mathcal{R}$ and $h \in \mathcal{H}(\mathcal{R})$. It can be easily seen that $\delta_x(h) = [x, h]$ is a derivation on $\mathcal{R}$. If $\delta_x(h)$ is nonzero, then it follows from [18, Theorem 2] that $\mathcal{R}$ satisfies s_4, the standard polynomial identity in four variables. On the other hand, if $\delta_x(h) = 0$, then the involution is of the first kind. This completes the proof.

A very immediate corollary of Proposition 2.4 is the following result.

Corollary 2.5 Let $\mathcal{R}$ be a s_4- free prime ring with involution of characteristic different from two. Then, any centralizing (resp. commuting) involution is of the first kind.

The following result is interesting in itself.

Proposition 2.6 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If $x \circ x^* \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4.
Proof. Direct linearization to the given hypotheses gives

$$x \circ y^* + y \circ x^* \in \mathcal{Z}(\mathcal{R}) \tag{2.15}$$

for all $x, y \in \mathcal{R}$. The last expression can be written as

$$[x \circ y, r] + [y^* \circ x^*, r] = 0 \tag{2.16}$$

for all $x, y, r \in \mathcal{R}$. Substitution $x = y$ in (2.16) and using the fact that char($\mathcal{R}$) $\neq 2$, we obtain

$$[x^2, r] + [x^{*2}, r] = 0 \tag{2.17}$$

for all $x, r \in \mathcal{R}$. Replacing x by x^2 and taking $y \in \mathcal{Z}(\mathcal{R}) \setminus \{0\}$ in (2.16), we find

$$[x^2, r]y + [x^{*2}, r]y^* = 0 \tag{2.18}$$

for all $x, r \in \mathcal{R}$ and $y \in \mathcal{Z}(\mathcal{R})$. Right multiplication by y^* to (2.17)yields

$$[x^2, r]y^* + [x^{*2}, r]y^* = 0 \tag{2.19}$$

for all $x, r \in \mathcal{R}$ and $y \in \mathcal{Z}(\mathcal{R})$. Combining last two relations, we get

$$[x^2, r]R(y - y^*) = 0 \tag{2.20}$$

for all $x, r \in \mathcal{R}$ and $y \in \mathcal{Z}(\mathcal{R})$. Invoking the primeness of $\mathcal{R}$, we find either $[x^2, r] = 0$ or $(y - y^*) = 0$. First we consider the case $[x^2, r] = 0$ for all $x, r \in \mathcal{R}$. Substituting $x + z$ for x, where $z \in \mathcal{Z}(\mathcal{R}) \setminus \{0\}$ in the last relation and using the fact that char($\mathcal{R}$) $\neq 2$, we get $[xz, r] = 0$ for all $x, r \in \mathcal{R}$. Since $z \in \mathcal{Z}(\mathcal{R}) \setminus \{0\}$, we conclude that $[x, r] = 0$ for all $x, r \in \mathcal{R}$. This implies that $[x, x^*] = 0$ for all $x \in \mathcal{R}$. Application of Proposition 2.4 yields the required result. On the other hand, if $y - y^* = 0$ for $y \in \mathcal{Z}(\mathcal{R}) \setminus \{0\}$. Then, in view of relation (2.15), we conclude that $(x + x^*)y \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. The primeness of $\mathcal{R}$ gives $(x + x^*) \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. This further implies that $[x, x^*] = 0$ for all $x \in \mathcal{R}$. Hence, the involution is of the first kind or $\mathcal{R}$ satisfies s_4 by Proposition 2.4. This proves the proposition.

Corollary 2.7 Let $\mathcal{R}$ be a s_4- free prime ring with involution of characteristic different from two. If $x \circ x^* \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind.

Now, we are ready to prove our main results:

Proof of Theorem 2.1. By the assumption, we have

$$[\psi(x), \psi(x^*)] - [x, x^*] \in \mathcal{Z}(\mathcal{R}) \tag{2.21}$$

for all $x \in \mathcal{R}$. In particular,

$$[\psi(x), \psi(x^*)] = [x, x^*] + \gamma \tag{2.22}$$

for all $x \in \mathcal{R}$ and for some $\gamma \in \mathcal{Z}(\mathcal{R})$. Linearization of (2.21) gives

$$[\psi(x), \psi(y^*)] + [\psi(y), \psi(x^*)] - [x, y^*] - [y, x^*] \in \mathcal{Z}(\mathcal{R}) \tag{2.23}$$

for all $x, y \in \mathcal{R}$. Taking y as x^*x in above relation, we have

$$[\psi(x), \psi(x^*)]\psi(x) + \psi(x^*)[\psi(x), \psi(x^*)] - [x, x^*]x - x^*[x, x^*] \in \mathcal{Z}(\mathcal{R}) \tag{2.24}$$

for all $x \in \mathcal{R}$. Application of (2.22) yields

$$[x, x^*]\psi(x) + \gamma\psi(x) + \psi(x^*)[x, x^*] + \gamma\psi(x^*) - [x, x^*]x - x^*[x, x^*] \in \mathcal{Z}(\mathcal{R})$$

for all $x \in \mathcal{R}$. For $x = h \in \mathcal{H}(\mathcal{R})$, we get $\gamma\psi(h) \in \mathcal{Z}(\mathcal{R})$ for all $h \in \mathcal{H}(\mathcal{R})$. Since $\mathcal{R}$ is a prime ring, so either $\gamma = 0$ or $\psi(h) \in \mathcal{Z}(\mathcal{R})$ for all $h \in \mathcal{H}(\mathcal{R})$. Since xx^* and $x^*x \in \mathcal{H}(\mathcal{R})$, so one can observe that $[\psi(x), \psi(x^*)] \in \mathcal{Z}(\mathcal{R})$. In view of (2.21), we have $[x, x^*] \in \mathcal{Z}(\mathcal{R})$. Thus the result follows form Proposition 2.4. Now consider the situation when $\gamma = 0$ and assume the involution is of the second kind. This reduces (2.22) into

$$[\psi(x), \psi(x^*)] - [x, x^*] = 0 \tag{2.25}$$

for all $x \in \mathcal{R}$. Reasoning as above, we get

$$[x, x^*]\psi(x) + \psi(x^*)[x, x^*] - [x, x^*]x - x^*[x, x^*] = 0 \tag{2.26}$$

for all $x \in \mathcal{R}$. That is

$$[x, x^*](\psi(x) - x) + (\psi(x^*) - x^*)[x, x^*] = 0 \tag{2.27}$$

for all $x \in \mathcal{R}$. Replacing x by $h + k$ in (2.26), where $h \in \mathcal{H}(\mathcal{R})$ and $k \in \mathcal{S}(\mathcal{R})$, we have

$$[k, h](\psi(h) - h) + [k, h](\psi(k) - k) = (\psi(h) - h)[h, k] - (\psi(k) - k)[h, k] \tag{2.28}$$

for all $h \in \mathcal{H}(\mathcal{R})$ and $k \in \mathcal{S}(\mathcal{R})$. Taking k as $-k$ in above expression and combine with (2.28), we obtain

$$[k,h](\psi(h)-h) = (\psi(h)-h)[h,k] \tag{2.29}$$

for all $h \in \mathcal{H}(\mathcal{R})$ and $k \in \mathcal{S}(\mathcal{R})$. Now we want to show that $\psi(x) - x \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. We claim that

$$[x,h](\psi(h)-h) = (\psi(h)-h)[h,x] \tag{2.30}$$

for all $x \in \mathcal{R}$ and $h \in \mathcal{H}(\mathcal{R})$. Indeed, if $x \in \mathcal{S}(\mathcal{R})$, then the relation (2.30) is true for all $x \in \mathcal{R}$. Now if $x \in \mathcal{H}(\mathcal{R})$, then take xk_1 instead of k in (2.29), where $k_1 \in \mathcal{S}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$ and using the fact that the involution is of the second kind. But then, since every $x \in \mathcal{R}$ can be represented as $2x = h + k$, $h \in \mathcal{H}(\mathcal{R})$, $k \in \mathcal{S}(\mathcal{R})$, it follows that (2.30) holds for all $x \in \mathcal{R}$. Next fix h. It is straightforward to show that the map defined as $x \mapsto [x,h]$, is a derivation. In view of [15, Theorem 1], we observe that either $h \in \mathcal{Z}(\mathcal{R})$ for all $h \in \mathcal{H}(\mathcal{R})$ or $\psi(h) - h \in \mathcal{Z}(\mathcal{R})$ for all $h \in \mathcal{H}(\mathcal{R})$. First case implies that the involution is of the first kind, a contradiction. Therefore, we assume that $\psi(h) - h \in \mathcal{Z}(\mathcal{R})$ for all $h \in \mathcal{H}(\mathcal{R})$. One can easily obtain from above that $\psi(k) - k \in \mathcal{Z}(\mathcal{R})$ for all $k \in \mathcal{S}(\mathcal{R})$. Reasoning as above, we get $\psi(x) - x \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. Henceforward, we conclude that $[\psi(x), x] = 0$ for all $x \in \mathcal{R}$. This completes the proof of the theorem.

Proof of Theorem 2.2. We are given that ψ is a non-trivial endomorphism on $\mathcal{R}$ and satisfying the relation

$$[\psi(x), \psi(x^*)] + x \circ x^* \in \mathcal{Z}(\mathcal{R}) \tag{2.31}$$

for all $x \in \mathcal{R}$. Substituting x^* for x in (2.31) and using the facts that $[x^*, x] = -[x, x^*]$ and $x^* \circ x = x \circ x^*$, we acquire

$$-[\psi(x), \psi(x^*)] + x \circ x^* \in \mathcal{Z}(\mathcal{R}) \tag{2.32}$$

for all $x \in \mathcal{R}$. Combining the relations (2.31) and (2.32), we get $x \circ x^* \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. Application of Proposition 2.6 yields the required result.

By symmetry, we also have the following results whose proofs parallels to that of Theorems 2.1 & 2.2.

Theorem 2.8 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(x^*] + (x \circ x^*) \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4.

Theorem 2.9 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $\psi(x) \circ \psi(x^*) + [x, x^*] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4.

Corollary 2.10 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $\psi(x)\psi(x^*) - xx^* \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4.

Proof. We assume that $\psi(x)\psi(x^*) - xx^* \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. This can be rewritten as $\psi(x^*)\psi(x) - x^*x \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. Combining the obtained expressions, we get $[\psi(x), \psi(x^*)] - (x \circ x^*) \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. In view of Theorem 2.2, we conclude the required result.

Corollary 2.11 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $\psi(x) \circ \psi(x^*) - xx^* \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4.

Corollary 2.12 Let $\mathcal{R}$ be a prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(x^*)] - xx^* \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind or $\mathcal{R}$ satisfies s_4.

Corollary 2.13 Let $\mathcal{R}$ be a s_4- free prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(x^*)] - xx^* \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then the involution is of the first kind.

Corollary 2.14 Let $\mathcal{R}$ be a s_4- free prime ring with involution of characteristic different from two. If ψ is a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(y)] - xy \in \mathcal{Z}(\mathcal{R})$ for all $x, y \in \mathcal{R}$, then the involution is of the first kind.

In view of the above discussions, we conclude our paper with the following open problems.

Open Problem 2.15 Let $\mathcal{R}$ be a semiprime ring with involution and with suitable characteristic. Is any centralizing involution on $\mathcal{R}$ of the first kind or $\mathcal{R}$ satisfies s_4, the standard polynomial identity in four variables ?

Open Problem 2.16 Let $\mathcal{R}$ be a semiprime ring with involution and with suitable characteristic. Next, ψ be a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(x^*)] - [x, x^*] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. Then what we can say about the behaviour of involution or the structure of $\mathcal{R}$?

Open Problem 2.17 Let $\mathcal{R}$ be a semiprime ring with involution and with suitable characteristic. Next, ψ be a non-trivial endomorphism on $\mathcal{R}$ such that $[\psi(x), \psi(x^*)] - (x \circ x^*) \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. Then what we can say about the behaviour of involution or the structure of $\mathcal{R}$?

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Conflict of interest

The authors declare no conflict of interest.

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