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Research article

Fundamental units for real quadratic fields determined by continued fraction conditions

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Abstract: The aim of this paper is to obtain the real quadratic fields $\mathbb{Q}\left(\sqrt{d}\right)$ including

$$\omega_d = \left[a_0;;\underbrace{\overline{\gamma,\gamma,\ldots,\gamma,a_l}}_{l-1}\right]$$

where $l = l(d)$ is the period length and γ is a positive odd integer. Moreover, we have considered a new perspective to determine the fundamental units ϵ_d and got important results on Yokoi's invariants n_d and m_d [since they satisfy necessary and sufficient conditions related to Ankeny-Artin-Chowla conjecture (A.A.C.C), give bounds for fundamental units and so on...] for such types of fields.

Keywords: continued fraction expansion; real quadratic number field; fundamental unit; special sequence; Yokoi's invariants
Mathematics Subject Classification: 11A55, 11R11, 11R27, 11R29, 11Y55, 11K83

1. Introduction

Zhang and Yue [33], established some congruence relations on the coefficients of the fundamental units for the real quadratic fields $\mathbb{Q}(\sqrt{d})$ while class number is odd. Similarly [28], Williams and Buck compared the lengths of the continued fraction expansions of $\sqrt{d}$ and $\frac{1}{2}\left(1+\sqrt{d}\right)$ and worked on Eisenstein's problem. Tomita and Yamamuro [27] obtained new sharpener lower bound for ϵ_d and determined continued fraction expansion of integral basis element using the Fibonacci integer sequence. Tomita also described explicitly ϵ_d by determining the continued fraction expansion of ω_d where the period length is equal to 3 in the reference [26]. Chakraborty et al. proved a new alternative class number formula for real quadratic fields with discriminant $\triangle \equiv 5(mod 8)$ [4]. In the reference [9], the author Jeangho investigated a real quadratic integer ξ with fixed norm in the real quadratic field of $\mathbb{Q}(\xi)$ and proved real equation's solvability as well as the quadratic progressions for fundamental units. He also demonstrated that prime ideals lying over fixed norm are principal by considering fixed norm equals to -1 or prime.

In recent years, Benamar and co-authors gave the lower bound of the number of special polynomials with a stable period continued fraction expansion in the reference [3]. Badziahin and Shallit [2] considered the some real transcendental numbers σ with the explicit type of continued fraction expansions. Tomita and Kawamoto [10] showed a relation between the real quadratic fields of class number one and a mysterious behavior of the simple continued fraction expansion of certain quadratic irrationals. Besides, some authors expressed important results for several kinds of continued fractions and the real quadratic number fields in their papers. So, one may consider [1, 5–8, 12–14, 23–25] references for getting more information about the continued fraction expansions and the fundamental units.

The authors Kim and Ryu [11] worked on the special class circular units ϵ_k of $\mathbb{Q}\left(\sqrt{pq}\right)$ and investigated unit group of such real quadratic field. They also proved class number of quartic fields $\mathbb{Q}\left(\sqrt{-p}, \sqrt{-q}\right)$ by using Sinnott's index formula. In generally, papers include specifying a very short period length but McLaughlin [13] took the very reasonable step of restricting to the case where the period need not be short, but with the constraint that all. By the way, Sasaki [24] and Mollin [14] studied bounds of ϵ_d for real quadratic fields $\mathbb{Q}\left(\sqrt{d}\right)$ and obtained useful significant results for that. Yokoi defined several invariants to solve class number problems and solutions of the Diophantine equations in the terms of coefficients of fundamental unit in the references [29–32]. Furthermore, the author of this paper proved some special and significant results in the references [15–22].

This paper determines the fundamental unit problem. The following has been identified; Continued fraction expansions have individual elements equal to each other's and also written as γs (except the last digit of the period) for a γ positive odd integer where d is square free integer and equivalent to 1 for modulo 4. We also state the fact that, infinitely countable values have been classified and generalized as of d having all γs in the symmetric part of period length of ω_d. By using our newly identified results, we have concluded that the general forms of the following structures are found out for such real quadratic number fields;

(1) Radicand d
(2) Fundamental units ϵ_d
(3) Coefficients of fundamental unit t_d, u_d
(4) Yokoi's invariants n_d, m_d

Also, additional results on fundamental units, Yokoi's invariants, continued fraction expansions and period lengths are empirically revealed and validated with this paper as well as related published papers.

2. Preliminaries

In this part, we give some basic and useful informations. Throughout this paper, we fix some notations and provide them as follows:

Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic number field for $d > 1$ square-free integer, the integral basis element ω_d is a part of $Z[\omega_d]$ and $l(d)$ is the period length of continued fraction expansion of ϵ_d, the fundamental unit ϵ_d of real quadratic number field is denoted by $\epsilon_d = \frac{t_d+u_d\sqrt{d}}{2} > 1$ such that $N(\epsilon_d) = (-1)^{l(d)}$ in this paper.

Note: The set $I(d)$ contains all quadratic irrational numbers in $k = \mathbb{Q}(\sqrt{d})$. α in $I(d)$ is defined as reduced if $\alpha > 1$, $1 < \alpha' < 0$ (α' is the conjugate of α with respect to $\mathbb{Q}$). Also, $R(d)$ is the set of all reduced quadratic irrational numbers in $I(d)$. Besides, it is well known that any number α in $R(d)$ has purely periodic continued fraction expansion and the denominator of its modular automorphism is equal to fundamental unit ϵ_d of $k = \mathbb{Q}(\sqrt{d})$. Yokoi's invariants defined by H. Yokoi are determined by the coefficients of the fundamental unit $\epsilon_d = \frac{t_d+\ u_d\sqrt{d}}{2}$ as $m_d = \lfloor \frac{u_d^2}{t_d} \rfloor$ and $n_d = \lfloor \frac{t_d}{u_d^2} \rfloor$ where $\lfloor x \rfloor$ represents the floor of x.

Definition 2.1. Let's fix a positive odd integer γ and define $\{c_i\}$ sequence with the recurrence relation

$$c_i = \gamma c_{i-1} + c_{i-2}$$

for $i \geq 2$ with the seed values $c_0 = 0$ and $c_1 = 1$.

Remark 2.1. If $\{c_i\}$ sequence is defined as Definition 2.1, then we state the following congruence for $\gamma \equiv 1(mod4)$:

$$c_n \equiv \begin{cases} 0\,(mod4)\,, & n \equiv 0\,(mod6)\,; \\ 1\,(mod4)\,, & n \equiv 1,2,5\ \ (mod6)\,; \\ 2\,(mod4)\,, & n \equiv 3\,(mod6)\,; \\ 3\,(mod4)\,, & n \equiv 4\,(mod6)\,. \end{cases}$$

and also, for $\gamma \equiv 3\,(mod4)$:

$$c_n \equiv \begin{cases} 0\,(mod4)\,, & n \equiv 0\,(mod6)\,; \\ 1\,(mod4)\,, & n \equiv 1,4,5\ \ (mod6)\,; \\ 3\,(mod4)\,, & n \equiv 2\,(mod6)\,; \\ 2\,(mod4)\,, & n \equiv 3\,(mod6)\,. \end{cases}$$

is satisfied where $n \geq 0$.

Lemma 2.1. Let d be a square-free positive integer. If we put $\omega_d = \frac{\sigma-1+\sqrt{d}}{\sigma}$, $a_0 = \lfloor \omega_d \rfloor$ into the ω_R, then we get $\omega_d \notin R(d)$ but $\omega_R \in R(d)$. Also, let $\omega_R = \frac{P_l\omega_R+P_{l-1}}{Q_l\omega_R+Q_{l-1}}$ be an image of the ω_R under the particular automorphism, then the fundamental unit ϵ_d of $\mathbb{Q}(\sqrt{d})$ is determined as follows:

(i) If d is congruent to 1 modulo 4, then $\sigma = 2$ with

$$\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$$

and

$$t_d = (2a_0 - 1)\,.Q_{l(d)} + 2Q_{l(d)-1}, u_d = Q_{l(d)}$$

(ii) If d is congruent to 2 or 3 modulo 4 then $\sigma = 1$ as well as

$$\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = \omega_R.Q_{l(d)} + Q_{l(d)-1} > 1,$$

and

$$t_d = 2a_0 Q_{l(d)} + 2Q_{l(d)-1}, u_d = 2Q_{l(d)}$$

Since the denominator of the particular automorphism is the fundamental unit of $\mathbb{Q}(\sqrt{d})$ with norm $(-1)^{l(d)}$ where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{(i+1)} = a_i Q_i + Q_{i-1}$, $(i \geq 1)$.

Proof: (i) Proof of the part in Lemma 2.1. was demonstrated in the reference [26, Lemma 1, pp 41].
(ii) It is trivial that second part of Lemma 2.1 can be proven with the consideration of proof (i).

3. Main theorems and results

Our theorems and results are given as follows:

Theorem 3.1. Let d be square-free positive integer and $l > 1$ be a positive integer.

(1) We suppose

$$d = (2tc_l + \gamma)^2 + 8tc_{l-1} + 4$$

for $t > 0$ positive integer. In this case, we obtain that $d \equiv 1(mod4)$ and

$$\omega_d = \left[tc_l + \frac{\gamma+1}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, 2tc_l + \gamma}\right]$$

and $l = l(d)$. Moreover, we obtain

$$t_d = 2tc_l{}^2 + \gamma c_l + 2c_{l-1}, u_d = c_l$$

for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$. Conversely, $d \equiv 1(mod4)$ is written in the terms of $\{c_i\}$ sequence, γ and t if

$$\omega_d = \left[tc_l + \frac{\gamma+1}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, 2tc_l + \gamma}\right]$$

holds.

(2) Let l be divisible by 3. if we suppose

$$d = (tc_l + \gamma)^2 + 4tc_{l-1} + 4$$

for $t > 0$ positive odd integer, then we have $d \equiv 1(mod4)$ and

$$\omega_d = \left[\frac{t}{2}c_l + \frac{\gamma+1}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, tc_l + \gamma}\right]$$

with $l = l(d)$. Furthermore, in this case

$$t_d = tc_l^2 + \gamma c_l + 2c_{l-1}, u_d = c_l$$

hold for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$. On the contrary, $d \equiv 1(mod4)$ is defined in the terms of $\{c_i\}$ sequence, γ and t if ω_d is determined as above;

$$\omega_d = \left[\frac{t}{2}c_l + \frac{\gamma+1}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, tc_l + \gamma}\right].$$

Proof: (1) Let the parameterization of d be $d = (2tc_l + \gamma)^2 + 8tc_{l-1} + 4$. Since γ is positive odd integer $(2tc_l + \gamma)^2$ is positive odd integer. So, we get $d \equiv 1(mod4)$. From (i) in Lemma 2.1, we know that $\omega_d = \frac{1+\sqrt{d}}{2}$, $a_0 = \lfloor \omega_d \rfloor$ and $\omega_R = a_0 - 1 + \omega_d$. By using these equations, we obtain

$$\omega_R = \frac{(2tc_l \ + \gamma - 1)}{2} + \left[tc_l + \frac{\gamma+1}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, 2tc_l + \gamma}\right]$$

so, we get

$$\omega_R = (2tc_l + \gamma) + \frac{1}{\gamma} \ + \cdots + \ \frac{1}{\omega_R}$$

By a straightforward induction argument, we obtain

$$\omega_R^2 - (2tc_l + \gamma)\omega_R - (1 + 2tc_{l-1}) = 0.$$

This requires that $\omega_R = \frac{(2tc_l+\gamma)+\sqrt{d}}{2}$ since $\omega_R > 0$. If we consider again (i) in Lemma 2.1, we get

$$\omega_R = \left[tc_l + \frac{\gamma+1}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, 2tc_l + \gamma}\right]$$

and $l = l(d)$.

Now, we have to determine ϵ_d, t_d and u_d by using (i) in Lemma 2.1. So, we get

$$Q_1 = 1 = c_1 \ , Q_2 = a_1 Q_1 + Q_0 \Rightarrow Q_2 = \gamma = c_2 \ ,$$
$$Q_3 = a_2 \ .Q_2 + Q_1 = \gamma.c_2 + c_1 = \gamma^2 + 1 = c_3 \ , Q_4 = c_4, \ldots$$

So, this implies that $Q_i = c_i$ using induction for $\forall i \geq 0$. If we substitute these values of sequence into the $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (\omega_R) \, .Q_{l(d)} + Q_{l(d)-1} > 1$ and rearrange, we have t_d and u_d as follows:

$$t_d = 2tc_l^2 + \gamma c_l + 2c_{l-1}, u_d = c_l$$

using (i) in Lemma 2.1 for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$.

Conversely, assume that $d \equiv 1 \, (mod4) \, , \omega_d = \left[tc_l + \frac{\gamma+1}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, 2tc_l + \gamma}\right]$ holds.

Since $\omega_d = \frac{1+\sqrt{d}}{2}$ and (i) in Lemma 2.1, we obtain $d = (2tc_l + \gamma)^2 + 8tc_{l-1} + 4$ which completes the proof of (1).

(2) If we assume that $l \equiv 0(mod3)$ and the parametrization of d is given as follows:

$$d = (tc_l + \gamma)^2 + 4tc_{l-1} + 4$$

for $t > 0$ positive odd integer, then we have $d \equiv 1\,(mod4)$ since c_l is even integer. By substituting $\frac{t}{2}$ instead of t into the case (1), we get

$$\omega_d = \left[\frac{t}{2}c_l + \frac{\gamma+1}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, tc_l + \gamma}\right]$$

and $l = l(d)$. Furthermore,

$$t_d = tc_l{}^2 + \gamma c_l + 2c_{l-1}, u_d = c_l$$

hold for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$.

On the other hand, $d = (tc_l + \gamma)^2 + 4tc_{l-1} + 4$ is obtained by putting $\frac{t}{2}$ instead of t into the case (1). So, the proof of (2) is also completed.

Corollary 3.1. Assume that $d \equiv 1\,(mod4)$. If d satisfies the conditions in Theorem 3.1, then Yokoi's invariant n_d is nonzero (and hence $m_d = 0$).

Proof: In the case of (1) in Theorem 3.1,

$$n_d = \left\lfloor \frac{t_d}{u_d{}^2} \right\rfloor = \left\lfloor \frac{2tc_l{}^2 + \gamma.c_l + 2c_{l-1}}{c_l{}^2} \right\rfloor = 2t + \left\lfloor \frac{\gamma.c_l + 2c_{l-1}}{c_l{}^2} \right\rfloor$$

Since $t > 0$ is positive integer and $\{c_i\}$ is increasing sequence, we get $n_d \neq 0$ for $l > 1$. In a similar way, for the case of (2) in the Theorem 3.1, we obtain

$$n_d = \left\lfloor \frac{t_d}{u_d{}^2} \right\rfloor = t + \left\lfloor \frac{\gamma c_l + 2c_{l-1}}{c_l{}^2} \right\rfloor$$

we have $n_d \neq 0$ since $t > 0$ and $t_d > u_d{}^2$.

Remark 3.1.

(i) Let d be a square free positive integer concerning the case of (1) in the Theorem 3.1. As a result of the theorem, ϵ_d, ω_d and $n_d - m_d$ also have been calculated for $\gamma = 1, 3, 5, 7, 9$ and $t = 1, 2$ with some values of $2 \leq l(d)$ in the published papers.

(ii) Let d be the square free positive integer concerning the case of (2) in the Theorem 3.1. As a result of the theorem, ϵ_d, ω_d and $n_d - m_d$ also have been calculated for $\gamma = 1, 3, 5$ and $t = 1, 3$ with $3 \leq l(d) \leq 12$.

Proof: Readers can see all numerical results with tables in the references [15–22].

Remark 3.2. It is easily seen that the present paper has got the most general theorems for such types of real quadratic number fields since d is written by countable infinite integers $t > 0$ and $\gamma > 0$ (but γ is odd) as well as in the terms of increasing integer sequence $\{c_i\}$.

Theorem 3.2. Let d be the square free positive integer and $l > 1$ be a positive integer not divisible by 3. We assume that parameterization of d is

$$d = \frac{(\gamma + (2t+1)c_l)^2}{4} + (2t+1)c_{l-1} + 1$$

for $t > 0$ positive integer. Assume that $\gamma \equiv 1(mod4)$ is any positive odd integer, then we have the followings:

(1) If $l \equiv 1(mod6)$ and $t \equiv 0(mod2)$ are positive integers, then $d \equiv 2(mod4)$ holds.
(2) If $l \equiv 2(mod6)$ and $t \equiv 0(mod2)$ are positive integers, then $d \equiv 3(mod4)$ holds.
(3) If $l \equiv 4(mod6)$ and $t \equiv 0(mod2)$ are positive integers, then $d \equiv 3(mod4)$ holds.
(4) If $l \equiv 5(mod6)$ and $t \equiv 1(mod2)$ are positive integers, then $d \equiv 2(mod4)$ holds.

Also, suppose that $\gamma \equiv 3(mod4)$ is any positive odd integer, then we have the followings:

(1*) If $l \equiv 1(mod6)$ and $t \equiv 1(mod2)$ are positive integers then $d \equiv 2(mod4)$ holds.
(2*) If $l \equiv 2(mod6)$ and $t \equiv 0(mod2)$ are positive integers then $d \equiv 3(mod4)$ holds.
(3*) If $l \equiv 4(mod6)$ and $t \equiv 0(mod2)$ are positive integers then $d \equiv 3(mod4)$ holds.
(4*) If $l \equiv 5(mod6)$ and $t \equiv 0(mod2)$ are positive integers then $d \equiv 2(mod4)$ holds.

Then, we obtain

$$\omega_d = \left[\frac{(2t+1)\,c_l + \gamma}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, (2t{+}1)c_l + \gamma}\right]$$

and $l = l(d)$. Moreover, we have the following equalities:

$$\epsilon_d = \left(\frac{(2t+1)c_l^2}{2} + \frac{\gamma c_l}{2} + c_{l-1}\right) + c_l\sqrt{d}$$

$$t_d = (2t{+}1)c_l^2 + \gamma c_l + 2c_{l-1}, u_d = 2c_l,$$

for ϵ_d, t_d and u_d. Conversely is also true.

Proof: It is clear that d is not integral while $l \equiv 0(mod3)$ considering Remark 2.1. So, we assume that d is not divisible by 3 in order to get integer d. We first assume that $l \equiv 1(mod4)$ positive odd integer, $l \equiv 1(mod6)$, $l > 1$ and $t \equiv 0(mod2)$ positive integer. So, we get $d \equiv 2(mod4)$ by substituting the equivalent results into the parameterization of d. We obtain the other cases in a similar way.

By using (ii) in Lemma 2.1, we have

$$\omega_R = \frac{(2t+1)\,c_l + \gamma}{2} + \left[\frac{(2t+1)\,c_l + \gamma}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, (2t{+}1)c_l + \gamma}\right],$$

So, we get

$$\omega_R = ((2t+1)c_l + \gamma) + \frac{1}{\gamma \ + \cdots + \ \frac{1}{\omega_R}}$$

By a straightforward induction algorithm, we get

$$\omega_R = ((2t+1)\,c_l + \gamma) + \frac{c_{l-1}\omega_R + c_{l-2}}{c_l\omega_R + c_{l-1}}$$

By rearranging the Definition 2.1 into above equality, we obtain

$$\omega_R{}^2 - ((2t+1)c_l \ + \gamma)\omega_R - (1 + (2t+1)c_{l-1}) = 0$$

This requires that $\omega_R = \frac{(2t+1)c_l}{2} + \sqrt{d}$ since $\omega_R > 0$. If we consider (ii) in Lemma 2.1, we get

$$\omega_d = \sqrt{d} = \left[\frac{(2t+1)\,c_l + \gamma}{2}; \overline{\underbrace{\gamma, \gamma, \ldots, \gamma}_{l-1}, (2t+1)c_l + \gamma}\right]$$

and $l = l\,(d)$. This shows that the part of continued fraction expansions is completed.

Now, we should determine ϵ_d, t_d and u_d by using (ii) in Lemma 2.1. Thereby, we have

$$Q_1 = 1 = c_1\ , Q_2 = a_1 Q_1 + Q_0 \Rightarrow Q_2 = \gamma = c_2,$$

$$Q_3 = a_2\ .Q_2 + Q_1 = \gamma c_2 + c_1 = \gamma^2 + 1 = c_3, Q_4 = c_4, \ldots$$

So, this implies that $Q_i = c_i$ by using mathematical induction for $\forall i \geq 0$. If we substitute these values of sequence into the $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (\omega_R)\,.Q_{l(d)} + Q_{l(d)-1} > 1$ and rearranged, we have t_d and u_d using (ii) in Lemma 2.1 as follows:

$$\epsilon_d = \left(\frac{(2t+1)c_l{}^2}{2} + \frac{\gamma c_l}{2} + c_{l-1}\right) + c_l\sqrt{d}$$

$$t_d = (2t+1)c_l{}^2 + \gamma c_l + 2c_{l-1}, u_d = 2c_l,$$

for ϵ_d, t_d and u_d which complete the proof of Theorem 3.2. Conversely, it is also true and trivial that if we consider (ii) in Lemma 2.1 and the definition of $\omega_d = \sqrt{d}$.

Remark 3.3. Let d be the square free positive integer concerning Theorem 3.2. As a result of the theorem ω_d, ϵ_d and $n_d - m_d$ also have been calculated for $\gamma = 1, 3, 5, 7$ and $t = 0, 1, 2, 3$ with some values of $2 \leq l(d)$.

Proof: Readers can see all numerical results with tables in the references [15–22].

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Conflict of interest

The author declares that there is no conflict of interests.

References

1. J. Buchmann, T. Tagaki, U. Vollmer, *Number field cryptography*, Fields Institute Communication, **41** (2004), 111–121.
2. D. Badziahin, J. Shallit, *An unusual continued fraction*, Proc. Amer. Math. Soc., **144** (2016), 1887–1896.
3. H. Benamar, A. Chandoul, M. Mkaouar, *On the continued fraction expansion of fixed period in finite fields*, Canada Math. Bull., **58** (2015), 704–712.
4. K. Chakraborty, S. Kanemitsu, T. Kuzumaki, *On the class number formula of certain real quadratic fields*, Hardy Ramanujan J., **36** (2013), 1–7.
5. L. E. Clemens, K. D. Merill, D. W. Roeder, *Continued fractions and series*, J. Number Theory, **54** (1995), 309–317.
6. N. Elezovic, *A note on continued fractions of quadratic irrationals*, Math. Commun., **2** (1997), 27–33.
7. C. Friesen, *On continued fraction of given period*, Proc. Amer. Math. Soc., **10** (1988), 9–14.
8. F. Halter-Koch, *Continued fractions of given symmetric period*, Fibonacci Quart., **29** (1991), 298–303.
9. P. Jeangho, *Notes on the quadratic integer and real quadratic number field*, Osaka J. Math., **53** (2016), 983–1002.
10. F. Kawamoto, K. Tomita, *Continued fraction and certain real quadratic fields of minimal type*, J. Math. Soc., **60** (2008), 865–903.
11. J. M. Kim, J. Ryu, *On the class number and fundamental unit of the quadratic field* $k = \mathbb{Q}\left(\sqrt{p.q}\right)$, Bull. Aus. Math. Soc., **85** (2012), 359–370.
12. S. Louboutin, *Continued fraction and real quadratic fields*, J. Number Theory, **30** (1988), 167–176.
13. J. McLaughlin, *Polynomial Solutions to Pell's Equation and Fundamental Units in Real Quadratic Fields*, J. London Math. Soc., **67** (2003), 16–28.
14. R. A. Mollin, *Quadratics*, 3^{rd} Ed, CRC Press, Boca Raton, FL, 1996.
15. Ö. Özer, *On Real Quadratic Number Fields Related With Specific Type of Continued Fractions*, J. Anal. Number Theory, **4** (2016), 85–90.
16. Ö. Özer, *Notes On Especial Continued Fraction Expansions and Real Quadratic Number Fields*, J. Eng. Sci., **2** (2016), 74–89.
17. Ö. Özer, *Fibonacci Sequence and Continued Fraction Expansions in Real Quadratic Number Fields*, Malays. J. Math. Sci., **11** (2017), 97–118.

18. Ö. Özer, *On The Fundamental Units of Certain Real Quadratic Number Fields*, Karaelmas Sci. Eng. J., **7** (2017), 160–164.
19. Ö. Özer, A. B. M. Salem, *A Computational Technique For Determining Fundamental Unit in Explicit Type of Real Quadratic Number Fields*, Int. J. Adv. Appl. Sci., **4** (2017), 22–27.
20. Ö. Özer, *A Study On The Fundamental Unit of Certain Real Quadratic Number Fields*, Turk. J. Anal. Number Theory (TJANT), **6** (2018), 1–8.
21. Ö. Özer, B. Djamel, *Some Results on Special Types of Real Quadratic Fields*, Bulletin of the Karaganda Univeristy-Mathematics, **3** (2019), 48–58.
22. Ö. Özer, *A Handy Technique For Fundamental Unit in Specific Type of Real Quadratic Fields*, Appl. Math. Nonlinear Sci. (AMNS), **4** (2019), 1–4 .
23. O. Perron, Die Lehre von den Kettenbrchen, Band I: Elementare Kettenbrche, 1977.
24. R. Sasaki, *A characterization of certain real quadratic fields*, Proc. Japan Acad. Series A, **62** (1986), 97–100.
25. W. Sierpinski, *Elementary Theory of Numbers*, Warsaw, Monogra Matematyczne, 1964.
26. K. Tomita, *Explicit representation of fundamental units of some quadratic fields*, Proc. Japan Acad. Series A, **71** (1995), 41–43.
27. K. Tomita, K. Yamamuro, *Lower bounds for fundamental units of real quadratic Fields*, Nagoya Math. J., **166** (2002), 29–37.
28. K. S. Williams, N. Buck, *Comparison of the lengths of the continued fractions of $\sqrt{d}$ and $\frac{1+\sqrt{d}}{2}$*, Proc. Amer. Math. Soc., **120** (1994), 995–1002.
29. H. Yokoi, *The fundamental unit and class number one problem of real quadratic fields with prime discriminant*, Nagoya Math. J., **120** (1990), 51–59.
30. H. Yokoi, *A note on class number one problem for real quadratic fields*, Proc. Japan Acad. Series A, **69** (1993), 22–26.
31. H. Yokoi, *The fundamental unit and bounds for class numbers of real quadratic fields*, Nagoya Math. J., **124** (1991), 181–197.
32. H. Yokoi, *New invariants and class number problem in real quadratic fields*, Nagoya Math., J., **132** (1993), 175–197.
33. Z. Zhang, Q. Yue, *Fundamental units of real quadratic fields of odd class number*, J. Number Theory, **137** (2014), 122–129.

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