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Research article

Caputo–Hadamard fractional differential equations with nonlocal fractional integro-differential boundary conditions via topological degree theory

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Abstract: This article aims to prove the existence and uniqueness of solutions to a nonlinear boundary value problem of fractional differential equations involving the Caputo–Hadamard fractional derivative with nonlocal fractional integro-differential boundary conditions. The concerned results are obtained employing topological degree for condensing maps via a priori estimate method and the Banach contraction principle fixed point theorem. Besides, two illustrative examples are presented.

Keywords: fractional differential equations; Caputo–Hadamard derivative; fractional integral conditions; existence; uniqueness; topological degree theory; condensing maps
Mathematics Subject Classification: 26A33, 34A08

1. Introduction

Fractional calculus generalizes the integer-order integration and differentiation concepts to an arbitrary(real or complex) order. Fractional calculus is the most well known and valuable branch of mathematics which gives a good framework for biological and physical phenomena, mathematical modeling of engineering, etc. To get a couple of developments about the theory of fractional differential equations, one can allude to the monographs of Hilfer [31], Kilbas et al [36], Miller and Ross [39], Oldham [40], Pudlubny [41], Sabatier et al [42], Tarasov [48] and the references therein.

At the present day, there are many results on the existence of solutions for fractional differential equations. For more details, the readers are referred to the previous studies [14, 16, 25, 29, 37] and the references therein. But here, we focus on which that uses the topological degree. This method is a powerful tool for the existence of solutions to BVPs of many mathematical models that arise in applied nonlinear analysis. Very recently F. Isaia [32] proved a new fixed theorem that was obtained via coincidence degree theory for condensing maps. To see more applications about the usefulness of coincidence degree theory approach for condensing maps in the study for the existence of solutions of certain integral equations, the reader can be referred to [7, 8, 12, 32, 34, 43–47, 50]. However, it has been observed that most of the work on the topic involves either RiemannLiouville or Caputo-type fractional derivative. Besides these derivatives, Hadamard fractional derivative is another kind of fractional derivatives that was introduced by Hadamard in 1892 [30]. This fractional derivative differs from the other ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains the logarithmic function of arbitrary exponent. Good overviews and applications of where the Hadamard derivative and the Hadamard integral arise can be found in the papers by Butzer et al [19–21]. Other important results dealing with Hadamard fractional calculus and Hadamard differential equations can be found in [10, 15, 17, 28, 35, 38, 49], as well as in the monograph by Kilbas et al [36]. In recent years, Jarad et al [33] modified the Hadamard fractional derivative into a more suitable one having physical interpretable initial conditions similar to the singles in the Caputo setting and called it Caputo–Hadamard fractional derivative. Details and properties of the modified derivative can be found in [33]. To the best of our knowledge, few results can be found in the literature concerning boundary value problems for Caputo–Hadamard fractional differential equations [1, 9, 11, 18]. On the other hand, it is well known that the nonlocal condition is more appropriate than the local condition (initial and /or boundary) to describe correctly certain features of applied mathematics and physics such as blood flow problems, chemical engineering, thermo-elasticity, population dynamics and so on [2, 4–6, 13, 22–24].

No contributions exist, as far as we know, concerning the Caputo–Hadamard fractional differential equations via topological degree theory. As a result, the goal of this paper is to enrich this academic area. Our proposed method is essentially based on the result given by F. Isaia [32] to study the existence of solutions for a class of fractional differential equations via topological degree theory. More specifically, we pose the following Caputo–Hadamard fractional differential equation of the form

$$\begin{cases} {}^{C}_{H}\mathcal{D}^{\alpha}_{1}u(t) = f(t,u(t)), \ \ t \in \mathrm{J} := [1,T], \\ a_1 u(1) + b_1 {}^{C}_{H}\mathcal{D}^{\gamma}_{1}u(1) = \lambda_1 \ {}^{H}\mathcal{I}^{\sigma_1}_{1}u(\eta_1), \ 1 < \eta_1 < T, \ \sigma_1 > 0, \\ a_2 u(T) + b_2 {}^{C}_{H}\mathcal{D}^{\gamma}_{1}u(T) = \lambda_2 \ {}^{H}\mathcal{I}^{\sigma_2}_{1}u(\eta_2), \ 1 < \eta_1 < T, \ \sigma_2 > 0, \end{cases} \tag{1.1}$$

where ${}^{C}_{H}\mathcal{D}^{\mu}_{1}$ is the Caputo–Hadamard fractional derivative order $\mu \in \{\alpha,\gamma\}$ such that $1 < \alpha \leq 2, 0 < \gamma \leq 1$, ${}^{H}\mathcal{I}^{\theta}_{1}$ is the Hadamard fractional integral of order $\theta > 0, \theta \in \{\sigma_1,\sigma_2\}$ and $f : \mathrm{J}\times\mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, $a_i, b_i, \lambda_i, i = 1,2$ are suitably chosen real constants such that

$$\begin{aligned} \Delta = &\left(a_1 - \frac{\lambda_1 (\log\eta_1)^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)\left(a_2\log T + \frac{b_2(\log T)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{\lambda_2(\log\eta_2)^{\sigma_2+1}}{\Gamma(\sigma_2+2)}\right) \\ &+ \frac{\lambda_1(\log\eta_1)^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\left(a_2\log T - \frac{\lambda_2(\log\eta_2)^{\sigma_2}}{\Gamma(\sigma_2+1)}\right) \neq 0. \end{aligned} \tag{1.2}$$

The remaining part of this manuscript is distinguished as follows: Section 2, in which we describe some basic notations of fractional derivatives and integrals, definitions of differential calculus, and important results that will be used in subsequent parts of the paper. In Section 3, based on the coincidence degree theory for condensing maps, we establish a theorem on the existence of solutions for problem (1.1) next by using the Banach contraction principle fixed point theorem, we give a uniqueness results for problem (1.1). Additionally, Section 4 provides a couple of examples to illustrate the applicability of the results developed. Finally, the paper is concluded in Section 5.

2. Preliminaries

We start this section by introducing some necessary definitions and basic results required for further developments.

Consider the space of real and continuous functions $\mathcal{U} = C([1,T],\mathbb{R})$ with topological norm $\|u\|_\infty = \sup\{|u(t)|, t \in \mathrm{J}\}$ for $u \in \mathcal{U}$. $\mathfrak{M}_\mathcal{U}$ represents the class of all bounded mappings in $\mathcal{U}$.

We state here the results given below from [3,26].

Definition 2.1. The mapping $\kappa : \mathfrak{M}_\mathcal{U} \to [0,\infty)$ for Kuratowski measure of non-compactness is defined as:

$$\kappa(B) = \inf\Big\{\varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter} \le \varepsilon\Big\}.$$

Properties 2.1. The Kuratowski measure of noncompactness satisfies some properties.

(1) $A \subset B \Rightarrow \kappa(A) \le \kappa(B)$,

(2) $\kappa(A) = 0$ if and only if A is relatively compact,

(3) $\kappa(A) = \kappa(\overline{A}) = \kappa(conv(A))$, where $\overline{A}$ and conv(A) represent the closure and the convex hull of A respectively,

(4) $\kappa(A + B) \le \kappa(A) + \kappa(B)$,

(5) $\kappa(\lambda A) = |\lambda|\kappa(A), \lambda \in \mathbb{R}$.

Definition 2.2. Let $\mathcal{T} : A \longrightarrow \mathcal{U}$ be a continuous bounded map and $A \subset \mathcal{U}$. The operator $\mathcal{T}$ is said to be κ-Lipschitz if we can find a constant $\ell \ge 0$ satisfying the following condition,

$$\kappa(\mathcal{T}(B)) \le \ell\kappa(B), \text{ for every } B \subset A.$$

Moreover, $\mathcal{T}$ is called strict κ-contraction if $\ell < 1$.

Definition 2.3. The function $\mathcal{T}$ is called κ-condensing if

$$\kappa(\mathcal{T}(B)) < \kappa(B),$$

for every bounded and nonprecompact subset B of A.
In other words,

$$\kappa(\mathcal{T}(B)) \ge \kappa(B), \text{ implies } \kappa(B) = 0.$$

Further we have $\mathcal{T} : A \longrightarrow \mathcal{U}$ is Lipschitz if we can find $\ell > 0$ such that

$$\|\mathcal{T}(u) - \mathcal{T}(v)\| \le \ell\|u - v\|, \text{ for all } u, v \in A,$$

if $\ell < 1$, $\mathcal{T}$ is said to be strict contraction.

For the following results, we refer to [32].

Proposition 2.4. *If $\mathcal{T}, \mathcal{S} : A \longrightarrow \mathcal{U}$ are κ-Lipschitz mapping with constants ℓ_1 and ℓ_2 respectively, then $\mathcal{T} + \mathcal{S} : A \longrightarrow \mathcal{U}$ are κ-Lipschitz with constants $\ell_1 + \ell_2$.*

Proposition 2.5. *If $\mathcal{T} : A \longrightarrow \mathcal{U}$ is compact, then $\mathcal{T}$ is κ-Lipschitz with constant $\ell = 0$.*

Proposition 2.6. *If $\mathcal{T} : A \longrightarrow \mathcal{U}$ is Lipschitz with constant ℓ, then $\mathcal{T}$ is κ-Lipschitz with the same constant ℓ.*

Isaia [32] present the following results using topological degree theory.

Theorem 2.7. *Let $\mathcal{K} : A \longrightarrow \mathcal{U}$ be κ-condensing and*

$$\Theta = \{u \in \mathcal{U} : \text{ there exist } \xi \in [0,1] \text{ such that } x = \xi\mathcal{K}u\}.$$

If Θ is a bounded set in $\mathcal{U}$, so there exists $r > 0$ such that $\Theta \subset \mathrm{B}_r(0)$, then the degree

$$\deg(I - \xi\mathcal{K}, \mathrm{B}_r(0), 0) = 1, \text{ for all } \xi \in [0,1].$$

Consequently, $\mathcal{K}$ has at least one fixed point and the set of the fixed points of $\mathcal{K}$ lies in $\mathrm{B}_r(0)$.

Now, we give some results and properties from the theory of of fractional calculus. We begin by defining Hadamard fractional integrals and derivatives. In what follows,

Definition 2.8. ([36]) The Hadamard fractional integral of order $\alpha > 0$, for a function $u \in L^1(\mathrm{J})$, is defined as

$$\left({}^H\mathcal{I}_1^\alpha u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s)\frac{\mathrm{ds}}{s}, \quad \alpha > 0,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}\mathrm{dt}, \quad \alpha > 0.$$

Set

$$\delta = t\frac{\mathrm{d}}{\mathrm{dt}}, \quad \alpha > 0, \; n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of α. Define the space

$$AC_\delta^n[1,T] := \left\{u : [1,T] \longrightarrow \mathbb{R} : \delta^{n-1}u(t) \in AC([1,T])\right\}.$$

Definition 2.9. ([36]) The Hadamard fractional derivative of order $\alpha > 0$ applied to the function $u \in AC_\delta^n[1,T]$ is defined as

$$\left({}^H\mathcal{D}_1^\alpha u\right)(t) = \delta^n \left({}^H\mathcal{I}_1^{n-\alpha} u\right)(t).$$

Definition 2.10. ([33, 36]) The Caputo–Hadamard fractional derivative of order $\alpha > 0$ applied to the function $u \in AC_\delta^n[1,T]$ is defined as

$$\left({}_H^C\mathcal{D}_1^\alpha u\right)(t) = \left({}^H\mathcal{I}_1^{n-\alpha} \delta^n u\right)(t).$$

Lemmas of the following type are rather standard in the study of fractional differential equations.

Lemma 2.11 ([33, 36]). *Let $\alpha > 0$, $r > 0$, $n = [\alpha] + 1$, and $a > 0$, then the following relations hold*

- ${}^H\mathcal{I}_1^\alpha \left(\log \frac{t}{a}\right)^{r-1} = \frac{\Gamma(r)}{\Gamma(\alpha+r)} \left(\log \frac{t}{a}\right)^{\alpha+r-1}$,

•

$$ {}^{C}_{H}\mathcal{D}^{\alpha}_{1}\left(\log\frac{t}{a}\right)^{r-1} = \begin{cases} \frac{\Gamma(r)}{\Gamma(r-\alpha)}\left(\log\frac{t}{a}\right)^{r-\alpha-1}, & (r>n), \\ 0, \quad r\in\{0,\dots,n-1\}. \end{cases} $$

Lemma 2.12 ([27,36]). *Let $\alpha>\beta>0$, and $u\in AC^{n}_{\delta}[1,T]$. Then we have:*

- ${}^{H}\mathcal{I}^{\alpha}_{1}\,{}^{H}\mathcal{I}^{\beta}_{1}u(t) = {}^{H}\mathcal{I}^{\alpha+\beta}_{1}u(t)$,
- ${}^{C}_{H}\mathcal{D}^{\alpha}_{1}\,{}^{H}\mathcal{I}^{\alpha}_{1}u(t) = u(t)$,
- ${}^{C}_{H}\mathcal{D}^{\beta}_{1}\,{}^{H}\mathcal{I}^{\alpha}_{1}u(t) = {}^{H}\mathcal{I}^{\alpha-\beta}_{1}u(t)$.

Lemma 2.13 ([33,36]). *Let $\alpha\geq 0$, and $n=[\alpha]+1$. If $u\in AC^{n}_{\delta}[1,T]$, then the Caputo–Hadamard fractional differential equation*

$$ \left({}^{C}_{H}\mathcal{D}^{\alpha}_{1}u\right)(t)=0, $$

has a solution:

$$ u(t)=\sum_{j=0}^{n-1}c_j\left(\log(t)\right)^j, $$

and the following formula holds:

$$ {}^{H}\mathcal{I}^{\alpha}_{1}\left({}^{C}_{H}\mathcal{D}^{\alpha}_{1}u(t)\right)=u(t)+\sum_{j=0}^{n-1}c_j\left(\log(t)\right)^j, $$

where $c_j\in\mathbb{R}$, $j=0,1,2,\dots,n-1$.

3. Main results

Before starting and proving our main result we introduce the following auxiliary lemma.

Lemma 3.1. *For a given $h\in C(\mathrm{J},\mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$ \begin{cases} {}^{C}_{H}\mathcal{D}^{\alpha}_{1}u(t)=h(t), \ \ t\in \mathrm{J}:=[1,T], \\ a_1u(1)+b_1{}^{C}_{H}\mathcal{D}^{\gamma}_{1}u(1)=\lambda_1\ {}^{H}\mathcal{I}^{\sigma_1}_{1}u(\eta_1),\ 1<\eta_1<T,\ \sigma_1>0, \\ a_2u(T)+b_2{}^{C}_{H}\mathcal{D}^{\gamma}_{1}u(T)=\lambda_2\ {}^{H}\mathcal{I}^{\sigma_2}_{1}u(\eta_2),\ 1<\eta_1<T,\ \sigma_2>0, \end{cases} \tag{3.1} $$

is given by

$$ u(t)={}^{H}\mathcal{I}^{\alpha}_{1}h(t)+\mu_1(t){}^{H}\mathcal{I}^{\sigma_1+\alpha}_{1}h(\eta_1)+\mu_2(t)(\lambda_2{}^{H}\mathcal{I}^{\sigma_2+\alpha}_{1}h(\eta_2)-(a_2{}^{H}\mathcal{I}^{\alpha}_{1}h(T)+b_2{}^{H}\mathcal{I}^{\alpha-\gamma}_{1}h(T))), \tag{3.2} $$

where

$$ \begin{aligned} \mu_1(t) &= \lambda_1(\Delta_1-\Delta_2 t), \quad \mu_2(t)=\lambda_1\Delta_3+\Delta_4 t, \\ \Delta_1 &= \frac{1}{\Delta}\left(a_2\log T+\frac{b_2(\log T)^{1-\gamma}}{\Gamma(2-\gamma)}-\frac{\lambda_2(\log\eta_2)^{\sigma_2+1}}{\Gamma(\sigma_2+2)}\right); \quad \Delta_2=\frac{1}{\Delta}\left(a_2\log T-\frac{\lambda_2(\log\eta_2)^{\sigma_2}}{\Gamma(\sigma_2+1)}\right) \\ \Delta_3 &= \frac{\lambda_1(\log\eta_1)^{\sigma_1+1}}{\Delta\Gamma(\sigma_1+2)}; \quad \Delta_4=\frac{1}{\Delta}\left(a_1-\frac{\lambda_1(\log\eta_1)^{\sigma_1}}{\Gamma(\sigma_1+1)}\right), \end{aligned} \tag{3.3} $$

and Δ is given by (1.2).

Proof. By applying Lemma 2.13, we may reduce (3.1) to an equivalent integral equation

$$u(t) = {}^{H}\mathcal{I}_1^{\alpha} h(t) + k_0 + k_1 \log(t), \quad k_0, k_1 \in \mathbb{R}. \tag{3.4}$$

Applying the boundary conditions (3.1) in (3.4) we may obtain

$$\begin{aligned} {}^{H}\mathcal{I}_1^{\sigma_i} x(\eta_i) &= {}^{H}\mathcal{I}_1^{\sigma_i+\alpha} h(\eta_i) + k_0 \frac{(\log \eta_i)^{\sigma_i}}{\Gamma(\sigma_i + 1)} + k_1 \frac{(\log \eta_i)^{\sigma_i+1}}{\Gamma(\sigma_i + 2)}, \ i = 1, 2. \\ {}^{C}_{H}\mathcal{D}_1^{\gamma} x(T) &= {}^{H}\mathcal{I}_1^{\alpha-\gamma} h(T) + k_1 \frac{\Gamma(2)}{\Gamma(2-\gamma)} (\log T)^{1-\gamma}. \end{aligned}$$

After collecting the similar terms in one part, we have the following equations:

$$\left(a_1 - \frac{\lambda_1 (\log \eta_1)^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) k_0 - \frac{\lambda_1 (\log \eta_1)^{\sigma_1+1}}{\Gamma(\sigma_1 + 2)} k_1 = \lambda_1 {}^{H}\mathcal{I}_1^{\sigma_1+\alpha} h(\eta_1). \tag{3.5}$$

$$\begin{aligned} &\left(a_2 \log T - \frac{\lambda_2 (\log \eta_2)^{\sigma_2}}{\Gamma(\sigma_2 + 1)}\right) k_0 + \left(a_2 \log T + \frac{b_2 (\log T)^{1-\gamma}}{\Gamma(2-\gamma)}\right. \\ &\left. - \frac{\lambda_2 (\log \eta_2)^{\sigma_2+1}}{\Gamma(\sigma_2 + 2)}\right) k_1 = \lambda_2 {}^{H}\mathcal{I}_1^{\sigma_2+\alpha} h(\eta_2) - (a_2 {}^{H}\mathcal{I}_1^{\alpha} h(T) + b_2 {}^{H}\mathcal{I}_1^{\alpha-\gamma} h(T)). \end{aligned} \tag{3.6}$$

Therefore, we get

$$\begin{aligned} k_0 &= \frac{\lambda_1}{\Delta}\left(a_2 \log T + \frac{b_2 (\log T)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{\lambda_2 (\log \eta_2)^{\sigma_2+1}}{\Gamma(\sigma_2 + 2)}\right) {}^{H}\mathcal{I}_1^{\alpha+\sigma_1} h(\eta_1) \\ &+ \frac{\lambda_1 (\log \eta_1)^{\sigma_1+1}}{\Delta\Gamma(\sigma_1 + 2)} (\lambda_2 {}^{H}\mathcal{I}_1^{\sigma_2+\alpha} h(\eta_2) - (a_2 {}^{H}\mathcal{I}_1^{\alpha} h(T) + b_2 {}^{H}\mathcal{I}_1^{\alpha-\gamma} h(T))), \\ k_1 &= \frac{\lambda_1}{\Delta}\left(a_1 - \frac{\lambda_1 (\log \eta_1)^{\sigma_1}}{\Gamma(\sigma_1 + 1)}\right) (\lambda_2 I_{0^+}^{\sigma_2+\alpha} h(\eta_2) - (a_2 {}^{H}\mathcal{I}_1^{\alpha} h(T) + b_2 {}^{H}\mathcal{I}_1^{\alpha-\gamma} h(1))), \\ &- \frac{\lambda_1}{\Delta}\left(a_2 \log T - \frac{\lambda_2 (\log \eta_2)^{\sigma_2}}{\Gamma(\sigma_2 + 1)}\right) I_{0^+}^{\sigma_1+\alpha} h(\eta_1). \end{aligned}$$

Substituting the value of k_0, k_1 in (3.4) we get (3.2), which completes the proof. □

We use the following assumptions in the proofs of our main results.

(H1) There exist constant $L > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq L\|u - v\|, \text{ for each } t \in \mathrm{J} \text{ and for each } u, v \in \mathcal{U}. \tag{3.7}$$

(H2) The functions f satisfy the following growth conditions for constants $M, N > 0, p \in [0, 1)$.

$$\|f(t, u)\| \leq M\|u\|^p + N \text{ for each } t \in \mathrm{J} \text{ and each } u \in \mathcal{U}. \tag{3.8}$$

In the following, we set an abbreviated notation for the Hadamard fractional integral of order $\alpha > 0$, for a function with two variables as

$$^{H}\mathcal{I}_{1}^{\alpha} f_u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, u(s)) \frac{\mathrm{d}s}{s}.$$

Moreover, for computational convenience we put

$$\omega = \frac{\tilde{\mu_1} (\log \eta_1)^{\sigma_1+\alpha}}{\Gamma(\sigma_1 + \alpha + 1)} + \tilde{\mu_2} \left[\frac{|\lambda_2| (\log \eta_2)^{\sigma_2+\alpha}}{\Gamma(\sigma_2 + \alpha + 1)} + \frac{|a_2| (\log T)^{\alpha}}{\Gamma(\alpha + 1)} + \frac{|b_2| (\log T)^{\alpha-\gamma}}{\Gamma(\alpha - \gamma + 1)} \right], \tag{3.9}$$

where $\tilde{\mu_1} = |\lambda_1|(|\Delta_1| + |\Delta_2|T),\ \tilde{\mu_2} = |\lambda_1\Delta_3| + |\Delta_4|T$,

$$\bar{\omega} = \frac{|\lambda_1||\Delta_2| (\log \eta_1)^{\sigma_1+\alpha}}{\Gamma(\sigma_1 + \alpha + 1)} + |\Delta_4| \left[\frac{|\lambda_2| (\log \eta_2)^{\sigma_2+\alpha}}{\Gamma(\sigma_2 + \alpha + 1)} + \frac{|a_2| (\log T)^{\alpha}}{\Gamma(\alpha + 1)} + \frac{|b_2| (\log T)^{\alpha-\gamma}}{\Gamma(\alpha - \gamma + 1)} \right]. \tag{3.10}$$

In view of Lemma 3.1, we consider two operators $\mathcal{T}, \mathcal{S} : \mathcal{U} \longrightarrow \mathcal{U}$ as follows:

$$\mathcal{T} u(t) = {}^{H}\mathcal{I}_{1}^{\alpha} f_u(t),\ t \in \mathrm{J},$$

and

$$\mathcal{S}u(t) = \mu_1(t) {}^{H}\mathcal{I}_{1}^{\sigma_1+\alpha} f_u(\eta_1) + \mu_2(t)(\lambda_2 {}^{H}\mathcal{I}_{1}^{\sigma_2+\alpha} f_u(\eta_2) - (a_2 {}^{H}\mathcal{I}_{1}^{\alpha} f_u(T) + b_2 {}^{H}\mathcal{I}_{1}^{\alpha-\gamma} f_u(T))),\ t \in \mathrm{J}.$$

Then the integral equation (3.2) in Lemma 3.1 can be written as an operator equation

$$\mathcal{K}u(t) = \mathcal{T} u(t) + \mathcal{S}u(t), \quad t \in \mathrm{J},$$

The continuity of f show that the operator $\mathcal{K}$ is well define and fixed points of the operator equation are solutions of the integral equations (3.2) in Lemma 3.1.

Lemma 3.2. *$\mathcal{T}$ is Lipschitz with constant $\ell_f = \frac{L(\log T)^{\alpha}}{\Gamma(\alpha+1)}$. Moreover, $\mathcal{T}$ satisfies the growth condition given below*

$$\|\mathcal{T} u\| \leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)} (M\|u\|^p + N),$$

for every $u \in \mathcal{U}$.

Proof. To show that the operator $\mathcal{T}$ is Lipschitz with constant ℓ_f. Let $u, v \in \mathcal{U}$, then we have

$$\begin{aligned} |\mathcal{T} u(t) - \mathcal{T} v(t)| &= \left| {}^{H}\mathcal{I}_{1}^{\alpha} f_u(t) - {}^{H}\mathcal{I}_{1}^{\alpha} f_v(t) \right| \\ &\leq {}^{H}\mathcal{I}_{1}^{\alpha} |f_u - f_v|(t) \\ &\leq {}^{H}\mathcal{I}_{1}^{\alpha}(1)(T) L\|u - v\| \\ &= \frac{L (\log T)^{\alpha}}{\Gamma(\alpha + 1)} \|u - v\|, \end{aligned}$$

for all $t \in \mathrm{J}$. Taking supremum over t, we obtain

$$\|\mathcal{T} u - \mathcal{T} v\| \leq \frac{L (\log T)^{\alpha}}{\Gamma(\alpha + 1)} \|u - v\|.$$

Hence, $\mathcal{T} : \mathcal{U} \longrightarrow \mathcal{U}$ is a Lipschitzian on $\mathcal{U}$ with Lipschitz constant $\ell_f = \frac{L(\log T)^\alpha}{\Gamma(\alpha+1)}$. By Proposition 2.6, $\mathcal{T}$ is κ–Lipschitz with constant ℓ_f. Moreover, for growth condition, we have

$$\begin{aligned}|\mathcal{T}u(t)| &\le {}^H\mathcal{I}_1^\alpha |f_u|(t)\\ &\le (M\|u\|^p + N)\,{}^H\mathcal{I}_1^\alpha(1)(T)\\ &= \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(M\|u\|^p + N).\end{aligned}$$

Hence it follows that

$$\|\mathcal{T}u\| \le \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(M\|u\|^p + N).$$

□

Lemma 3.3. *$\mathcal{S}$ is continuous and satisfies the growth condition given as below,*

$$\|\mathcal{S}u\| \le (M\|u\|^p + N)\omega, \text{ for every } u \in \mathcal{U},$$

where ω is given by (3.9).

Proof. To prove that $\mathcal{S}$ is continuous. Let $\{u_n\}, u \in \mathcal{U}$ with $\lim_{n\to+\infty} \|u_n - u\| \to 0$. It is trivial to see that $\{u_n\}$ is a bounded subset of $\mathcal{U}$. As a result, there exists a constant $r > 0$ such that $\|u_n\| \le r$ for all $n \ge 1$. Taking limit, we see $\|u\| \le r$. It is easy to see that $f(s, u_n(s)) \to f(s, u(s))$, as $n \to +\infty$. due to the continuity of f. On the other hand taking (H2) into consideration we get the following inequality:

$$\frac{\left(\log \frac{t}{s}\right)^{\alpha-1}}{s}\|f(s, u_n(s)) - f(s, u(s))\| \le 2\frac{\left(\log \frac{t}{s}\right)^{\alpha-1}}{s}(Mr^p + N).$$

We notice that since the function $s \mapsto 2\frac{\left(\log \frac{t}{s}\right)^{\alpha-1}}{s}(Mr^p + N)$ is Lebesgue integrable over $[1, t]$, the functions

$$\begin{aligned}s &\mapsto 2\frac{(Mr^p + N)}{s}\left(\log \frac{\eta_i}{s}\right)^{\sigma_i+\alpha-1}, i = 1, 2,\\ s &\mapsto 2\frac{(Mr^p + N)}{s}\left(\log \frac{T}{s}\right)^{\alpha-1},\\ s &\mapsto 2\frac{(Mr^p + N)}{s}\left(\log \frac{T}{s}\right)^{\alpha-\gamma-1},\end{aligned}$$

are also. This fact together with the Lebesgue dominated convergence theorem implies that

$$\begin{aligned}&{}^H\mathcal{I}_1^{\sigma_1+\alpha}|f_{u_n} - f_u|(\eta_1) \to 0 \text{ as } n \to +\infty,\\ &{}^H\mathcal{I}_1^{\sigma_2+\alpha}|f_{u_n} - f_u|(\eta_2) \to 0 \text{ as } n \to +\infty,\\ &{}^H\mathcal{I}_1^{\alpha}|f_{u_n} - f_u|(T) \to 0 \text{ as } n \to +\infty,\\ &{}^H\mathcal{I}_1^{\alpha-\gamma}|f_{u_n} - f_u|(T) \to 0 \text{ as } n \to +\infty.\end{aligned}$$

It follows that $\|\mathcal{S}u_n - \mathcal{S}u\| \to 0$ as $n \to +\infty$. Which implies the continuity of the operator $\mathcal{S}$.

For the growth condition, using the assumption (H2) we have

$$\begin{aligned}
|\mathcal{S}u(t)| &\le \tilde{\mu_1}\, {}^H\mathcal{I}_1^{\sigma_1+\alpha}|f_u|(\eta_1) + \tilde{\mu_2}\Big(|\lambda_2|\, {}^H\mathcal{I}_1^{\sigma_2+\alpha}|f_u|(\eta_2) \\
&\quad + |a_2|\, {}^H\mathcal{I}_1^{\alpha}|f_u|(T) + |b_2|\, {}^H\mathcal{I}_1^{\alpha-\gamma}|f_u|(T)\Big) \\
&\le (M\|u\|^p + N)\tilde{\mu_1}\, {}^H\mathcal{I}_1^{\sigma_1+\alpha}(1)(\eta_1) + (M\|u\|^p + N)\tilde{\mu_2}\Big(|\lambda_2|\, {}^H\mathcal{I}_1^{\sigma_2+\alpha}(1)(\eta_2) \\
&\quad + |a_2|\, {}^H\mathcal{I}_1^{\alpha}(1)(T) + |b_2|\, {}^H\mathcal{I}_1^{\alpha-\gamma}(1)(T)\Big) \\
&\le (M\|u\|^p + N)\left(\frac{\tilde{\mu_1}\,(\log\eta_1)^{\sigma_1+\alpha}}{\Gamma(\sigma_1+\alpha+1)} + \tilde{\mu_2}\left[\frac{|\lambda_2|\,(\log\eta_2)^{\sigma_2+\alpha}}{\Gamma(\sigma_2+\alpha+1)} + \frac{|a_2|\,(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{|b_2|\,(\log T)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right]\right).
\end{aligned}$$

Therefore,

$$\|\mathcal{S}u\| \le (M\|u\|^p + N)\omega, \tag{3.11}$$

where ω is given by (3.9). This completes the proof of Lemma 3.3. □

Lemma 3.4. *The operator $\mathcal{S} : \mathcal{U} \longrightarrow \mathcal{U}$ is compact. Consequently, $\mathcal{S}$ is κ-Lipschitz with zero constant.*

Proof. In order to show that $\mathcal{S}$ is compact. Let us take a bounded set $\Omega \subset B_r$. We are required to show that $\mathcal{S}(\Omega)$ is relatively compact in $\mathcal{U}$. For arbitrary $u \in \Omega \subset B_r$, then with the help of the estimates (3.11) we can obtain

$$\|\mathcal{S}u\| \le (Mr^p + N)\omega,$$

where ω is given by (3.9), which shows that $\mathcal{S}(\Omega)$ is uniformly bounded. Furthermore, for arbitrary $u \in \mathcal{U}$ and $t \in$ J. From the definition of $\mathcal{S}$ using the notations given by (3.3) and $(H2)$, we can obtain

$$\begin{aligned}
|(\mathcal{S}u)'(t)| &\le |\mu_1'(t)|\, {}^H\mathcal{I}_1^{\sigma_1+\alpha}|f_u|(\eta_1) + |\mu_2'(t)|\Big(|\lambda_2|\, {}^H\mathcal{I}_1^{\sigma_2+\alpha}|f_u|(\eta_2) + |a_2|\, {}^H\mathcal{I}_1^{\alpha}|f_u|(T) + |b_2|\, {}^H\mathcal{I}_1^{\alpha-\gamma}|f_u|(T)\Big) \\
&\le (M\|u\|^p + N)\Big(|\lambda_1||\Delta_2|\, {}^H\mathcal{I}_1^{\sigma_1+\alpha}(1)(\eta_1) + |\Delta_4|\Big(|\lambda_2|\, {}^H\mathcal{I}_1^{\sigma_2+\alpha}(1)(\eta_2) + |a_2|\, {}^H\mathcal{I}_1^{\alpha}(1)(T) \\
&\quad + |b_2|\, {}^H\mathcal{I}_1^{\alpha-\gamma}(1)(T)\Big)\Big) \\
&\le (M\|u\|^p + N)\left(\frac{|\lambda_1||\Delta_2|\,(\log\eta_1)^{\sigma_1+\alpha}}{\Gamma(\sigma_1+\alpha+1)}|\Delta_4|\left[\frac{|\lambda_2|\,(\log\eta_2)^{\sigma_2+\alpha}}{\Gamma(\sigma_2+\alpha+1)} + \frac{|a_2|\,(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right.\right. \\
&\quad \left.\left. + \frac{|b_2|\,(\log T)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right]\right) \\
&= \bar{\omega}(M\|u\|^p + N),
\end{aligned}$$

where $\bar{\omega}$ is given by (3.10). Now, for equi-continuity of $\mathcal{S}$ take $t_1, t_2 \in$ J with $t_1 < t_2$, and let $u \in \Omega$. Thus, we get

$$|\mathcal{S}u(t_2) - \mathcal{S}u(t_1)| \le \int_{t_1}^{t_2} |(\mathcal{S}u)'(s)|\,\mathrm{ds} \le \bar{\omega}(Mr^p + N)(t_2 - t_1).$$

From the last estimate, we deduce that $\|(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)\| \to 0$ when $t_2 \to t_1$. Therefore, $\mathcal{S}$ is equicontinuous. Thus, by Ascoli–Arzelà theorem, the operator $\mathcal{S}$ is compact and hence by Proposition 2.5. $\mathcal{S}$ is κ–Lipschitz with zero constant. □

Theorem 3.5. *Suppose that (H1)–(H2) are satisfied, then the BVP* (1.1) *has at least one solution* $u \in C(\mathrm{J}, \mathbb{R})$ *provided that* $\ell_f < 1$ *and the set of the solutions is bounded in* $C(\mathrm{J}, \mathbb{R})$.

Proof. Let $\mathcal{T}, \mathcal{S}, \mathcal{K}$ are the operators defined in the start of this section. These operators are continuous and bounded. Moreover, by Lemma 3.2, $\mathcal{T}$ is κ–Lipschitz with constant ℓ_f and by Lemma 3.4, $\mathcal{S}$ is κ–Lipschitz with constant 0. Thus, $\mathcal{K}$ is κ–Lipschitz with constant ℓ_f. Hence $\mathcal{K}$ is strict κ–contraction with constant ℓ_f. Since $\ell_f < 1$, so $\mathcal{K}$ is κ-condensing.

Now consider the following set

$$\Theta = \{u \in \mathcal{U} : \text{ there exist } \xi \in [0, 1] \text{ such that } x = \xi \mathcal{K} u\}.$$

We will show that the set Θ is bounded. For $u \in \Theta$, we have $u = \xi \mathcal{K} u = \xi(\mathcal{T}(u) + S(u))$, which implies that

$$\begin{aligned} \|u\| &\leq \xi(\|\mathcal{T} u\| + \|\mathcal{S} u\|) \\ &\leq \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \omega\right](M\|u\|^{p} + N), \end{aligned}$$

where ω is given by (3.9). From the above inequalities, we conclude that Θ is bounded in $C(\mathrm{J}, \mathbb{R})$. If it is not bounded, then dividing the above inequality by $a := \|u\|$ and letting $a \to \infty$, we arrive at

$$1 \leq \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] \lim_{a\to\infty} \frac{Ma^{p} + N}{a} = 0,$$

which is a contradiction. Thus the set Θ is bounded and the operator $\mathcal{K}$ has at least one fixed point which represent the solution of BVP (1.1). □

Remark 3.6. If the growth condition (H2) is formulated for $p = 1$, then the conclusions of Theorem 3.5 remain valid provided that

$$\left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] M < 1.$$

To end this section, we give an existence and uniqueness result.

Theorem 3.7. *Under assumption (H1) the BVP* (1.1) *has a unique solution if*

$$\left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] L < 1. \tag{3.12}$$

Proof. Let $u, v \in C(\mathrm{J}, \mathbb{R})$ and $t \in \mathrm{J}$, then we have

$$\begin{aligned} |\mathcal{K}u(t) - \mathcal{K}v(t)| \leq\ & {}^{H}\mathcal{I}_{1}^{\alpha}|f_u - f_v|(t) \\ &+ |\mu_1(t)|^{H}\mathcal{I}_{1}^{\sigma_1+\alpha}|f_u - f_v|(\eta_1) \\ &+ |\mu_2(t)|\lambda_2{}^{H}\mathcal{I}_{1}^{\sigma_2+\alpha}|f_u - f_v|(\eta_2) \\ &+ |a_2\mu_2(t)|^{H}\mathcal{I}_{1}^{\alpha}|f_u - f_v|(T) \\ &+ |b_2\mu_2(t)|^{H}\mathcal{I}_{1}^{\alpha-\gamma}|f_u - f_v|(T) \end{aligned}$$

$$\leq L\|u-v\|\Big[{}^{H}\mathcal{I}_{1}^{\alpha}(1)(T)+\tilde{\mu_1}{}^{H}\mathcal{I}_{1}^{\sigma_1+\alpha}(1)(\eta_1)$$
$$+\tilde{\mu_2}\Big(|\lambda_2|{}^{H}\mathcal{I}_{1}^{\sigma_2+\alpha}(1)(\eta_2)+|a_2|{}^{H}\mathcal{I}_{1}^{\alpha}(1)(T)+|b_2|{}^{H}\mathcal{I}_{1}^{\alpha-\gamma}(1)(T)\Big)\Big]$$
$$\leq L\|u-v\|\Bigg(\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\tilde{\mu_1}(\log\eta_1)^{\sigma_1+\alpha}}{\Gamma(\sigma_1+\alpha+1)}$$
$$+\tilde{\mu_2}\left[\frac{|\lambda_2|(\log\eta_2)^{\sigma_2+\alpha}}{\Gamma(\sigma_2+\alpha+1)}+\frac{|a_2|(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{|b_2|(\log T)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right]\Bigg)$$
$$=\left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\omega\right]L\|u-v\|.$$

In view of the given condition $\left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\omega\right]L<1$, it follows that the mapping $\mathcal{K}$ is a contraction. Hence, by the Banach fixed point theorem, $\mathcal{K}$ has a unique fixed point which is a unique solution of problem (1.1). This completes the proof. □

4. Examples

In this section, in order to illustrate our results, we consider two examples.

Example 4.1. Let us consider problem (1.1) with specific data:

$$\begin{aligned}&\alpha=\frac{3}{2},\gamma=\frac{1}{2}=1,T=e\\&a_1=b_1=a_2=b_2=1;\lambda_1=\lambda_2=0,\\&\sigma_1=\frac{1}{2},\sigma_2=\frac{3}{2},\eta_1=\frac{5}{4},\eta_2=\frac{3}{2}.\end{aligned}\tag{4.1}$$

Using the given values of the parameters in (3.3) and (3.9), by the Matlab program, we find that

$$\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\omega=3.5284.\tag{4.2}$$

In order to illustrate Theorem 3.5, we take

$$f(t,u(t))=\frac{1}{e^{(t-1)}+9}\left(\frac{|u(t)|}{1+|u(t)|}\right)+\log(t),\tag{4.3}$$

in (1.1) and note that

$$\begin{aligned}|f(t,u)-f(t,v)|&=\frac{1}{e^{(t-1)}+9}\left(\left|\frac{|u|}{1+|u|}-\frac{|v|}{1+|v|}\right|\right)\\&\leq\frac{1}{e^{(t-1)}+9}\left(\frac{|u-v|}{(1+|u|)(1+|v|)}\right)\\&\leq\frac{1}{10}|u-v|.\end{aligned}$$

Hence the condition (H1) holds with $L=\frac{1}{10}$. Further from the above given data it is easy to calculate

$$\ell_f=\frac{L(\log T)^{\alpha}}{\Gamma(\alpha+1)}=0.0752$$

On the other hand, for any $t \in \mathrm{J}, u \in \mathbb{R}$ we have

$$|f(t,u)| \leq \frac{1}{10}|u| + 1.$$

Hence condition (H2) holds with $M = \frac{1}{10}, p = N = 1$. In view of Theorem 3.5,

$$\Theta = \{u \in \mathcal{U} : \text{ there exist } \xi \in [0,1] \text{ such that } x = \xi \mathcal{K} u\},$$

is the solution set; then

$$\begin{aligned}\|u\| &\leq \xi(\|\mathcal{T}u\| + \|\mathcal{S}u\|)\\ &\leq \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \omega\right](M\|u\| + N).\end{aligned}$$

From which, we have

$$\|u\| \leq \frac{\left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \omega\right]N}{1 - \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \omega\right]M} = 5.4522,$$

by Theorem 3.5 the BVP (1.1) with the data (4.1) and (4.3) has at least a solution u in $C(\mathrm{J} \times \mathbb{R}, \mathbb{R})$. Furthermore $\left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \omega\right]L = 0.3528 < 1$. Hence by Theorem 3.7 the boundary value problem (1.1) with the data (4.1) and (4.3) has a unique solution.

Example 4.2. Consider the following boundary value problem of a fractional differential equation:

$$\begin{cases} {}^{C}_{H}\mathcal{D}_1^{\frac{7}{4}} u(t) = \frac{1}{2(t+1)^2}\left(u(t) + \sqrt{1+u^2(t)}\right), \quad t \in \mathrm{J} := [1,2],\\ u(1) = {}^{H}\mathcal{I}_1^{\frac{3}{2}} u(\frac{3}{2}),\\ u(2) = {}^{H}\mathcal{I}_1^{\frac{5}{2}} u(\frac{7}{4}), \end{cases} \tag{4.4}$$

Note that, this problem is a particular case of BVP (1.1), where

$$\begin{aligned}&\alpha = \frac{7}{4}, T = 2\\ &a_1 = a_2 = \lambda_1 = \lambda_2 = 1, a_2 = b_2 = 0\\ &\sigma_1 = \frac{3}{2}, \sigma_2 = \frac{5}{2}, \eta_1 = \frac{3}{2}, \eta_2 = \frac{7}{4},\end{aligned}$$

and $f : \mathrm{J} \times \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(t,u) = \frac{1}{2(t+1)^2}\left(u + \sqrt{1+u^2}\right), \quad \text{for } t \in \mathrm{J}, u \in \mathbb{R}.$$

It is clear that the function f is continuous. On the other hand, for any $t \in \mathrm{J}, u, v \in \mathbb{R}$ we have

$$|f(t,u) - f(t,v)| = \frac{1}{(t+1)^2}\left|\frac{1}{2}\left(u - v + \sqrt{1+u^2} - \sqrt{1+v^2}\right)\right|$$

$$
\begin{aligned}
&= \frac{1}{(t+1)^2}\left|\frac{1}{2}(u-v)\left(1+\frac{u+v}{\sqrt{1+u^2}+\sqrt{1+v^2}}\right)\right| \\
&\le \frac{1}{4}|u-v|.
\end{aligned}
$$

Hence condition (H1) holds with $L = \frac{1}{4}$. We shall check that condition (3.12) is satisfied. Indeed using the Matlab program, we can find

$$
\left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \omega\right] L = 0.3417 < 1,
$$

Hence by Theorem 3.7 the boundary value problem (4.4) has a unique solution.

5. Conclusion

We have presented the existence and uniqueness of solutions to a nonlinear boundary value problem of fractional differential equations involving the Caputo–Hadamard fractional derivative. The proof of the existence results is based on a fixed point theorem due to Isaia [32], which was obtained via coincidence degree theory for condensing maps, while the uniqueness of the solution is proved by applying the Banach contraction principle. Moreover, two examples are presented for the illustration of the obtained theory. Our results are not only new in the given configuration but also correspond to some new situations associated with the specific values of the parameters involved in the given problem.

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Conflict of interest

The authors declare no conflict of interest.

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