AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(3): 2376–2387.
DOI:10.3934/math.2020157
Received: 30 December 2019
Accepted: 25 February 2020
Published: 04 March 2020

Research article

On $\mathscr{M}$-convex functions

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Abstract: In this article, we introduce the notion of $\mathscr{M}$-convex functions, log-$\mathscr{M}$-convex functions and the notion of quasi $\mathscr{M}$-convex functions. We derive some new analogues of Hermite-Hadamard like inequalities associated with $\mathscr{M}$-convex functions by using the concepts of ordinary, fractional and quantum calculus. The main results of this paper may be useful where bounds for natural phenomena described by integrals such as mechanical work are frequently required. These results are also helpful in the field of numerical analysis where error analysis is required.

Keywords: convex; $\mathscr{M}$-convex; log-$\mathscr{M}$-convex; quasi $\mathscr{M}$-convex; Hermite-Hadamard inequality; fractional; quantum
Mathematics Subject Classification: 05A30, 26A33, 26A51, 26D15

1. Introduction and preliminaries

Means of different types play significant role in different fields of sciences through their applications. For instance it has been observed harmonic means have applications in electrical circuits theory. To be more precise, the total resistance of a set of parallel resistors is just half of harmonic means of the total resistors, for details, see [3]. Recently many researchers have extensively utilized different types of means in theory of convexity. Consequently a number of new and novel extensions of classical convexity have been proposed in the literature. For some recent studies, see [4, 5, 21, 22]. We now recall some preliminary concepts and results pertaining to convexity and for its other extensions.

Definition 1.1 ([18]). (AA-convex functions) A function $\mathcal{X} : C \subseteq \mathbb{R} \to \mathbb{R}$ is said to be AA-convex, if

$$(1-\mu)\mathcal{X}(x) + \mu\mathcal{X}(y) \geq \mathcal{X}((1-\mu)x + ty), \quad \forall x, y \in C, \mu \in [0, 1],$$

where C is a convex set.

Definition 1.2 ([18])**.** (*GG*-convex functions) A function $\mathcal{X} : \mathcal{G} \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *GG*-convex, if

$$\mathcal{X}^{1-\mu}(x)\mathcal{X}^{\mu}(y) \geq \mathcal{X}(x^{1-\mu}y^{\mu}), \quad \forall x, y \in \mathcal{G}, \mu \in [0, 1],$$

where $\mathcal{G}$ is a geometric convex set.

Definition 1.3 ([13])**.** (*HH*-convex functions) A function $\mathcal{X} : \mathcal{H} \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be *HH*-convex, if

$$\frac{\mathcal{X}(x)\mathcal{X}(y)}{\mu\mathcal{X}(x) + (1-\mu)\mathcal{X}(y)} \geq \mathcal{X}\left(\frac{xy}{(1-\mu)x + ty}\right), \quad \forall x, y \in \mathcal{H}, \mu \in [0, 1],$$

where $\mathcal{H}$ is a harmonic convex set.

For some other useful details, see [18]. Convexity theory also played significant role in the development of theory of inequalities. Many known results are obtained directly using the functions having convexity property. Hermite and Hadamard presented independently a result which now a days known as Hermite-Hadamard's inequality. This result is very simple in nature but very powerful, as it provides us a necessary and sufficient condition for a function to be convex. It reads as: Let $\mathcal{X} : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\mathcal{X}\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \mathcal{X}(x)\mathrm{d}x \leq \frac{\mathcal{X}(c) + \mathcal{X}(d)}{2}.$$

Dragomir et al. [8] written a very interesting detailed monograph on Hermite-Hadamard's inequality and its applications. Interested readers may find useful details in it. In recent years several famously known researchers from all over the world have studied the result of Hermite and Hadamard intensively. For more details, see [4, 6, 7, 9, 10, 17, 20]. This result has also been generalized for other classes of convex functions, for instance, see [8, 11, 12, 14, 18, 22].

Fractional calculus [15, 16] has played an important role in various scientific fields since it is a good tool to describe long-memory processes. Sarikaya et al. [24] used the concepts of fractional calculus and obtained new refinements of fractional Hermite-Hadamard like inequalities. This article of Sarikaya et al. opened a new venue of research. Consequently several new generalizations of Hermite-Hadamard's inequality have been obtained using the fractional calculus concepts.

Recently many authors have shown their special interest in utilizing the concepts of quantum calculus for obtaining q-analogues of different integral inequalities. For some basic definitions and recent studies, see [1, 2, 19, 23, 25, 26]. The main objective of this article is to introduce the notion of $\mathscr{M}$-convex functions. This class can be viewed as novel extension of the classical definition of convexity. We link this class with Hermite-Hadamard's inequality and obtain several new variants of this famous result. We also obtain the fractional and quantum analogues of the obtained results. We expect that the results of this paper may stimulate further research in this direction.

2. $\mathscr{M}$-convexity

In this section, we introduce the notions of $\mathscr{M}$-convex functions, log-$\mathscr{M}$-convex and quasi $\mathscr{M}$-convex functions. First of all for the sake of simplicity, we take $\mathcal{G} = \sqrt{cd}$ and $\mathcal{A} = \frac{c+d}{2}$.

Definition 2.1. A function $X : D \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is said to be $\mathscr{M}$-convex function, if

$$X((1-\mu)\mathcal{G} + \mu\mathcal{A}) \leq (1-\mu)X(\mathcal{G}) + tX(\mathcal{A}),$$
$$\forall c, d \in D, \mu \in [0,1].$$

Definition 2.2. A function $X : D \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is said to be log-$\mathscr{M}$-convex function, if

$$X((1-\mu)\mathcal{G} + \mu\mathcal{A}) \leq X^{1-\mu}(\mathcal{G})X^{\mu}(\mathcal{A}), \forall c, d \in D, \mu \in [0,1].$$

Definition 2.3. A function $X : D \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is said to be quasi $\mathscr{M}$-convex function, if

$$X((1-\mu)\mathcal{G} + \mu\mathcal{A}) \leq \max\{X(\mathcal{G}), X(\mathcal{A})\}, \forall c, d \in D, \mu \in [0,1].$$

3. Hermite-Hadamard like inequalities using $\mathscr{M}$-convex functions

We now derive a new auxiliary result which play a key role in the development of our coming results.

Lemma 3.1. *Let $X : I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on I°, $c, d \in I^\circ$ with $c < d$. If $X' \in L[c, d]$, then*

$$\frac{X(\mathcal{G}) + X(\mathcal{A})}{2} - \frac{2}{(\sqrt{d} - \sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}} X(x)\mathrm{d}x = \frac{(\sqrt{d} - \sqrt{c})^2}{4}\int_0^1 (1-2\mu)X'(\mu\mathcal{G} + (1-\mu)\mathcal{A})\,\mathrm{d}\mu.$$

Proof. It suffices to show that

$$\int_0^1 (1-2\mu)X'(\mu\mathcal{G} + (1-\mu)\mathcal{A})\,\mathrm{d}\mu = 2\frac{X(\mathcal{G}) + X(\mathcal{A})}{(\sqrt{d} - \sqrt{c})^2} - \frac{8}{(\sqrt{d} - \sqrt{c})^4}\int_{\mathcal{G}}^{\mathcal{A}} X(x)\mathrm{d}x.$$

This implies

$$\frac{(\sqrt{d} - \sqrt{c})^2}{4}\int_0^1 (1-2\mu)X'(\mu\mathcal{G} + (1-\mu)\mathcal{A})\,\mathrm{d}\mu = \frac{X(\mathcal{G}) + X(\mathcal{A})}{2} - \frac{2}{(\sqrt{d} - \sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}} X(x)\mathrm{d}x.$$

This completes the proof. □

Now utilizing Lemma 3.1, we derive our next results.

Theorem 3.2. *Let $X : I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on I°, $c, d \in I^\circ$ with $c < d$ and $X' \in L[c, d]$. If $|X'|$ is $\mathscr{M}$-convex function, then*

$$\left|\frac{X(\mathcal{G}) + X(\mathcal{A})}{2} - \frac{2}{(\sqrt{d} - \sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}} X(x)\mathrm{d}x\right| \leq \frac{(\sqrt{d} - \sqrt{c})^2}{16}\left[|X'(\mathcal{G})| + |X'(\mathcal{A})|\right].$$

Proof. Using Lemma 3.1, property of the modulus and the fact that $|\mathcal{X}'|$ is $\mathscr{M}$-convex function, we have

$$\begin{aligned}
&\left|\frac{\mathcal{X}(\mathcal{G})+\mathcal{X}(\mathcal{A})}{2}-\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(x)\mathrm{d}x\right| \\
&\leq \frac{(\sqrt{d}-\sqrt{c})^2}{4}\int_0^1 |1-2\mu|\,|\mathcal{X}'(\mu\mathcal{G}+(1-\mu)\mathcal{A})|\,\mathrm{d}\mu \\
&\leq \frac{(\sqrt{d}-\sqrt{c})^2}{4}\int_0^1 |1-2\mu|\left[\mu|\mathcal{X}'(\mathcal{G})|+(1-\mu)\,|\mathcal{X}'(\mathcal{A})|\right]\mathrm{d}\mu \\
&= \frac{(\sqrt{d}-\sqrt{c})^2}{16}\left[|\mathcal{X}'(\mathcal{G})|+|\mathcal{X}'(\mathcal{A})|\right].
\end{aligned}$$

This completes the proof. □

If we apply Theorem 3.2 for log-$\mathscr{M}$-convex functions, then

Theorem 3.3. *Let $\mathcal{X}: I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on I°, $c, d \in I^\circ$ with $c < d$ and $\mathcal{X}' \in L[c,d]$, If $|\mathcal{X}'|$ is decreasing and* log-*$\mathscr{M}$-convex function, then*

$$\left|\frac{\mathcal{X}(\mathcal{G})+\mathcal{X}(\mathcal{A})}{2}-\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(x)\mathrm{d}x\right| \leq \frac{(\sqrt{d}-\sqrt{c})^2}{4}\left[\frac{-2+4\sqrt{w}-2w-\log w+w\log w}{\log w^2}\right],$$

where $w=\frac{|\mathcal{X}'(\mathcal{G})|}{\left|\mathcal{X}'(\mathcal{A})\right|}$.

Theorem 3.4. *Let $\mathcal{X}: I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on I°, $c, d \in I^\circ$ with $c < d$ and $\mathcal{X}' \in L[c,d]$. If $|\mathcal{X}'|^q$, where $\frac{1}{p}+\frac{1}{q}=1$ is $\mathscr{M}$-convex function, then*

$$\begin{aligned}
&\left|\frac{\mathcal{X}(\mathcal{G})+\mathcal{X}(\mathcal{A})}{2}-\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(x)\mathrm{d}x\right| \\
&\leq \frac{(\sqrt{d}-\sqrt{c})^2}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{X}'(\mathcal{G})|^q+|\mathcal{X}'(\mathcal{A})|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Using Lemma 3.1, Holder's inequality and the fact that $|\mathcal{X}'|^q$ is $\mathscr{M}$-convex functions, we have

$$\begin{aligned}
&\left|\frac{\mathcal{X}(\mathcal{G})+\mathcal{X}(\mathcal{A})}{2}-\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(x)\mathrm{d}x\right| \\
&\leq \frac{(\sqrt{d}-\sqrt{c})^2}{4}\left(\int_0^1 |1-2\mu|^p\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1 |\mathcal{X}'(\mu\mathcal{G}+(1-\mu)\mathcal{A})|\,\mathrm{d}\mu\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \frac{(\sqrt{d}-\sqrt{c})^2}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 [\mu|\mathcal{X}'(\mathcal{G})|^q+(1-\mu)|\mathcal{X}'(\mathcal{A})|^q]\,\mathrm{d}\mu\right)^{\frac{1}{q}}$$
$$= \frac{(\sqrt{d}-\sqrt{c})^2}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{X}'(\mathcal{G})|^q+|\mathcal{X}'(\mathcal{A})|^q}{2}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 3.5. *Let $\mathcal{X}: I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on I°, $c, d \in I^\circ$ with $c < d$ and $\mathcal{X}' \in L[c, d]$. If $|\mathcal{X}'|^q$, where $q \geq 1$ is $\mathscr{M}$-convex function, then*

$$\left|\frac{\mathcal{X}(\mathcal{G})+\mathcal{X}(\mathcal{A})}{2}-\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(x)\mathrm{d}x\right| \leq \frac{(\sqrt{d}-\sqrt{c})^2}{8}\left(\frac{|\mathcal{X}'(\mathcal{G})|^q+|\mathcal{X}'(\mathcal{A})|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 3.1, power mean inequality and the fact that $|\mathcal{X}'|$ is $\mathscr{M}$-convex functions, we have

$$\left|\frac{\mathcal{X}(\mathcal{G})+\mathcal{X}(\mathcal{A})}{2}-\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(x)\mathrm{d}x\right|$$
$$\leq \frac{(\sqrt{d}-\sqrt{c})^2}{4}\left(\int_0^1|1-2\mu|\mathrm{d}\mu\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2\mu|\,|\mathcal{X}'(\mu\mathcal{G}+(1-\mu)\mathcal{A})|\,\mathrm{d}\mu\right)^{\frac{1}{q}}$$
$$\leq \frac{(\sqrt{d}-\sqrt{c})^2}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2\mu|\,[\mu|\mathcal{X}'(\mathcal{G})|^q+(1-\mu)|\mathcal{X}'(\mathcal{A})|^q]\,\mathrm{d}\mu\right)^{\frac{1}{q}}$$
$$= \frac{(\sqrt{d}-\sqrt{c})^2}{8}\left(\frac{|\mathcal{X}'(\mathcal{G})|^q+|\mathcal{X}'(\mathcal{A})|^q}{2}\right)^{\frac{1}{q}}.$$

This completes the proof. □

4. Fractional estimates

In this section, we derive some fractional estimates of Hermite-Hadamard like inequalities using $\mathscr{M}$-convex functions. Before that we recall basic definition of Riemann-Liouville fractional integrals.

Definition 4.1 ([15])**.** Let $\mathcal{X} \in L[c, d]$, where $c \geq 0$. The Riemann-Liouville integrals $J^\nu_{c+}\mathcal{X}$ and $J^\nu_{d-}\mathcal{X}$, of order $\nu > 0$, are defined by

$$J^\nu_{c+}\mathcal{X}(x) = \frac{1}{\Gamma(\nu)}\int_c^x (x-\mu)^{\nu-1}\mathcal{X}(\mu)\mathrm{d}\mu, \text{ for } x > c$$

and

$$J^\nu_{d-}\mathcal{X}(x) = \frac{1}{\Gamma(\nu)}\int_x^d (\mu-x)^{\nu-1}\mathcal{X}(\mu)\mathrm{d}\mu, \text{ for } x < d,$$

respectively. Here, $\Gamma(\nu) = \int_0^\infty e^{-\mu}\mu^{\nu-1}\mathrm{d}\mu$ is the Gamma function. We also make the convention

$$J^0_{c+}\mathcal{X}(x) = J^0_{d-}\mathcal{X}(x) = \mathcal{X}(x).$$

We now derive a new auxiliary result utilizing the definition of Riemann-Liouville fractional integrals.

Lemma 4.1. *Let $\mathcal{X} : I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function. If $\mathcal{X}' \in L[c,d]$, then*

$$\frac{\mathcal{X}(\mathcal{G}) + \mathcal{X}(\mathcal{A})}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sqrt{d}-\sqrt{c})^2}\left[J^\alpha_{(\mathcal{A})^-}\mathcal{X}(\mathcal{G}) + J^\alpha_{(\mathcal{G})^+}\mathcal{X}(\mathcal{A})\right]$$
$$= \frac{(\sqrt{d}-\sqrt{c})^2}{4}\int_0^1 [(1-\mu)^\alpha - \mu^\alpha]\mathcal{X}'(\mu\mathcal{G} + (1-\mu)\mathcal{A})\,\mathrm{d}\mu.$$

Proof. It suffices to show that

$$\begin{aligned}
\mathcal{I} &= \int_0^1 [(1-\mu)^\alpha - \mu^\alpha]\mathcal{X}'(\mu\mathcal{G} + (1-\mu)\mathcal{A})\,\mathrm{d}\mu \\
&= \int_0^1 (1-\mu)^\alpha\mathcal{X}'(\mu\mathcal{G} + (1-\mu)\mathcal{A})\,\mathrm{d}\mu - \int_0^1 \mu^\alpha\mathcal{X}'(\mu\mathcal{G} + (1-\mu)\mathcal{A})\,\mathrm{d}\mu \\
&= \mathcal{I}_1 - \mathcal{I}_2. \qquad (4.1)
\end{aligned}$$

Now using change of variable technique and definition of Riemann-Liouville fractional integrals, we have

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 (1-\mu)^\alpha\mathcal{X}'(\mu\mathcal{G} + (1-\mu)\mathcal{A})\,\mathrm{d}\mu \\
&= \frac{2}{(\sqrt{d}-\sqrt{c})^2}\mathcal{X}(\mathcal{A}) \\
&\quad - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\sqrt{d}-\sqrt{c})^{2(\alpha+1)}}\frac{1}{\Gamma(\alpha)}\int_{\mathcal{G}}^{\mathcal{A}}(x-\mathcal{G})^{\alpha-1}\mathcal{X}(x)\mathrm{d}x \\
&= \frac{2}{(\sqrt{d}-\sqrt{c})^2}\mathcal{X}(\mathcal{A}) - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\sqrt{d}-\sqrt{c})^{2(\alpha+1)}}J^\alpha_{(\mathcal{A})^-}\mathcal{X}(\mathcal{G}). \qquad (4.2)
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathcal{I}_2 &= \int_0^1 \mu^\alpha\mathcal{X}'(\mu\mathcal{G} + (1-\mu)\mathcal{A})\,\mathrm{d}\mu \\
&= -\frac{2}{(\sqrt{d}-\sqrt{c})^2}\mathcal{X}(\mathcal{G}) + \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\sqrt{d}-\sqrt{c})^{2(\alpha+1)}}J^\alpha_{(\mathcal{G})^+}\mathcal{X}(\mathcal{A}). \qquad (4.3)
\end{aligned}$$

Combining (4.1), (4.2) and (4.3) completes the proof. □

Now using Lemma 4.1, we derive our next results.

Theorem 4.2. *Let $\mathcal{X} : I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function and $\mathcal{X}' \in L[c,d]$. If $|\mathcal{X}'|$ is $\mathscr{M}$-convex function, then*

$$\left|\frac{\mathcal{X}(\mathcal{G})+\mathcal{X}(\mathcal{A})}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sqrt{d}-\sqrt{c})^2}\left[J^{\alpha}_{(\mathcal{A})^-}\mathcal{X}(\mathcal{G})+J^{\alpha}_{(\mathcal{G})^+}\mathcal{X}(\mathcal{A})\right]\right|$$
$$\leq \frac{(\sqrt{d}-\sqrt{c})^2}{4(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[|\mathcal{X}'(a)|+|\mathcal{X}'(b)|].$$

Proof. Using Lemma 4.1 and the property of modulus, we have

$$\left|\frac{\mathcal{X}(\mathcal{G})+\mathcal{X}(\mathcal{A})}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sqrt{d}-\sqrt{c})^2}\left[J^{\alpha}_{(\mathcal{A})^-}\mathcal{X}(\mathcal{G})+J^{\alpha}_{(\mathcal{G})^+}\mathcal{X}(\mathcal{A})\right]\right|$$
$$\leq \int_0^1 \frac{(\sqrt{d}-\sqrt{c})^2}{4}|(1-\mu)^\alpha-\mu^\alpha|\,|\mathcal{X}'(\mu\mathcal{G}+(1-\mu)\mathcal{A})|\,\mathrm{d}\mu.$$

Since it is given that $|\mathcal{X}'|$ is $\mathscr{M}$-convex function, so we have

$$\left|\frac{\mathcal{X}(\mathcal{G})+\mathcal{X}(\mathcal{A})}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sqrt{d}-\sqrt{c})^2}\left[J^{\alpha}_{(\mathcal{A})^-}\mathcal{X}(\mathcal{G})+J^{\alpha}_{(\mathcal{G})^+}\mathcal{X}(\mathcal{A})\right]\right|$$
$$\leq \int_0^1 \frac{(\sqrt{d}-\sqrt{c})^2}{4}|(1-\mu)^\alpha-\mu^\alpha|\left[\mu|\mathcal{X}'(\mathcal{G})|+(1-\mu)\left|\mathcal{X}'(\mathcal{A})\right|\right]\mathrm{d}\mu$$
$$=\frac{(\sqrt{d}-\sqrt{c})^2}{4}\left[|\mathcal{X}'(\mathcal{G})|\int_0^1\mu|(1-\mu)^\alpha-\mu^\alpha|\mathrm{d}\mu+\left|\mathcal{X}'(\mathcal{A})\right|\int_0^1(1-\mu)|(1-\mu)^\alpha-\mu^\alpha|\mathrm{d}\mu\right]$$
$$=\frac{(\sqrt{d}-\sqrt{c})^2}{4(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[|\mathcal{X}'(a)|+|\mathcal{X}'(b)|].$$

This completes the proof. □

5. Quantum estimates

In this section, we derive some quantum analogues of Hermite-Hadamard like inequalities using $\mathscr{M}$-convex functions. Before proceeding, let us recall some basics of quantum calculus. Tariboon et al. [25] defined the q-integral as follows:

Definition 5.1 ([25])**.** Let $\mathcal{X} : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as:

$$\int_a^x \mathcal{X}(\mu)\,{}_a\mathrm{d}_q\mu = (1-q)(x-a)\sum_{n=0}^{\infty} q^n\mathcal{X}(q^n x+(1-q^n)a), \tag{5.1}$$

for $x \in J$.

The following result will play significant role in main results of the section.

Lemma 5.1 ([25]). *Let $\alpha \in \mathbb{R} \setminus \{-1\}$, then*

$$\int_a^x (\mu - a)^\alpha \ {}_a\mathrm{d}_q\mu = \Big(\frac{1-q}{1-q^{\alpha+1}}\Big)(x-a)^{\alpha+1}.$$

Lemma 5.2. *Let $\mathcal{X} : I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a q-differentiable function on I°, $c, d \in I^\circ$ with $c < d$. If $\mathrm{D}_q\mathcal{X}$ is an integrable function with $0 < q < 1$, then*

$$\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}} \mathcal{X}(\mu)\mathrm{d}_q\mu - \frac{qf(\mathcal{G}) + \mathcal{X}(\mathcal{A})}{1+q} = \frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)}\int_0^1 (1-(1+q)\mu)\mathrm{D}_q\Big((1-\mu)\mathcal{G}+\mu\mathcal{A}\Big)\mathrm{d}_q\mu.$$

Proof. It suffices to show that

$$\begin{aligned}
&\int_0^1 (1-(1+q)\mu)\mathrm{D}_q\Big((1-\mu)\mathcal{G}+\mu\mathcal{A}\Big)\mathrm{d}_q\mu \\
&= \frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_0^1 \left(\frac{\mathcal{X}((1-\mu)\mathcal{G}+\mu\mathcal{A}) - \mathcal{X}((1-q\mu)\mathcal{G}+q\mu\mathcal{A})}{(1-q)\mu}\right)\mathrm{d}_q\mu \\
&\quad - \frac{2(1+q)}{(\sqrt{d}-\sqrt{c})^2}\int_0^1 \mu\left(\frac{\mathcal{X}((1-\mu)\mathcal{G}+\mu\mathcal{A}) - \mathcal{X}((1-q\mu)\mathcal{G}+q\mu\mathcal{A})}{(1-q)\mu}\right)\mathrm{d}_q\mu \\
&= \frac{2}{(\sqrt{d}-\sqrt{c})^2}\left[\sum_{n=0}^{\infty}\mathcal{X}\Big((1-q^n)\mathcal{G}+q^n\mathcal{A}\Big) - \sum_{n=0}^{\infty}\mathcal{X}\Big((1-q^{n+1})\mathcal{G}+q^{n+1}\mathcal{A}\Big)\right] \\
&\quad - \frac{2(1+q)}{(\sqrt{d}-\sqrt{c})^2}\left[\sum_{n=0}^{\infty}q^n\mathcal{X}\Big((1-q^n)\mathcal{G}+q^n\mathcal{A}\Big) - \sum_{n=0}^{\infty}q^n\mathcal{X}\Big((1-q^{n+1})\mathcal{G}+q^{n+1}\mathcal{A}\Big)\right] \\
&= \frac{2}{(\sqrt{d}-\sqrt{c})^2}[\mathcal{X}(\mathcal{A}) - \mathcal{X}(\mathcal{G})] - \frac{2(1+q)}{(\sqrt{d}-\sqrt{c})^2}\sum_{n=0}^{\infty}q^n\mathcal{X}\Big((1-q^n)\mathcal{G}+q^n\mathcal{A}\Big) \\
&\quad + \frac{2(1+q)}{q(\sqrt{d}-\sqrt{c})^2}\sum_{n=1}^{\infty}q^n\mathcal{X}\Big((1-q^n)\mathcal{G}+q^n\mathcal{A}\Big) \\
&= \frac{2}{(\sqrt{d}-\sqrt{c})^2}[\mathcal{X}(\mathcal{A}) - \mathcal{X}(\mathcal{G})] - \frac{2(1+q)}{(\sqrt{d}-\sqrt{c})^2}\sum_{n=0}^{\infty}q^n\mathcal{X}\Big((1-q^n)\mathcal{G}+q^n\mathcal{A}\Big) \\
&\quad + \frac{2(1+q)}{q(\sqrt{d}-\sqrt{c})^2}\left[\mathcal{X}(\mathcal{A}) - \mathcal{X}(\mathcal{A}) + \sum_{n=1}^{\infty}q^n\mathcal{X}\Big((1-q^n)\mathcal{G}+q^n\mathcal{A}\Big)\right] \\
&= -\frac{2}{q(\sqrt{d}-\sqrt{c})^2}[qf(\mathcal{G}) + \mathcal{X}(\mathcal{A})] + \frac{4(1+q)}{q(\sqrt{d}-\sqrt{c})^4}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(\mu)\mathrm{d}_q\mu.
\end{aligned}$$

This completes the proof. □

Now using Lemma 5.2, we derive our next results.

Theorem 5.3. *Let $\mathcal{X} : I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a q-differentiable function on I°, $c, d \in I^\circ$ with $c < d$ and $\mathrm{D}_q\mathcal{X}$ is an integrable function with $0 < q < 1$. If $|\mathrm{D}_q\mathcal{X}|$ is $\mathscr{M}$-convex, then*

$$\left|\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(\mu)\mathrm{d}_q\mu - \frac{qf(\mathcal{G})+\mathcal{X}(\mathcal{A})}{1+q}\right|$$
$$\leq \frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)^4(1+q+q^2)}\left\{(1+3q^2+2q^3)|\mathrm{D}_q\mathcal{X}(\mathcal{G})| + (1+4q+q^2)\Big|\mathrm{D}_q\mathcal{X}(\mathcal{A})\Big|\right\}.$$

Proof. Using Lemma 5.2 and the given hypothesis of the theorem, we have

$$\left|\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(\mu)\mathrm{d}_q\mu - \frac{qf(\mathcal{G})+\mathcal{X}(\mathcal{A})}{1+q}\right|$$
$$= \left|\frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)}\int_0^1(1-(1+q)\mu)\mathrm{D}_q\mathcal{X}\Big((1-\mu)\mathcal{G}+\mu\mathcal{A}\Big)\mathrm{d}_q\mu\right|$$
$$\leq \frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)}\int_0^1|1-(1+q)\mu|\left[(1-\mu)|\mathrm{D}_q\mathcal{X}(\mathcal{G})| + \mu\mathrm{D}_q\Big|\mathrm{D}_q\mathcal{X}(\mathcal{A})\Big|\right]\mathrm{d}_q\mu$$
$$= \frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)}\left\{|\mathrm{D}_q\mathcal{X}(\mathcal{G})|\int_0^1(1-\mu)|1-(1+q)\mu|\mathrm{d}_q\mu + \Big|\mathrm{D}_q\mathcal{X}(\mathcal{A})\Big|\int_0^1\mu|1-(1+q)\mu|\mathrm{d}_q\mu\right\}$$
$$= \frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)^4(1+q+q^2)}\left\{(1+3q^2+2q^3)|\mathrm{D}_q\mathcal{X}(\mathcal{G})| + (1+4q+q^2)\Big|\mathrm{D}_q\mathcal{X}(\mathcal{A})\Big|\right\}.$$

This completes the proof. □

Theorem 5.4. *Let $\mathcal{X} : I^\circ \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a q-differentiable function on I°, $c, d \in I^\circ$ with $c < d$ and $\mathrm{D}_q\mathcal{X}$ is an integrable function with $0 < q < 1$. If $|\mathrm{D}_q\mathcal{X}|^r$ is $\mathscr{M}$-convex, where $r > 1$, then*

$$\left|\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(\mu)\mathrm{d}_q\mu - \frac{qf(\mathcal{G})+\mathcal{X}(\mathcal{A})}{1+q}\right|$$
$$\leq \frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\left(\frac{q(1+3q^2+2q^3)}{(1+q)^3(1+q+q^2)}|\mathrm{D}_q\mathcal{X}(\mathcal{G})|^r + \frac{q(1+4q+q^2)}{(1+q)^3(1+q+q^2)}\Big|\mathrm{D}_q\mathcal{X}(\mathcal{A})\Big|^r\right)^{\frac{1}{r}}.$$

Proof. Using Lemma 5.2, power-mean inequality and the given hypothesis of the theorem, we have

$$\left|\frac{2}{(\sqrt{d}-\sqrt{c})^2}\int_{\mathcal{G}}^{\mathcal{A}}\mathcal{X}(\mu)\mathrm{d}_q\mu - \frac{qf(\mathcal{G})+\mathcal{X}(\mathcal{A})}{1+q}\right|$$

$$= \left| \frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)} \int_0^1 (1-(1+q)\mu) \mathrm{D}_q \mathcal{X}\Big((1-\mu)\mathcal{G}+\mu\mathcal{A}\Big) \mathrm{d}_q\mu \right|$$

$$\leq \frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)} \left(\int_0^1 |1-(1+q)\mu| \mathrm{d}_q\mu \right)^{1-\frac{1}{r}}$$

$$\times \left(\int_0^1 |1-(1+q)\mu| \left[(1-\mu)|\mathrm{D}_q\mathcal{X}(\mathcal{G})|^r + \mu \Big|\mathrm{D}_q\mathcal{X}(\mathcal{A})\Big|^r \right] \mathrm{d}_q\mu \right)^{\frac{1}{r}}$$

$$= \frac{q(\sqrt{d}-\sqrt{c})^2}{2(1+q)} \left(\frac{2q}{(1+q)^2} \right)^{1-\frac{1}{r}} \left(\frac{q(1+3q^2+2q^3)}{(1+q)^3(1+q+q^2)} |\mathrm{D}_q\mathcal{X}(\mathcal{G})|^r + \frac{q(1+4q+q^2)}{(1+q)^3(1+q+q^2)} \Big|\mathrm{D}_q\mathcal{X}(\mathcal{A})\Big|^r \right)^{\frac{1}{r}}.$$

This completes the proof. □

6. Conclusion

In this article, we have introduced the notions of $\mathscr{M}$-convex functions, log-$\mathscr{M}$-convex functions and quasi $\mathscr{M}$-convex functions. We have discussed these classes in context with integral inequalities of Hermite-Hadamard type. We have also obtained some new fractional and quantum versions of these results. It is worth to mention here that essentially using the techniques of this article one can easily obtain extensions of Iynger type inequalities using the class of quasi $\mathscr{M}$-convex functions. We hope that the ideas and techniques of this paper will inspire interested readers working in the field.

Acknowledgments

Authors are thankful to the editor and anonymous referees for their valuable comments and suggestions. First and second authors are thankful for the support of HEC project (No. 8081/Punjab/NRPU/R&D/HEC/2017).

Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Alp, M. Z. Sarikaya, M. Kunt, et al. *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, J. King Saud Univ., Sci., **30** (2018), 193–203.
2. M. U. Awan, G. Cristescu, M. A. Noor, et al. *Upper and lower bounds for Riemann type quantum integrals of preinvex and preinvex dominated functions*, UPB Sci. Bull., Ser. A., **79** (2017), 33–44.
3. F. Al-Azemi, O. Calin, *Asian options with harmonic average*, Appl. Math. Inf. Sci., **9** (2015), 1–9.

4. G. Cristescu, M. A. Noor, M. U. Awan, *Bounds of the second degree cumulative frontier gaps of functions with generalized convexity*, Carpath. J. Math., **31** (2015), 173–180.

5. G. Cristescu, L. Lupsa, *Non-connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, Holland, 2002.

6. S. S. Dragomir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett., **11** (1998), 91–95.

7. S. S. Dragomir, B. Mond, *Integral inequalities of Hadamard's type for* log*-convex functions*, Demonstration Math., **2** (1998), 354–364, .

8. S. S. Dragomir, C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, Victoria University, Australia, 2000.

9. T. S. Du, J. G. Liao, Y. J. Li, *Properties and integral inequalities of Hadamard–Simpson type for the generalized* (s, m)*-preinvex functions*, J. Nonlinear Sci. Appl., **9** (2016), 3112–3126.

10. T. S. Du, Y. J. Li, Z. Q. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended* (s, m)*-convex functions*, Appl. Math. Comput., **293** (2017), 358–369.

11. A. Guessab, G. Schmeisser, *Sharp integral inequalities of the Hermite-Hadamard type*, J. Approx. Theory, **115** (2002), 260–288.

12. A. Guessab, G. Schmeisser, *Convexity results and sharp error estimates in approximate multivariate integration*, Math. Comp., **73** (2004), 1365–1384.

13. I. Iscan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe J. Math. Stat., **43** (2014), 935–942.

14. M. Khaled, P. Agarwal, *New Hermite-Hadamard type integral inequalities for convex functions and their applications*, J. Comput. Appl. Math., **350** (2019), 274–285.

15. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional dierential equations*, Elsevier B.V., Amsterdam, Netherlands, 2006.

16. K. Liu, J. R. Wang, D. O'Regan, *On the Hermite-Hadamard type inequality for* ψ*-Riemann-Liouville fractional integrals via convex functions*, J. Inequal. Appl., **2019** (2019), 27.

17. M. V. Mihai, M. A. Noor, K. I. Noor, et al. *Some integral inequalities for harmonic h-convex functions involving hypergeometric functions*, Appl. Math. Comput., **252** (2015), 257–262.

18. C. P. Niculescu, L. E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, 2 Eds. CMS Books in Mathematics, vol. 23. Springer, New York, 2018.

19. M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum estimates for Hermite-Hadamard inequalities*, Appl. Math. Comput., **251** (2015), 675–679.

20. M. A. Noor, K. I. Noor, M. U. Awan, et al. *Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions*, Appl. Math. Inf. Sci., **8** (2014), 2865–2872.

21. C. E. M. Pearce, J. Pecaric, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett., **13** (2000), 51–55.

22. J. E. Pecaric, F. Prosch, Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, NewYork, 1992.

23. L. Riahi, M. U. Awan, M. A. Noor, *Some complementary q-bounds via different classes of convex functions*, UPB Sci. Bull., Ser. A, **79** (2017), 171–182.

24. M. Z. Sarikaya, E. Set, H. Yaldiz, et al. *Hermite-Hadamards inequalities for fractional in- tegrals and related fractional inequalities*, Math. Comput. Model., **57** (2013), 2403–2407.

25. J. Tariboon, S. K. Ntouyas, *Quantum integral inequalities on finite intervals*, J. Inequal. Appl., **2014** (2014), 121.

26. Y. Zhang, T. S. Du, H. Wang, et al. *Different types of quantum integral inequalities via* (α, m)*-convexity*, J. Inequal. Appl., **2018** (2018), 264.