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Research article

On some ψ Caputo fractional Čebyšev like inequalities for functions of two and three variables

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Abstract: In this paper we obtain some ψ Caputo fractional Čebyšev like inequalities. Some new Čebyšev type inequalities involving functions of two and three variables using ψ Caputo fractional derivatives definition are obtained.

Keywords: Cebysev inequality; ψ Caputo fractional
Mathematics Subject Classification: 26A33, 26D10, 26D15

1. Introduction

P.L Čebyšev in the year 1882 has proved the following interesting inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right|$$
$$\leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty .$$

where f, g are absolutely continuous functions defined on $[a, b]$ and $f', g' \in L_\infty[a, b]$. The left hand side of the above equation is denoted by $T(f, g)$ is called Cebysev Functional if the integral exists. The applications of above type of inequalities can be found in the field of coding theory, statistics and other branches of mathematics.

In last few decades many researchers have obtained various extensions and generalizations of above inequalities using various techniques see [1, 2]. Study of inequalities have attracted the attention of researchers from various fields due to its wide applications in various fields [3, 4].

During last few years the subject of Fractional Calculus has been developed rapidly due to the applications in various fields of science and engineering. Various new definitions of fractional derivatives and integrals have been obtained by various researchers depending on the applications such as Riemann liouville, Caputo, Saigo, Hilfer, Hadmard, Katugampola and others See [5–8]. Many results on study of mathematical inequalities using various new fractional definitions such as Conformable and generalized fractional integral were obtained in [9, 10]. Recently in [11–15] the authors have obtained the results on Cebysev inequalities using various fractional integral and derivatives definitions.

In [7] authors have given definations of fractional derivative and integrals of a functions with respect to another functions. Recently in [16, 17] authors have studied the ψ Caputo and ψ Hilfer fractional derivative of a function with respect to another functions and its applications. The ψ fractional and integral definations are more generalized and it reduces to Riemann Liouville, Hadmard and Erdelyi-Kober fractional definitions for different values of ψ.

Motivated from the above mentioned literature the aim of this paper is to obtain ψ Caputo fractional Čebyšev inequalities involving functions of two and three variables.

2. Preliminaries

Now in this section we give some basic definitions and properties which are useful in our subsequent discussions. In [7, 8] the authors have defined the fractional integrals and fractional derivative of a function with respect to another function as follows.

Definition 2.1 [7, 16]. Let $I = [a, b]$ be an interval, $\alpha > 0$, f is an integrable function defined on I and $\psi \in C^1(I)$ an increasing function such that $\psi'(x) \neq 0$ for all $x \in I$ then fractional derivative and integral of f is given by

$$I_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t) \, dt$$

and

$$\begin{aligned} D_{a+}^{\alpha,\psi} f(x) &= \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n I_{a+}^{n-\alpha,\psi} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \int_a^x \psi'(t) (\psi(x) - \psi(t))^{n-\alpha-1} f(t) \, dt, \end{aligned}$$

respectively. Similarly right fractional integral and right fractional derivative are given by

$$I_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t) \, dt$$

and

$$D_{b-}^{\alpha,\psi} f(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{b-}^{n-\alpha,\psi} f(x)$$
$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x \psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1} f(t)\,dt.$$

In [16] Almedia has considered a Caputo type fractional derivative with respect to another function.

Definition 2.2 [16] Let $\alpha > 0$, $n \in \mathbb{N}$, I is the interval $-\infty \le a < b \le \infty$, $f, \psi \in C^n(I)$ two functions such that ψ is increasing and $\psi'(x) \ne 0$ for all $x \in I$. The left ψ-Caputo fractional derivative of f of order α is given by

$$^{C}D_{a+}^{\alpha,\psi} f(x) = I_{a+}^{n-\alpha,\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x),$$

and the right ψ-Caputo fractional derivative of f is given by

$$^{C}D_{b-}^{\alpha,\psi} f(x) = I_{b-}^{n-\alpha,\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x).$$

For given $\alpha \notin \mathbb{N}$

$$^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1} f_\psi^{[n]}(t)\,dt$$

and

$$^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1}(-1)^n f_\psi^{[n]}(t)\,dt.$$

In particular when $\alpha \in (0,1)$ then

$$^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x (\psi(x)-\psi(t))^{-\alpha} f'(t)\,dt$$

and

$$^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x (\psi(t)-\psi(x))^{-\alpha} f'(t)\,dt.$$

In [18] the author has defined the ψ fractional partial integral with respect to another functions as

Definition 2.3 Let $\theta = (a, b)$ and $\alpha = (\alpha_1, \alpha_2)$ where $0 \leq \alpha_1, \alpha_2 \leq 1$. Also put $I = [a, k] \times [b, m]$ where a, b and k, m are positive constants. Also let $\psi(.)$ be an increasing positive monotone function on $(a, k] \times (b, m]$ having continuous derivative $\psi'(.)$ on $(a, k] \times (b, m]$. Then the fractional partial integral is

$$I_{\theta}^{\alpha;\psi} u(x, y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^x \int_b^y \psi'(s)\psi'(t)$$
$$(\psi(x) - \psi(s))^{\alpha_1 - 1} (\psi(y) - \psi(t))^{\alpha_2 - 1} f(s, t) dt ds.$$

The Caputo fractional partial derivative is defined as follows

Definition 2.4 Let $\theta = (a, b)$ and $\alpha = (\alpha_1, \alpha_2)$ where $0 \leq \alpha_1, \alpha_2 \leq 1$. Also put $I = [a, k] \times [b, m]$ where a, b and a, b are positive constants. Also let $\psi(.)$ be an increasing function on $(a, k] \times (b, m]$ and $\psi'(.) \neq 0$ on $(a, k] \times (b, m]$. The ψ Caputo fractional partial derivative of functions of two variables of order α is given by

$${}^{C}D_{\theta}^{\alpha;\psi} u(x, y) = I_{\theta}^{2-\alpha;\psi} \left(\frac{1}{\psi'(s)\psi'(t)} \frac{\partial^2 \alpha}{\partial y \partial x} \right) u(x, y).$$

We use the following notation:

$${}^{C}D_{\theta}^{\alpha;\psi} u(x, y) = \frac{\partial_{\psi}^{2\alpha} u}{\partial_{\psi} y^{\alpha} \partial_{\psi} x^{\alpha}} (x, y).$$

We define the norm for a function of two variables as follows

$$\left\| {}^{C}D_{\theta}^{\alpha;\psi} f \right\|_{\infty} = \sup \left| {}^{C}D_{\theta}^{\alpha;\psi} f(x, y) \right|.$$

Similarly as in Definition (2.3) and (2.4) we define the ψ fractional partial integral with respect to another functions and ψ Caputo fractional partial derivative of functions of three variables as follows:

Definition 2.5 Let $\Theta = (a, b, c)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1$. Also put $I = [a, k] \times [b, m] \times [c, n]$ where a, b, c and k, m, n are positive constants. Also let $\psi(.)$ be an increasing positive monotone function on $(a, k] \times (b, m] \times [c, n]$ having continuous derivative $\psi'(.)$ on $(a, k] \times (b, m] \times (c, n]$.

Then the fractional partial integral is

$$I_{\Theta}^{\alpha;\psi} u(x, y, z) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^x \int_b^y \int_c^z \psi'(s)\psi'(t)\psi'(r)$$
$$\times (\psi(x) - \psi(s))^{\alpha_1 - 1} (\psi(y) - \psi(t))^{\alpha_2 - 1} (\psi(z) - \psi(r))^{\alpha_3 - 1} f(s, t, r) dr dt ds.$$

Definition 2.6 Let $\theta = (a, b, c)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1$. Also put $I = [a, k] \times [b, m] \times [c, n]$ where a, b, c and k, m, n are positive constants. Also let $\psi(.)$ be an increasing function on $(a, k) \times (b, m) \times (c, n]$ and $\psi'(.) \neq 0$ on $(a, k] \times (b, m] \times (c, n]$. The ψ Caputo fractional partial derivative of functions of three variables of order α is given by

$$ {}^{C}D_{\Theta}^{\alpha;\psi} u(x, y, z) = I_{\Theta}^{3-\alpha;\psi} \left(\frac{1}{\psi'(s)\psi'(t)\psi'(r)} \frac{\partial^3}{\partial z \partial y \partial x} \right) u(x, y, z). $$

We use the following notation:

$$ {}^{C}D_{\Theta}^{\alpha;\psi} u(x, y, z) = \frac{\partial_{\psi}^{3\alpha} u}{\partial_{\psi} z^{\alpha} \partial_{\psi} y^{\alpha} x^{\alpha}} (x, y, z). $$

We define the norm for a function of three variables as follows

$$ \left\| {}^{C}D_{\Theta}^{\alpha;\psi} f \right\|_{\infty} = \sup \left| {}^{C}D_{\Theta}^{\alpha;\psi} f(x, y, z) \right|. $$

3. Čebyšev inequality involving functions of two variables

Now we give the ψ Caputo fractional Čebyšev inequality involving functions of two variables as follows:

Theorem 3.1 Let $f, g : [a, l] \times [b, m] \to R$ be a continuous function on $[a, l] \times [b, m]$ and $\frac{\partial^{2\alpha} f}{\partial_{\psi} y^{\alpha} \partial_{\psi} x^{\alpha}}$, $\frac{\partial^{2\alpha} g}{\partial_{\psi} y^{\alpha} \partial_{\psi} x^{\alpha}}$ exists continuous and bounded on $[a, l] \times [b, m]$ and $\alpha = (\alpha_1, \alpha_2)$. Then

$$ \left| \int_a^l \int_b^m \left[f(x, y)g(x, y) - \frac{1}{2} \left[G(f(x, y))g(x, y) + G(g(x, y))f(x, y) \right] dydx \right] \right| $$
$$ \leq \frac{1}{8} (\psi(l) - \psi(a)) (\psi(m) - \psi(b)) $$
$$ \int_a^l \int_b^m \left[|g(x, y)| \left\| D_{\theta}^{\alpha;\psi} f \right\|_{\infty} + g(x, y) \left\| D_{\theta}^{\alpha;\psi} g \right\|_{\infty} \right] dydx, \tag{3.1} $$

where

$$ G(f(x, y)) = \frac{1}{2} [f(a, y) + f(x, m) + f(x, b) + f(l, y)] - \frac{1}{4} [f(a, b) + f(a, m) + f(l, b) + f(l, m)] $$

and

$$ H\left(\frac{\partial^{2\alpha} f}{\partial_{\psi} y^{\alpha} \partial_{\psi} x^{\alpha}} (x, y) \right) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \times $$

$$\times\left[\int_a^x\int_b^y \psi'(t)\psi'(s)\,(\psi(x)-\psi(t))^{\alpha_1-1}\,\,(\psi(y)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\,dsdt\right.$$

$$-\int_a^x\int_y^m \psi'(t)\psi'(s)\,(\psi(x)-\psi(t))^{\alpha_1-1}\,\,(\psi(m)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\,dsdt$$

$$-\int_x^l\int_b^y \psi'(t)\psi'(s)\,(\psi(l)-\psi(t))^{\alpha_1-1}\,\,(\psi(y)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\,dsdt$$

$$\left.+\int_x^l\int_y^m \psi'(t)\psi'(s)\,(\psi(l)-\psi(t))^{\alpha_1-1}\,\,(\psi(m)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\,dsdt\right].$$

Proof. From the given hypotheses for $(x,y)\in[a,l]\times[b,m]$ we have

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_a^x\int_b^y \psi'(t)\psi'(s)\\
&\times(\psi(x)-\psi(t))^{\alpha_1-1}\,\,(\psi(y)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\,dsdt\\
&=\frac{1}{\Gamma(\alpha_1)}\int_a^x \psi'(s)\,(\psi(x)-\psi(t))^{\alpha_1-1}\left[\left.\frac{\partial^\alpha f}{\partial_\psi s^\alpha}(s,t)\right|_c^y\right]\\
&=\frac{1}{\Gamma(\alpha_1)}\int_a^x \psi'(s)\,(\psi(y)-\psi(t))^{\alpha_1-1}\left[\frac{\partial^\alpha f}{\partial_\psi s^\alpha}(t,y)-\frac{\partial^\alpha f}{\partial_\psi s^\alpha}(t,b)\right]\\
&=\,f(t,y)|_a^x-f(t,b)|_a^x\\
&=f(x,y)-f(a,y)-f(x,b)+f(a,b).
\end{aligned}\tag{3.2}$$

Similarly we have

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_a^x\int_y^m \psi'(t)\psi'(s)\\
&\times(\psi(x)-\psi(t))^{\alpha_1-1}\,\,(\psi(m)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\,dsdt\\
&=-f(x,y)-f(a,m)+f(x,m)+f(a,y),
\end{aligned}\tag{3.3}$$

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_x^l\int_b^y \psi'(t)\psi'(s)$$
$$\times(\psi(l)-\psi(t))^{\alpha_1-1}\,(\psi(y)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\,dsdt$$
$$=-f(x,y)-f(l,b)+f(x,b)+f(l,y), \tag{3.4}$$

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_x^l\int_y^m \psi'(t)\psi'(s)$$
$$\times(\psi(l)-\psi(t))^{\alpha_1-1}\,(\psi(m)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(s,t)\,dsdt$$
$$=f(x,y)+f(l,b)-f(x,b)-f(l,y). \tag{3.5}$$

Adding the above identities we have

$$4f(x,y)-2[f(a,y)+f(x,m)+f(x,b)+f(l,y)]$$
$$+[f(a,b)+f(a,m)+f(l,b)+f(l,m)]$$
$$=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$
$$\left[\int_a^x\int_b^y \psi'(t)\psi'(s)(\psi(x)-\psi(t))^{\alpha_1-1}(\psi(y)-\psi(s))^{\alpha_2-1}\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)dsdt\right.$$
$$-\int_a^x\int_y^d \psi'(t)\psi'(s)(\psi(x)-\psi(t))^{\alpha_1-1}(\psi(m)-\psi(s))^{\alpha_2-1}\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)dsdt$$
$$-\int_x^l\int_b^y \psi'(t)\psi'(s)(\psi(l)-\psi(t))^{\alpha_1-1}(\psi(y)-\psi(s))^{\alpha_2-1}\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)dsdt$$
$$\left.+\int_x^l\int_y^m \psi'(t)\psi'(s)(\psi(l)-\psi(t))^{\alpha_1-1}(\psi(m)-\psi(s))^{\alpha_2-1}\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)dsdt\right]. \tag{3.6}$$

From (3.6) we have

$$f(x,y)-G(f(x,y))=\frac{1}{4}H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right), \tag{3.7}$$

for $(x,y)\in[a,l]\times[b,m]$. Similarly we have

$$g(x,y)-G(g(x,y))=\frac{1}{4}H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right), \tag{3.8}$$

for $(x,y) \in [a,l] \times [b,m]$.

Multiplying (3.7) by $g(x,y)$, (3.8) by $f(x,y)$ adding them and Integrating over $(x,y) \in [a,l] \times [b,m]$ we get

$$\int_a^l \int_b^m [2f(x,y)g(x,y) - g(x,y)G(f(x,y)) - f(x,y)G(g(x,y))]dydx$$
$$= \frac{1}{8} \int_a^l \int_b^m \left[H\left(\frac{\partial^{2\alpha} f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y) \right) g(x,y) + \frac{1}{4} f(x,y) H\left(\frac{\partial^{2\alpha} g}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y) \right) \right]. \quad (3.9)$$

From the properties of modulus we have

$$\left| H\left(\frac{\partial^{2\alpha} f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y) \right) \right| \leq \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$
$$\int_a^l \int_b^m \psi'(t)\psi'(s)(\psi(l) - \psi(t))^{\alpha_1 - 1}(\psi(m) - \psi(s))^{\alpha_2 - 1} \left| \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s) \right| dsdt$$
$$\leq (\psi(l) - \psi(a))^{\alpha_1} (\psi(m) - \psi(b))^{\alpha_2} \left\| {}^c D_\theta^{\alpha;\psi} f \right\|_\infty, \quad (3.10)$$

$$\left| H\left(\frac{\partial^{2\alpha} g}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y) \right) \right| \leq \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$
$$\int_a^l \int_b^m \psi'(t)\psi'(s)(\psi(l) - \psi(t))^{\alpha_1 - 1}(\psi(m) - \psi(s))^{\alpha_2 - 1} \left| \frac{\partial^{2\alpha} g}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s) \right| dsdt$$
$$\leq (\psi(l) - \psi(a))^{\alpha_1} (\psi(m) - \psi(b))^{\alpha_2} \left\| {}^c D_\theta^{\alpha;\psi} g \right\|_\infty. \quad (3.11)$$

From (3.9), (3.10) and (3.11) we have

$$\left| \int_a^l \int_b^m \left[f(x,y)g(x,y) - \frac{1}{2}[G(f(x,y))g(x,y) + G(g(x,y))f(x,y)] \right] dydx \right|$$
$$\leq \frac{1}{8} \int_a^l \int_b^m \left[\left| H\left(\frac{\partial^{2\alpha} f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y) \right) \right| |g(x,y)| + \left| H\left(\frac{\partial^{2\alpha} g}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y) \right) \right| |f(x,y)| \right]$$
$$\leq \frac{1}{8} \int_a^l \int_b^m \{ |g(x,y)| \left[\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right.$$
$$\times \left[\int_a^l \int_b^m \psi'(t)\psi'(s)(\psi(l) - \psi(t))^{\alpha_1 - 1}(\psi(m) - \psi(s))^{\alpha_2 - 1} \left| \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s) \right| dsdt \right]$$
$$+ |f(x,y)|$$

$$\times\left[\int_a^l\int_b^m \psi'(t)\psi'(s)(\psi(l)-\psi(t))^{\alpha_1-1}(\psi(m)-\psi(s))^{\alpha_2-1}\right.$$
$$\left.\left.\left|\frac{\partial^{2\alpha}g}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\right|dsdt\right]\right\}dydx$$
$$\leq\frac{1}{8}(\psi(l)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}$$
$$\times\int_a^l\int_b^m\left[|g(x,y)|\left\|{}^cD_\theta^{\alpha;\psi}f\right\|_\infty+|f(x,y)|\left\|{}^cD_\theta^{\alpha;\psi}g\right\|_\infty\right]dydx, \tag{3.12}$$

which is required inequality.

Theorem 3.2 Let $f,g,G(f(x,y)),G(g(f(x,y)),\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha\partial_\psi x^\alpha},\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha\partial_\psi x^\alpha}$ be as in Theorem 3.1 then

$$\left|\int_a^l\int_b^m\{f(x,y)g(x,y)-[G(f(x,y))g(x,y)+G(g(x,y))f(x,y)\right.$$
$$-G(f(x,y))G(g(x,y))]\}dydx$$
$$\leq\frac{1}{16}\{(\psi(l)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}\}^2\left\|{}^cD_\theta^{\alpha;\psi}f\right\|_\infty\left\|{}^cD_\theta^{\alpha;\psi}g\right\|_\infty, \tag{3.13}$$

for $(x,y)\in[a,l]\times[b,m]$.

Proof. Multiplying left hand side and right hand side of (3.7) and (3.8) we have

$$f(x,y)g(x,y)-[f(x,y)G(g(x,y))+g(x,y)G(f(x,y))]$$
$$=\frac{1}{16}H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right)H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right). \tag{3.14}$$

Integrating (3.14) over $[a,l]\times[b,m]$ and from the properties of modulus we get

$$\left|\int_a^l\int_b^m\{f(x,y)g(x,y)-[G(g(x,y))f(x,y)+G(f(x,y))g(x,y)]\right.$$
$$-G(f(x,y))G(g(x,y))\}dydx|$$
$$\leq\frac{1}{16}\int_a^l\int_b^m\left|H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right)\right|\left|H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right)\right|dydx. \tag{3.15}$$

Now using (3.13), (3.14) in (3.19) we get required inequality (3.13).

4. Čebyšev inequality involving functions of three variables

Now in our result we give the ψ Caputo fractional Čebyšev inequality involving functions of three variables. We use some notations as follows:

$$\begin{aligned} A\left(p\left(u,v,w\right)\right) =& \frac{1}{8}\left[p\left(a,b,c\right)+p(k,m,n)\right] \\ &-\frac{1}{4}\left[p\left(u,b,c\right)+p\left(u,m,n\right)+p\left(u,m,c\right)+p\left(u,b,n\right)\right] \\ &-\frac{1}{4}\left[p\left(a,v,c\right)+p\left(k,v,n\right)+p\left(a,v,n\right)+p\left(k,v,c\right)\right] \\ &-\frac{1}{4}\left[p\left(a,b,w\right)+p\left(k,m,w\right)+p\left(k,b,w\right)+p\left(a,m,w\right)\right] \\ &+\frac{1}{2}\left[p\left(a,v,w\right)+p\left(k,v,w\right)\right] \\ &+\frac{1}{2}\left[p\left(u,b,w\right)+p\left(u,m,w\right)\right] \\ &+\frac{1}{2}\left[p\left(u,v,c\right)+p\left(u,v,n\right)\right] \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &B\left(\frac{\partial^{3\alpha}p}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}\left(u,v,w\right)\right) \\ &= \frac{1}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_3\right)}\int\limits_a^u\int\limits_b^v\int\limits_c^w \psi'\left(r\right)\psi'\left(s\right)\psi'\left(t\right)\left(\psi\left(u\right)-\psi\left(r\right)\right)^{\alpha_1-1} \\ &\times\left(\psi\left(v\right)-\psi\left(s\right)\right)^{\alpha_2-1}\left(\psi\left(w\right)-\psi\left(t\right)\right)^{\alpha_3-1}\frac{\partial^{3\alpha}p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}\left(r,s,t\right)dtdsdr \\ &- \frac{1}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_3\right)}\int\limits_a^u\int\limits_b^v\int\limits_c^n \psi'\left(r\right)\psi'\left(s\right)\psi'\left(t\right)\left(\psi\left(u\right)-\psi\left(r\right)\right)^{\alpha_1-1} \\ &\times\left(\psi\left(v\right)-\psi\left(s\right)\right)^{\alpha_2-1}\left(\psi\left(n\right)-\psi\left(t\right)\right)^{\alpha_3-1}\frac{\partial^{3\alpha}p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}\left(r,s,t\right)dtdsdr \\ &- \frac{1}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_3\right)}\int\limits_a^u\int\limits_v^m\int\limits_c^w \psi'\left(r\right)\psi'\left(s\right)\psi'\left(t\right)\left(\psi\left(u\right)-\psi\left(r\right)\right)^{\alpha_1-1} \\ &\times\left(\psi\left(m\right)-\psi\left(s\right)\right)^{\alpha_2-1}\left(\psi\left(w\right)-\psi\left(t\right)\right)^{\alpha_3-1}\frac{\partial^{3\alpha}p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}\left(r,s,t\right)dtdsdr \\ &- \frac{1}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_3\right)}\int\limits_u^k\int\limits_b^v\int\limits_c^w \psi'\left(r\right)\psi'\left(s\right)\psi'\left(t\right)\left(\psi\left(k\right)-\psi\left(r\right)\right)^{\alpha_1-1} \\ &\times\left(\psi\left(u\right)-\psi\left(s\right)\right)^{\alpha_2-1}\left(\psi\left(w\right)-\psi\left(t\right)\right)^{\alpha_3-1}\frac{\partial^{3\alpha}p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}\left(r,s,t\right)dtdsdr \end{aligned}$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_a^u\int_r^m\int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}$$
$$\times(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}p}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$
$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_u^k\int_v^m\int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$
$$\times(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}p}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$
$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_u^k\int_b^v\int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$
$$\times(\psi(v)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}p}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$
$$-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_u^k\int_v^m\int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$
$$\times(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}p}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)\,dtdsdr. \tag{4.2}$$

Now we give our next result as

Theorem 4.1 Let $f, g : [a,k]\times[b,m]\times[c,n] \to R$ be a continuous function on $[a,l]\times[b,m]$ and $\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}$, $\frac{\partial^{3\alpha}g}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}$ exists and continuous and bounded on $[a,k]\times[b,m]\times[c,n]$. Then

$$\int_a^k\int_b^m\int_c^n \Big[f(u,v,w)g(u,v,w)$$
$$-\frac{1}{2}\left[f(u,v,w)A(g(u,v,w))+g(u,v,w)A(f(u,v,w))\right]\Big]dwdvdu$$
$$\le\frac{1}{16}(\psi(k)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}(\psi(n)-\psi(c))^{\alpha_3}$$
$$\times\int_a^k\int_b^m\int_c^n\Big[|g(u,v,w)|\left\|{}^cD_\Theta^{\alpha;\psi}f\right\|_\infty+|f(u,v,w)|\left\|{}^cD_\Theta^{\alpha;\psi}g\right\|_\infty\Big]dwdvdu, \tag{4.3}$$

where A, B are as given in (4.1), (4.2).

Proof. From the hypotheses we have for $u, v, w \in [a,k] \times [b,m] \times [c,n]$

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_a^u \int_b^v \int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}$$
$$(\psi(v)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr$$
$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^u \int_b^v \psi'(r)\psi'(s)(\psi(u)-\psi(r))^{\alpha_1-1}$$
$$(\psi(v)-\psi(s))^{\alpha_2-1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\bigg|_c^w dsdr$$
$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^u \int_b^v \psi'(r)\psi'(s)(\psi(u)-\psi(r))^{\alpha_1-1}$$
$$(\psi(v)-\psi(s))^{\alpha_2-1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,w)dsdr$$
$$- \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^u \int_b^v \psi'(r)\psi'(s)(\psi(u)-\psi(r))^{\alpha_1-1}$$
$$(\psi(v)-\psi(s))^{\alpha_2-1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,c)dsdr$$
$$= \frac{1}{\Gamma(\alpha_1)} \int_a^u \psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1} \frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,s,w)\bigg|_b^v dr$$
$$- \frac{1}{\Gamma(\alpha_1)} \int_a^u \psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1} \frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,s,c)\bigg|_b^v dr$$
$$= \frac{1}{\Gamma(\alpha_1)} \int_a^u \psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1} \frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,v,w)dr$$
$$- \frac{1}{\Gamma(\alpha_1)} \int_a^u \psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1} \frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,b,w)dr$$
$$- \frac{1}{\Gamma(\alpha_1)} \int_a^u \psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1} \frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,v,c)dr$$
$$+ \frac{1}{\Gamma(\alpha_1)} \int_a^u \psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1} \frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,b,c)dr$$
$$= f(r,v,w)|_a^u - f(r,b,w)|_a^u - f(r,v,c)|_a^u + f(r,b,c)|_a^u$$

$$= f(u,v,w) - f(a,v,w) - f(u,b,w) + f(a,b,w)$$
$$- f(u,v,c) + f(a,v,c) + f(u,b,c) + f(a,b,c).$$

Thus we have

$$f(u,v,w) = f(a,v,w) + f(u,b,w) - f(a,b,w)$$
$$+ f(u,v,c) - f(a,v,c) - f(u,b,c) - f(a,b,c)$$
$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_a^u \int_b^v \int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(u) - \psi(r))^{\alpha_1 - 1}$$
$$(\psi(v) - \psi(s))^{\alpha_2 - 1}(\psi(w) - \psi(t))^{\alpha_3 - 1} \frac{\partial^{3\alpha} f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr, \tag{4.4}$$

Similarly we have

$$f(u,v,w) = f(u,v,n) + f(a,v,w) + f(u,b,w)$$
$$+ f(a,b,n) - f(a,b,w) - f(a,v,n) - f(v,b,n)$$
$$- \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_a^u \int_b^v \int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(u) - \psi(r))^{\alpha_1 - 1}$$
$$(\psi(v) - \psi(s))^{\alpha_2 - 1}(\psi(n) - \psi(t))^{\alpha_3 - 1} \frac{\partial^{3\alpha} f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr, \tag{4.5}$$

$$f(u,v,w) = f(u,m,w) + f(u,v,c) + f(a,m,c)$$
$$+ f(a,v,w) - f(u,m,c) - f(a,m,w) - f(a,v,c)$$
$$- \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_a^u \int_v^m \int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(u) - \psi(r))^{\alpha_1 - 1}$$
$$(\psi(m) - \psi(s))^{\alpha_2 - 1}(\psi(w) - \psi(t))^{\alpha_3 - 1} \frac{\partial^{3\alpha} f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr, \tag{4.6}$$

$$f(u,v,w) = f(k,s,t) + f(k,b,c) + f(u,v,c)$$
$$+ f(u,b,w) - f(k,v,c) - f(k,b,w) - f(u,b,c)$$
$$- \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_u^k \int_b^v \int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(k) - \psi(r))^{\alpha_1 - 1}$$
$$(\psi(v) - \psi(s))^{\alpha_2 - 1}(\psi(w) - \psi(t))^{\alpha_3 - 1} \frac{\partial^{3\alpha} f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr, \tag{4.7}$$

$$f(u,v,w) = f(u,m,w) + f(u,v,n) + f(a,m,n)$$

$$+ f(a,v,w) - f(u,m,n) - f(a,m,w) - f(a,v,n)$$
$$+ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_a^u \int_v^m \int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}$$
$$(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr, \tag{4.8}$$

$$f(u,v,w) = f(r,m,t) + f(u,v,c) + f(k,s,t)$$
$$+ f(k,m,c) - f(k,m,w) - f(k,v,c) - f(u,m,c)$$
$$+ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_u^k \int_v^m \int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$
$$(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr, \tag{4.9}$$

$$f(u,v,w) = f(k,v,w) + f(k,b,n) + f(u,v,n)$$
$$+ f(u,b,t) - f(k,v,n) - f(k,b,w) - f(u,b,n)$$
$$+ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_u^k \int_b^v \int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$
$$(\psi(v)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr \tag{4.10}$$

and

$$f(u,v,w) = f(k,m,n) + f(k,v,w) + f(u,m,w)$$
$$+ f(u,v,n) - f(k,m,w) - f(k,v,n) - f(u,m,n)$$
$$+ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_u^k \int_v^m \int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$
$$(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr. \tag{4.11}$$

Adding the above identities we have

$$f(u,v,w) - A(f(u,v,w)) = \frac{1}{8} B\left(\frac{\partial^{3\alpha} f}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right), \tag{4.12}$$

for $(u,v,w) \in [a,k] \times [b,m] \times [c,n]$.

Similarly we have

$$g(u,v,w) - A(g(u,v,w)) = \frac{1}{8} B\left(\frac{\partial^{3\alpha} g}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right), \tag{4.13}$$

for $(u,v,w) \in [a,k] \times [b,m] \times [c,n]$.

Now multiplying (4.12) and (4.13) by $g(u,v,w)$ and $f(u,v,w)$ respectively, adding them and Integrating over $[a,k]\times[b,m]\times[c,n]$ we have

$$\int_a^k\int_b^m\int_c^n [f(u,v,w)\,g(u,v,w)-\frac{1}{2}[g(u,v,w)\,A(f(u,v,w))$$

$$g(u,v,w)\,A(f(u,v,w))]]\,dwdvdu$$

$$=\frac{1}{16}\int_a^k\int_b^m\int_c^n\left[g(u,v,w)\,B\left(\frac{\partial^{3\alpha}f}{\partial_\psi w^\alpha\partial_\psi v^\alpha\partial_\psi u^\alpha}(u,v,w)\right)\right.$$

$$\left.+f(u,v,w)\,B\left(\frac{\partial^{3\alpha}g}{\partial_\psi w^\alpha\partial_\psi v^\alpha\partial_\psi u^\alpha}(u,v,w)\right)\right]. \tag{4.14}$$

From the properties of modulus we have

$$\left|B\left(\frac{\partial^{3\alpha}f}{\partial_\psi w^\alpha\partial_\psi v^\alpha\partial_\psi u^\alpha}(u,v,w)\right)\right|$$

$$\leq\int_a^k\int_b^m\int_c^n\psi'(r)\psi'(s)\,\psi'(t)\,(\psi(k)-\psi(r))^{\alpha_1-1}(\psi(m)-\psi(s))^{\alpha_2-1}$$

$$\times(\psi(n)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$

$$\leq(\psi(k)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}(\psi(n)-\psi(c))^{\alpha_3}\left\|{}^C D_\Theta^{\alpha;\psi}f\right\|_\infty, \tag{4.15}$$

$$\left|B\left(\frac{\partial^{3\alpha}g}{\partial_\psi w^\alpha\partial_\psi v^\alpha\partial_\psi u^\alpha}(u,v,w)\right)\right|$$

$$\leq\int_a^k\int_b^m\int_c^n\psi'(r)\psi'(s)\,\psi'(t)\,(\psi(k)-\psi(r))^{\alpha_1-1}(\psi(m)-\psi(s))^{\alpha_2-1}$$

$$\times(\psi(n)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}g}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$

$$\leq(\psi(k)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}(\psi(n)-\psi(c))^{\alpha_3}\left\|{}^C D_\Theta^{\alpha;\psi}g\right\|_\infty. \tag{4.16}$$

Now by substituting the values from equation (4.15) and (4.16) in (4.14) we get the required inequality (4.3).

Theorem 4.2 Let f, g, $\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}$ and $\frac{\partial^{3\alpha}g}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}$ be as in Theorem 4.1. Then

$$\left|\int_a^k\int_b^m\int_c^n [f(u,v,w)\,g(u,v,w)\;-\;[A(f(u,v,w))\,g(u,v,w)\right.$$

$$A(g(u,v,w))\,f(u,v,w)-A(f(u,v,w))\,A(g(u,v,w))|\,dwdvdu$$

$$\leq \frac{1}{64}\left\{(\psi(k)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}(\psi(n)-\psi(c))^{\alpha_3}\right\}^2$$
$$\left\|{}^C D_{\Theta}^{\alpha;\psi} f\right\|_{\infty}\left\|{}^C D_{\Theta}^{\alpha;\psi} g\right\|_{\infty}, \tag{4.17}$$

for $(r,s,t)\in[a,k]\times[b,m]\times[c,n]$ and A, B are as given in (4.1), (4.2).

Proof. Multiplying left hand and right hand side of equation (4.12) and (4.13) we have

$$f(u,v,w)\,g(u,v,w)-[f(u,v,w)\,A(g(u,v,w))$$
$$+\ g(u,v,w)\,A(f(u,v,w))-A(f(u,v,w))\,A(g(u,v,w))]$$
$$=\frac{1}{64}B\left(\frac{\partial^{3\alpha}f}{\partial_{\psi}w^{\alpha}\partial_{\psi}v^{\alpha}\partial_{\psi}u^{\alpha}}(u,v,w)\right)B\left(\frac{\partial^{3\alpha}g}{\partial_{\psi}w^{\alpha}\partial_{\psi}v^{\alpha}\partial_{\psi}u^{\alpha}}(u,v,w)\right). \tag{4.18}$$

Integrating over $[a,k]\times[b,m]\times[c,n]$ and from the properties of modulus we have

$$\left|\int_a^k\int_b^m\int_c^n [f(u,v,w)\ g(u,v,w)-[f(u,v,w)\,A(g(u,v,w))\right.$$
$$\left.+g(u,v,w)\,A(f(u,v,w))-A(f(u,v,w))\,A(g(u,v,w))]]\right|dwdvdu$$
$$\leq\frac{1}{64}\int_a^k\int_b^m\int_c^n\left|B\left(\frac{\partial^{3\alpha}f}{\partial_{\psi}w^{\alpha}\partial_{\psi}v^{\alpha}\partial_{\psi}u^{\alpha}}(u,v,w)\right)\right.$$
$$\left.B\left(\frac{\partial^{3\alpha}f}{\partial_{\psi}w^{\alpha}\partial_{\psi}v^{\alpha}\partial_{\psi}u^{\alpha}}(u,v,w)\right)\right|dwdvdu. \tag{4.19}$$

Using (4.15) and (4.16) in (4.19) we get the required inequality (4.17).

Remark: If we put different values for $\psi(x)$ as x, lnx, x^{σ}then it reduces to various types of fractional Čebyšev inequalities such as Riemann Liouville fractional, Hadmard Fractional and Erdelyi-Kober fractional inequalities respectively.

5. Conclusions

In this paper, we studied Čebyšev like inequalities. We proved some new ψ Caputo fractional Čebyšev type inequalities involving functions of two and three variables.

Conflict of interest

All authors declare no conflict of interest in this paper.

References

1. B. G. Pachpatte, *New Čebyšev type inequalities via Trapezoidal like Rules*, *J. Inequal. Pure Appl. Math.*, **7** (2006).
2. B. G. Pachpatte, *New Čebyšev type inequalities involving functions of two and three variables*, *Soochow J. Math.*, **33** (2007), 569–577.
3. P. Cerone, S. S. Dragomir, *Mathematical inequalities*, Springer, 2011.
4. B. G. Pachpatte, *Analytic inequalities, Recent Advances*, Atlantis press, 2012.
5. G. A. Anastassiou, *Fractional differentation inequalities*, Springer, 2009.
6. G. A. Anastassiou, *Advances on fractional inequalities*, Springer, 2011.
7. A. A. Kilbas, H. M. Srivastava, J. J. Trujilio, *Theory and applications of fractional differential equations*, North-Holland Math. stud., **204** (2006).
8. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach Science Publishers, 1993.
9. D. Baleanu, S. D. Purohit, *Chebyshev type integral inequalities involving the fractional hypergeometric operators*, Abstr. Appl. Anal., **2014** (2014), 1–10.
10. E. Set, I. Mumcu, S. DemirbaÅ, *Conformable fractional integral inequalities of Chebyshev type*, RACSAM, **113** (2019), 2253–2259.
11. K. Nisar, G. Rahman, K. Mehrez, *Chebyshev type inequalities via generalized fractional conformable integrals*, J. Inequal. Appl., **2019** (2019), 1–9.
12. S. D. Purohit, R. K. Raina, *Chebyshev type inequalities for the Saigo fractional integrals and their q-analogus,* J. Math. Inequal., **7** (2013), 239–249.
13. F. Qi, G. Rahman, M. Hussain, et al, *Some inequalities of Čebyšev type for conformable k-fractional integral operators*, Symmetry, **10** (2018), 614.
14. E. Set, Z. Dahmani, I. Mumcu, *New extensions of Chebyshev type inequalities using generalized Katugampola integrals via PolyaSzeg inequality*, Int. J. Optim. Control Theor. Appl., **8** (2018), 137–144.
15. E. Set, J. Choi, I. Mumcu, *Chebyshev type inequalities involving generalized Katugampola fractional integral operators*, Tamkang J. Math., **50** (4), 381–390.
16. R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simult., **44** (2017), 460–481.
17. J. Vanterler, C. Sousa, E. C. De Oliveira, *A gronwall inequality and the Cauchy type problem by means of ψ-Hilfer operator*, Differ. Equ. Appl., **11** (2019), 87–106.
18. J. Vanterler, C. Sousa, E. C. De Oliveira, *On the stability of a hyperbolic fractional partial differential Equation*, Differ. Equ. Dyn. Syst., (2019), 1–22.