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Research article

An accurate solution for the generalized Black-Scholes equations governing option pricing

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Abstract: Today industries related to finance are essentially implementing advanced mathematical tools. In 1973, Fisher Black and Myron Scholes developed an eminent stochastic model which later coined as Black-Scholes differential equations for option pricing. This paper illustrates a convenient time integration scheme based on the generalized trapezoidal formulas (GTF $[\alpha = \frac{1}{3}]$) introduced by Chawla et al. in 1996. GTF is applied for the temporal discretization along with the classical finite difference schemes in space direction. The proposed scheme yields the (uniform) stability employing the uniform bound of the inverse operator, as well as second-order spatial accuracy and third-order temporal accuracy under reasonable conditions. Finally, the numerical illustrations and comparison with existing schemes demonstrate the stability and accuracy of the method.

Keywords: Black-Scholes equation; option pricing; european options ; generalized trapezoidal formulas; uniform boundedness
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1. Introduction

Pricing of options is an amplified area of discussion among financial practitioners. An option is a bond between two parties in which the option buyer buys the right, not the obligation to buy or sell an underlying asset at a prefixed strike price from or to the option writer within a fixed period. According to the option rights, options are classified into Call and Put options. An option that brings the owner the right to buy at a specific price is known as a call; an option that brings the right of the owner to sell at a particular price is known as a put. Option styles are classified into American and European options. American options can be exercised at any time up to and including the expiry. European options can only be exercised on the day of expiration. Fischer Black and Myron Scholes [1, 2] have given a mathematics model under the assumptions:

1. The change in stock price dS of the underlying satisfies the stochastic differential equation

$$dS = (\mu - D)S\,dt + \sigma S\,dW,$$

where, μ is the drift rate , D is the dividend yield, σ is the market volatility and dW is the increment of a standard Wiener process.
2. The risk-free rate of return r, the drift μ, dividend yield D, and the market volatility σ are constants.
3. The market is arbitrage-free and frictionless.

In effect, the market should be complete in the sense that any financial derivative or commodity can be hedged with a portfolio of another commodity. Now using Ito's Lemma [3], we have

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}dt^2 + \frac{\partial^2 V}{\partial S\,\partial t}dS\,dt$$

and by eliminating the market randomness, one can derive the celebrated Black-Scholes partial differential equation as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0, \quad S \in (0, \infty), \quad t \in (0, T) \tag{1.1a}$$

with the terminal condition

$$V(S, T) = \max(S - E, 0), \tag{1.1b}$$

here, T is the expiry and E is the stike price of the commodity in the option.

The analytical solution for (1.1) in a closed form [1, 2, 4], can be obtained as

$$V(S, t) = S\exp(-D(T - t))N(d_1) - E\exp(-r(T - t))N(d_2). \tag{1.2a}$$

which is also known as the Black-Scholes formula for European options, where

$$d_1 = \frac{\ln S - \ln E + (r - D + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \quad d_2 = d_1 - \sigma\sqrt{T - t}, \tag{1.2b}$$

and $N(x)$ is the cumulative standard normal distribution function given as

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{1}{2}t^2\right)dt. \tag{1.2c}$$

But in the present scenario of the financial market, the parameters σ, r and D depend highly on the asset price S and the time τ. The analytical solution of generalized Black-Scholes model

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S, t)S^2\frac{\partial^2 V}{\partial S^2} + (r(S, t) - D(S, t))S\frac{\partial V}{\partial S} - r(S, t)V = 0, \quad S \in (0, \infty), \quad t \in (0, T), \tag{1.3}$$

with the same terminal condition, is often not available. The generalised Black-Scholes model (1.3) is numerically solved in [4–14]. Cubic B-spline collocation method [5, 6] have second-order accuracy in approximating the generalized Black-Scholes model. In [14], cubic polynomial spline method in space direction after application of implicit Euler method in the time direction, producing second-order accuracy in space. A simultaneous application of the HODIE [15] and backward differentiation formulas [16–18] result in a second-order converging scheme [13] for generalized Black-Scholes equation. Besides of these models, fractal behavior of a stochastic process inflames the fractional Ito calculus for stochastic models and financial theories like time fractional Black-Scholes model [19, 20]. Also numerical methods to solve these models are given in Zheng, Liu, Turner [21], R. D Staelen, Hendy [22].

In this paper, we fixated on method of lines (MOL) to the generalised linear Black-Scholes model for European call option, firstly by elementary finite difference schemes in space direction, and the generalized trapezoidal formulas (GTF [$\alpha = \frac{1}{3}$]) introduced by Chawla et al. [23–26], in the temporal discretization for the system of ordinary differential equations. Section 2 of this article portrays the terminal value problem of linear Black- Scholes model. In Section 3, semi-discretization of the parabolic partial differential equation along with initial and the artificial boundary conditions is done, and the numerical scheme is derived. Section 4 deals with convergence and stability analysis of the numerical scheme. Numerical experimentations and error comparison with existing numerical schemes are given in Section 5, and Section 6 concludes the paper.

2. Model equation

Let $r(S,t)$, $D(S,t)$, and $\sigma(S,t)$ be sufficiently smooth and bounded functions on the domain $((0,\infty)\times(0,T))$. Consider the generalized Black-Scholes differential equation [1, 3] for European call option

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2}+(r(S,t)-D(S,t))S\frac{\partial V}{\partial S}-r(S,t)V=0,\quad S\in(0,\infty),\quad t\in(0,T), \tag{2.1}$$

here, $V(S,t)$ is the value of the European call option at the the stock price S (spatial variable) and at time t. with $V(0,t) = V_0(t), V(S,t) \sim V_\infty(t)$ as S tends to ∞ and $V(S,T) = V_T(S)$. We proceed with the often case $V_0(t) = 0, V_\infty(t) = S$ and $V_T(S) = \max\{S - E, 0\}$. Here σ denotes a statistical measure of the volatility of the underlying commodity, E, the exercise or striking price, T, the expiry time, D, the dividend pay and r, the risk-free rate of return. The parameters r, D, and σ are constant functions in the case of classical Black-Scholes equation (1.1). The existence and uniqueness of a classical solution of (2.1) is well known [27–31]. Now, it can be seen that the above model is derived in an infinite domain $(0,\infty)\times(0,T)$, which makes difficulties in composing the numerical solutions. Thus we are insisted to consider the following model defined on a truncated domain $(0, S_{max})\times(0,T)$, where S_{max} is the suitably chosen positive number.

$$\begin{aligned}
&\frac{\partial \mathscr{V}}{\partial t}+\frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 \mathscr{V}}{\partial S^2}+(r(S,t)-D(S,t))S\frac{\partial \mathscr{V}}{\partial S}-r(S,t)\mathscr{V}=0;\quad (S,t)\in(0,S_{\max})\times(0,T)\\
&\mathscr{V}(S,T)=\max\{S-E,0\},\quad S\in[0,S_{\max}]\\
&\mathscr{V}(0,t)=0,\quad t\in[0,T]\\
&\mathscr{V}(S_{\max},t)=S_{\max}\cdot\exp\left(-\int_t^T D(S_{\max},s)\,ds\right)-E\cdot\exp\left(-\int_t^T r(S_{\max},s)\,ds\right),\quad t\in[0,T]
\end{aligned} \tag{2.2}$$

The existence and uniqueness of analytical solution of (2.2) can be found in [27–31]. Here, the boundary conditions are chosen according to [32]. Moreover, it is proved in [33] that if V and $\mathscr{V}$ are solutions of (2.1) and (2.2) respectively, then at every point $(S,t) \in (0, S_{\max}) \times [0,T]$ satisfying

$$\ln(\frac{S_{max}}{S}) \geq -d(T-t),$$

we have

$$|V(S,t) - \mathscr{V}(S,t)| \leq \|V - \mathscr{V}\|_{L^{\infty}(\Lambda\times(t,T))} \left(\exp\left(-\frac{\left(\ln \frac{S_{\max}}{S}\right)\left((T-t)\times \min\{0,d\} + \ln \frac{S_{\max}}{S}\right)}{2(T-t)\left(\min_{(S,t)\in[0,S_{\max}]\times[0,T]} \sigma^2(S,t)\right)} \right)\right)$$

where $d = \inf\left\{2D(S,t) - 2r(S,t) + \sigma^2(S,t) : (S,t) \in (0, S_{\max}) \times (0,T)\right\}$ and $\Lambda = \{0, S_{\max}\}$.

Since the pay-off is not differentiable at the striking price, the resulting solution is not differentiable for the convergence of numerical approximations [34]. We replace $\max\{S-E, 0\}$ in the terminal condition by a smooth function $\phi(S) = \varphi(S-E)$ [35] defined as

$$\varphi(x) = \begin{cases} x & \text{for } x \geq \epsilon \\ c_0 + c_1 x + c_2 x^2 + \ldots + c_9 x^9 & -\epsilon < x < \epsilon \\ 0 & \text{for } x \leq -\epsilon \end{cases}$$

where $\epsilon > 0$, a small constant and $c_i, i = 0, 1, \ldots, 9$ are the constant coefficients to be determined.

Applying the following conditions on the funtion $\varphi(x)$:

$$\varphi(-\epsilon) = \varphi'(-\epsilon) = \varphi''(-\epsilon) = \varphi'''(-\epsilon) = \varphi^{(4)}(-\epsilon) = 0$$

$$\varphi(\epsilon) = \epsilon, \varphi'(\epsilon) = 1, \varphi''(\epsilon) = \varphi'''(\epsilon) = \varphi^{(4)}(\epsilon) = 0$$

we can uniquely determine the unknown coefficients $c_i, i = 0, 1, \ldots, 9$ viz.:

$$c_0 = \frac{35}{256}\epsilon, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{35}{64\epsilon}, \quad c_4 = -\frac{35}{128\epsilon^3}$$

$$c_6 = \frac{7}{64\epsilon^5}, \quad c_8 = -\frac{5}{256\epsilon^7}, \quad c_3 = c_5 = c_7 = c_9 = 0$$

Figure 1 demonstrates the smoothening procedure of the pay-off terminal condition (European call option) by the function ϕ, wherein the value of ϵ is taken as 0.5 for the better view.

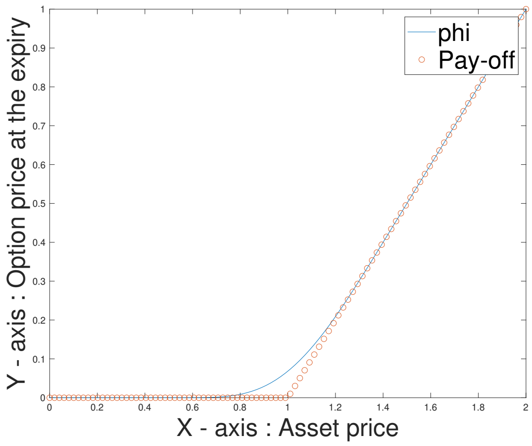

Figure 1. Smoothening of the terminal condition.

Thus we obtain another prototype in which $\mathscr{W}$ is treated as the worth or value of option,

$$\frac{\partial \mathscr{W}}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 \mathscr{W}}{\partial S^2} + (r(S,t) - D(S,t))S\frac{\partial \mathscr{W}}{\partial S} - r(S,t)\mathscr{W} = 0; \quad (S,t) \in (0, S_{\max}) \times (0,T) \quad (2.3)$$

with final condition

$$\mathscr{W}(S,T) = \phi(S), \quad S \in [0, S_{\max}]$$

and boundary conditions

$$\mathscr{W}(0,t) = 0, \; t \in [0,T]$$

$$\mathscr{W}(S_{\max}, t) = S_{\max} \exp\left(-\int_t^T D(S_{\max}, s)\, ds\right) - E \exp\left(-\int_t^T r(S_{\max}, s)\, ds\right), \quad t \in [0,T]$$

The existence and uniqueness of the analytical solution of (2.3) can be seen in [27], which also contains the proof of the following estimate:

There exists a positive constant $\mathcal{K}$ independent of $\phi(S)$ such that

$$|\mathscr{V}(S,\tau) - \mathscr{W}(S,\tau)| \leq \mathcal{K}\|\phi - \max(S-E,0)\|_{L^\infty}, \quad (S,\tau) \in [0, S_{\max}] \times [0,T]$$

To dispose of the degeneracy and backwardness of (2.3), we set $\tau = T - t, S = e^x$ and $\mathscr{W}(S,t) = u(x,\tau)$ to get

$$\frac{\partial u}{\partial \tau} = a_2(x,\tau)\frac{\partial^2 u}{\partial x^2} + a_1(x,\tau)\frac{\partial u}{\partial x} + a_0(x,\tau), \quad (x,\tau) \in \Omega = (x_{\min}, x_{\max}) \times (0,T) \quad (2.4a)$$

$$a_2(x,\tau) = \frac{1}{2}\hat{\sigma}^2(x,\tau), \; \hat{\sigma}(x,\tau) = \sigma(x, T-t)$$

$$a_1(x,\tau) = \hat{r}(x,\tau) - \hat{D}(x,\tau) - \frac{1}{2}\hat{\sigma}^2(x,\tau), \; \hat{r}(x,\tau) = r(x,T-t), \; \hat{D}(x,\tau) = D(x,T-t)$$

$$a_0(x,\tau) = -\hat{r}(x,\tau)$$

with

$$u(x,0) = \phi(x) \tag{2.4b}$$

and

$$u(x_{\min},\tau) = 0, \;\; u(x_{max},\tau) = \exp\left(x_{max} - \int_0^\tau \hat{D}(x_{max},q)dq\right) - E\exp\left(-\int_0^\tau \hat{r}(x_{max},q)dq\right) \tag{2.4c}$$

3. Numerical methods

A numerical approach to the generalized Black-Scholes equation becomes sensible in this transformed settings (2.4). Here, we execute a method of vertical lines (MOL) on (2.4) to get a system of ODEs. Thenceforth we employ the generalized trapezoidal formulas (GTF($\frac{1}{3}$)) as a numerical time integration.

3.1. Semi-discretization by Method of Lines (MOL)

Let $M+1$ be the number of price grid points $x_i = x_{\min}+ih,\ i = 0,1,...,M$, where $h = \dfrac{x_{\max} - x_{\min}}{M}$. For a positive integer N, Define temporal grid $\tau_j = jk,\ j = 0,1,...,N$, where $k = \dfrac{T}{N}$. Now let $u_{i,j} = u(x_i,\tau_j)$ and discretize spacial derivatives using classical central differences,

$$\begin{aligned}\frac{\partial u_i(\tau)}{\partial \tau} &= \frac{a_{2,i}(\tau)}{h^2}(u_{i+1}(\tau) - 2u_i(\tau) + u_{i-1}(\tau)) \\ &\quad + \frac{a_{1,i}(\tau)}{2h}(u_{i+1}(\tau) - u_{i-1}(\tau)) + a_{0,i}(\tau)u_i(\tau) \\ &= \left(\frac{a_{2,i}(\tau)}{h^2} + \frac{a_{1,i}(\tau)}{2h}\right)u_{i+1}(\tau) + \left(-\frac{2a_{2,i}(\tau)}{h^2} + a_{0,i}(\tau)\right)u_i(\tau) \\ &\quad + \left(\frac{a_{2,i}(\tau)}{h^2} - \frac{a_{1,i}(\tau)}{2h}\right)u_{i-1}(\tau), \quad i = 1,2,...,M-1\end{aligned} \tag{3.1}$$

Suppose $\vec{U}(\tau) = (u_1(\tau), u_2(\tau), ..., u_{M-1}(\tau))'$ and $\vec{U}_M(\tau) = (0,0,...,u_M(\tau))_{M-1\times 1}$, then (3.1) can be expressed as

$$\frac{\partial \vec{U}(\tau)}{\partial \tau} = A(\tau)\vec{U}(\tau) + B(\tau) \tag{3.2}$$

where

$$A = trid\left\{\frac{a_{2,i}}{h^2} - \frac{a_{1,i}}{2h},\ -\frac{2a_{2,i}}{h^2} + a_{0,i},\ \frac{a_{2,i}}{h^2} + \frac{a_{1,i}}{2h}\right\},\ i = 1,2,...,M-1$$

and

$$B(\tau) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \left(\frac{a_{2,M-1}((\tau))}{h^2} + \frac{a_{1,M-1}((\tau))}{2h}\right)u_M(\tau) \end{pmatrix},$$

Now the system (3.2) with the initial condition $\vec{U}(0) = (\phi(x_1), \phi(x_2), ..., \phi(x_{M-1}))'$ is an IVP in τ.

3.2. Full-discretization by Generalized Trapezoidal Formulas (GTF)

Let $F(\tau, \vec{U})$ be the right hand side of (3.2), we apply the generalized trapezoidal formulas [23–26] for the time integration of (3.2) to obtain,

$$\frac{\vec{U}_{j+1} - \vec{U}_j}{k} = \frac{1}{2}\left(\frac{2}{3}F_j + \frac{1}{3}\hat{F}_j + F_{j+1}\right)$$

where, $\vec{U}_j = \vec{U}(\tau_j)$, $F_j = F(\tau_j, \vec{U}_j)$ and $\hat{F}_j = F(\tau_j, \vec{U}_{j+1} - kF_{j+1})$, now by rearranging we have *the generalized trapezoidal formulas* for the generalised Black-Scholes equation,

$$\mathbf{A}_j \vec{U}_{j+1} = \mathbf{F}_j \tag{3.3}$$

with $\vec{U}_0 = (\phi_1, \phi_2, ..., \phi_{M-1})$, $\vec{U}_j(0) = u^j(x_{\min}) = u(x_{\min}, \tau_j)$ and $\vec{U}_j(M) = u^j(x_{\max}) = u(x_{\max}, \tau_j)$, where,

$$\mathbf{A}_j = \left(I - \frac{k}{6}A_j + \frac{k^2}{6}A_j A_{j+1} - \frac{k}{2}A_{j+1}\right)$$

and

$$\mathbf{F}_j = \left(I + \frac{k}{3}A_j\right)\vec{U}_j + \frac{k}{2}\left(B_j + B_{j+1}\right) - \frac{k^2}{6}A_j B_{j+1}$$

with

$$A_j = A(\tau_j) \text{ and } B_j = B(\tau_j).$$

4. Error analysis

Lemma 4.1. *Let $\hat{\sigma}$, $\hat{D}$, $\hat{r}$ be the parameters defined in (2.4), Assume that the spacial and temporal grid sizes h and k satisfies the conditions,*

$$i.\ \ h < \frac{\hat{\sigma}^2}{|(\hat{r} - \hat{D}) - \frac{1}{2}\hat{\sigma}^2|}, \tag{4.1}$$

$$ii.\ \ \frac{k^2}{6}\left(\left(\frac{a^j_{2,i+1}}{h^2} - \frac{a^j_{1,i+1}}{2h}\right)\left(\frac{a^{j+1}_{1,i}}{h} + a^{j+1}_{0,i} - \frac{2a^{j+1}_{2,i}}{h^2}\right) + \left(a^{j+1}_{0,i+1}\right)\left(a^j_{0,i+1} - \frac{2a^j_{2,i+1}}{h^2}\right)\right.$$
$$\left. + \left(\frac{a^j_{2,i+1}}{h^2} + \frac{a^j_{1,i+1}}{2h}\right)\left(\left(-\frac{a^{j+1}_{1,i+2}}{h}\right) + \left(a^{j+1}_{0,i+2} - \frac{2a^{j+1}_{2,i+2}}{h^2}\right)\right)\right) < 1 - \frac{k}{6}\left(a^j_{0,i+1} + 3a^{j+1}_{0,i+1}\right) \tag{4.2}$$

Then

$$\|\mathbf{A}_j^{-1}\|_\infty \leq \frac{1}{\alpha}$$

for some $\alpha > 1$, where, $\mathbf{A}_j$ is the matrix given by (3.3).

Proof. By the assumptions on h and k, the matrix $\mathbf{A}_j$ is a pentadiagonal diagonally dominant matrix with the minimum dominance

$$\alpha = \min_k \left(|a^j_{kk}| - \sum_{l \neq k} |a^j_{kl}| \right),$$

where $\mathbf{A}_j = (a^j_{kl})$ and the l_∞ bound of inverse is in agreement with Varah [36, 37]. □

Lemma 4.2. *Assume the conditions in lemma (4.1) for h and k, the operator L^k_h defined by*

$$L^k_h u^j_i = \mathbf{A}_j(i)\vec{U}_{j+1} = \mathbf{F}_j(i), \tag{4.3}$$

where $\mathbf{A}_j(i)$, $\mathbf{F}_j(i)$ are the i th rows of $\mathbf{A}_j$, $\mathbf{F}_j$ respectively, satisfies the consistency estimate

$$||L^k_h(u^j_i) - (Lu)^j_i|| \leq C(h^2 + k^3),\ i = 1, 2, ..., M,\ j = 1, 2, ...N.$$

for some constant $C > 0$.

Proof. The derivative $\frac{\partial u}{\partial \tau}\big|_{x=x_i}$ is given as

$$\frac{\partial u}{\partial \tau}\Big|_i = \frac{a_{2,i}}{2h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{a_{1,i}}{2h}(u_{i+1} - u_{i-1}) + a_{0,i}u_i + e^{(1)}_i(\tau) \tag{4.4}$$

where, it can be seen that (By Taylor's expansion)

$$e^{(1)}_i(\tau) = -\frac{h^2}{24}\left(a_{2,i}\frac{\partial^4}{\partial x^4} + 4a_{1,i}\frac{\partial^3}{\partial x^3} \right) u_i(\tau) + O(h^4) \tag{4.5}$$

To calculate error of the time integration scheme, we have

$$\hat{u}^j = u^{j+1} - k\frac{\partial u}{\partial \tau}\Big|^{j+1} = u^j - \frac{k^2}{2}\frac{\partial^2 u}{\partial \tau^2}\Big|^j + O(k^3) \tag{4.6}$$

and

$$\frac{u^{j+1} - u^j}{k} = \frac{1}{2}\left(\frac{2}{3}\frac{\partial u}{\partial \tau}\Big|^j + \frac{1}{3}\frac{\partial \hat{u}}{\partial \tau}\Big|^j + \frac{\partial u}{\partial \tau}\Big|^{j+1} \right) + e^j_{(2)}(x) \tag{4.7}$$

Use Taylor's expansion for u^{j+1}, $\frac{\partial u}{\partial \tau}\big|^{j+1}$ to give,

$$e^j_{(2)}(x) = \frac{k^3}{72}\frac{\partial^4 u}{\partial t^4}\Big|^j(x) + O(k^4) \tag{4.8}$$

Now, to calculate error of the scheme, we write (4.4) in the form

$$\frac{\partial u}{\partial \tau}\Big|_i = \psi_i(\tau) + e^{(1)}_i(\tau) \tag{4.9}$$

Then, an application of (4.7) gives

$$\frac{u^{j+1}_i - u^j_i}{k} = \frac{1}{2}\left(\frac{2}{3}(\psi^j_i + e^{(1),j}_i) + \frac{1}{3}\frac{\partial \hat{u}}{\partial \tau}\Big|^j_i + \psi^{j+1}_i + e^{(1),j+1}_i \right) + e^j_{(2),i} \tag{4.10}$$

Now under the assumption that the problem 2.4 satisfies sufficient regularity and compatibility conditions [27, 30], we have $\left|\frac{\partial^{m+n}u}{\partial x^m \partial \tau^n}\right| \leq C'$, for $0 \leq n \leq 3$ and $0 \leq m+n \leq 5$. Further, from the continuous problem (2.4), it can be seen that

$$\left|\frac{\partial^4 u}{\partial \tau^4}\right| \leq \left|\frac{\partial^3}{\partial \tau^3}(a_2(x,\tau)\frac{\partial^2 u}{\partial x^2})\right| + \left|\frac{\partial^3}{\partial \tau^3}(a_1(x,\tau)\frac{\partial u}{\partial x})\right| + \left|\frac{\partial^3}{\partial \tau^3}(a_0(x,\tau)u)\right| \leq C'' > 0$$

This gives

$$\begin{aligned}
\|L_h^k(u_i^j - \mathcal{U}_i^j)\| &= \|L_h^k(u_i^j) - (Lu)_i^j\| = \|\tfrac{1}{3}e_i^{(1),j} + \tfrac{1}{2}e_i^{(1),j+1} + e_{(2),i}^j\| \\
&\leq \left|\frac{h^2}{72}\left(a_{2,i}^j \frac{\partial^4}{\partial x^4} + 4a_{1,i}^j \frac{\partial^3}{\partial x^3}\right)u_i^j\right| + \left|\frac{h^2}{48}\left(a_{2,i}^{j+1} \frac{\partial^4}{\partial x^4} + 4a_{1,i}^{j+1} \frac{\partial^3}{\partial x^3}\right)u_i^{j+1}\right| + \left|\frac{k^3}{72}\left(\frac{\partial^4}{\partial \tau^4}\right)u_i^j\right| \\
&\leq C_1 h^2 + C_2 k^3 \\
&\leq C(h^2 + k^3),\ i = 1,2,...,M,\ j = 1,2,...N.
\end{aligned}$$

□

Theorem 4.3. *Let u be the solution of the problem (2.4) and $\mathcal{U}_i^j$ be the solution for the discrete problem (4.3), then*

$$\|u_i^j - \mathcal{U}_i^j\| \leq C(h^2 + k^3),\ i = 1,2,...,M,\ j = 1,2,...,N$$

for some $C > 0$.

Proof. The lemmas 4.1 and 4.2 says that the discretisation 4.3 is stable with

$$\|L_h^k(u_i^j) - (Lu)_i^j\| \leq C(h^2 + k^3),\ i = 1,2,...,M,\ j = 1,2,...N.$$

for some constant $C > 0$. Together with the uniform bound of $\mathbf{A}_j^{-1}$, we obtain

$$\|u_i^j - \mathcal{U}_i^j\| \leq C(h^2 + k^3),\ i = 1,2,...,M,\ j = 1,2,...,N$$

for some $C > 0$ [38–40]. □

5. Numerical illustrations

First we illustrate an example on the transformed Black-Scholes model (2.4) for which the closed form solution is available and is given by

$$u(x,t) = \exp(x - \hat{D}t)N\left(\hat{d}_1\right) - E\exp(-\hat{r}t)N\left(\hat{d}_2\right) \tag{5.1}$$

where

$$\hat{d}_1 = \frac{x - \ln E + \left(\hat{r} - \hat{D} + \frac{1}{2}\hat{\sigma}^2\right)t}{\hat{\sigma}\sqrt{t}} \text{ and } \hat{d}_2 = \hat{d}_1 - \hat{\sigma}\sqrt{t}$$

Let $\mathcal{U}_{i,j}^{M,N}$ be the numerical approximation with M and N points in space and time directions respectively, Compute the true L_∞ norm error (maximum absolute error, $e_{\max}^{M,N}$), L_2 norm error (root mean square error, $e_{rms}^{M,N}$) and corresponding orders of convergence $p_{max}^{M,N}$ and $p_{rms}^{M,N}$ as follows:

$$e_{\max}^{M,N} = \max_{0\leq m\leq M}\left|u(x_m, t_N) - \mathcal{U}_{m,N}^{M,N}\right|$$

$$e_{rms}^{M,N} = \sqrt{\frac{\sum_{m=0}^{M}\left(u\left(x_m, t_N\right) - \mathcal{U}_{m,N}^{M,N}\right)^2}{M+1}}$$

and

$$p_{\max}^{M,N} = log_2\left(\frac{e_{\max}^{M,N}}{e_{\max}^{2M,2N}}\right)$$

$$p_{rms}^{M,N} = log_2\left(\frac{e_{rms}^{M,N}}{e_{rms}^{2M,2N}}\right)$$

Example 1. *Here we consider the Black-Scholes equation (2.4) for European call option with* $\hat{\sigma} = 0.4, \hat{r} = 0.06$ $\hat{D} = 0.02, E = 1$ *and* $T = 1$. *For computational purpose, we assume that* $x_{\min} = -2,\ x_{\max} = +2$ and $\epsilon = 10^{-6}$. *The maximum and rms errors and their numerical order of convergence is given in Table 1. Table2 gives the the maximum norm error and their numerical order of convergence for HODIE scheme [13] for the Black-Scholes equation* ($\hat{\sigma} = 0.4, \hat{r} = 0.04$ $\hat{D} = 0.02, E = 1$ *and* $T = 1$). *The solution profile is given in Figure 2.*

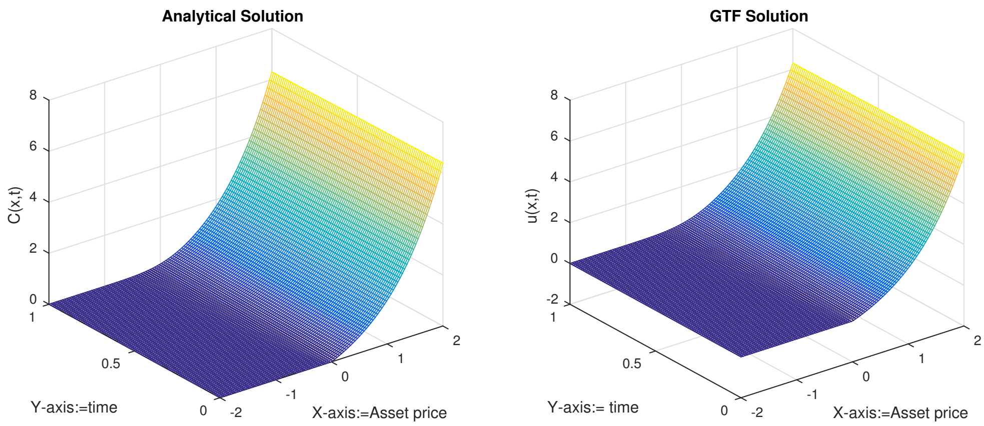

Figure 2. Exact solution and GTF solution for the example 1, $M = N = 100$.

Table 1. Example (1): Root mean square error $e_{rms}^{M,N}$,sup norm error $e_{\max}^{M,N}$ and corresponding orders of convergence $p_{rms}^{M,N}$, $p_{\max}^{M,N}$.

M	2^6	2^7	2^8	2^9	2^{10}	2^{10}
N	10×2^2	10×2^3	10×2^4	10×2^5	10×2^6	10×2^7
$e_{rms}^{M,N}$	5.6600$e-05$	1.4043$e-05$	3.4972$e-06$	8.7262$e-07$	2.1797$e-07$	5.4554$e-08$
$p_{rms}^{M,N}$		2.0109	2.0055	2.0027	2.0012	1.9984
$e_{\max}^{M,N}$	1.1602$e-04$	2.8566$e-05$	7.0855$e-06$	1.7643$e-06$	4.4024$e-07$	1.0995$e-07$
$p_{\max}^{M,N}$		2.0220	2.0113	2.0057	2.0027	2.0014

Table 2. Maximum norm error $e_{\max}^{M,N}$ and corresponding order of convergence $p_{\max}^{M,N}$ by the HODIE scheme [13].

M	2^6	2^7	2^8	2^9	2^{10}
N	10×2^2	10×2^3	10×2^4	10×2^5	10×2^6
$e_{\max}^{M,N}$	$1.7759e-03$	$4.4895e-04$	$1.1219e-04$	$2.8068e-05$	$7.0223e-06$
$p_{\max}^{M,N}$	2.0739	1.9839	2.0006	1.9989	1.9989

Due to the unavailability of exact solution data of the generalized Black-Scholes equation, we are supposed to use the double mesh principle for computing the root mean square error ($e_{rms}^{M,N}$), sup norm error ($e_{\max}^{M,N}$) and corresponding orders of convergence $p_{rms}^{M,N}$, $p_{\max}^{M,N}$ and it is given by

$$e_{rms}^{M,N} = \sqrt{\frac{\sum_{m=0}^{M}(\mathcal{U}_{m,N}^{M,N} - \mathcal{U}_{2m,2N}^{2M,2N})^2}{M+1}}$$

$$e_{\max}^{M,N} = \max_{0\le m\le M} |\mathcal{U}_{m,N}^{M,N} - \mathcal{U}_{2m,2N}^{2M,2N}|$$

and

$$p_{rms}^{M,N} = log_2\left(\frac{e_{rms}^{M,N}}{e_{rms}^{2M,2N}}\right), \quad p_{\max}^{M,N} = log_2\left(\frac{e_{\max}^{M,N}}{e_{\max}^{2M,2N}}\right).$$

In this example, we have illustrated the errors on the line $t = t_N$ which is a significant subdomain for the problem (2.4) in which the 'option premium' (Initial option price) is approximated by the generalised trapezoidal formulas.

Example 2. *Consider the generalized Black-Scholes equation for European call option price (2.4) with* $\hat{\sigma}(x,t) = 0.4(2 + t\sin(\exp(x)))$, $\hat{r}(x,t) = 0.06(1 + (T-t)\exp(-\exp(x)))$, $\hat{D}(x,t) = 0.02\exp(-t - \exp(x)), T = 1$ *and* $E = 1$. *Assume that* $x_{min} = -2$, $x_{max} = 2$ *and* $\epsilon = 10^{-6}$. *The numerics and the GTF solutions are displayed in Table 3 and Figure 3 respectively. Table 4 gives the maximum norm error and their numerical order of convergence for cubic B-spline collocation scheme combaining* $\theta-$*method [6].*

Table 3. Example (2): Root mean square error $e_{rms}^{M,N}$,sup norm error $e_{\max}^{M,N}$ and corresponding orders of convergence $p_{rms}^{M,N}$, $p_{\max}^{M,N}$.

M	10	20	40	80	160	320
N	10	20	40	80	160	320
$e_{\max}^{M,N}$	$4.1000e-03$	$1.0000e-03$	$2.4544e-04$	$6.0112e-05$	$1.4844e-05$	$3.6884e-06$
$p_{\max}^{M,N}$		2.0356	2.0265	2.0296	2.0177	2.0088
$e_{rms}^{M,N}$	$2.1000e-03$	$5.4491e-04$	$1.1902e-04$	$2.7020e-05$	$6.9777e-06$	$2.0405e-06$
$p_{rms}^{M,N}$		1.9482	2.0291	2.0167	2.0087	2.0044

Table 4. Example (2): Maximum norm error $e_{\max}^{M,N}$ and corresponding order of convergence $p_{\max}^{M,N}$ for the cubic B-spline collocation method [6].

M	10	20	40	80	160
N	10	20	40	80	160
$e_{\max}^{M,N}$ $(\theta = 1)$	$1.36957e-02$	$4.90074e-03$	$1.96168e-03$	$8.60762e-04$	$4.00390e-04$
$p_{\max}^{M,N}$		1.4827	1.3209	1.1884	1.1042
$e_{\max}^{M,N}$ $(\theta = \frac{1}{2})$	$9.71170e-03$	$2.42037e-03$	$6.05127e-04$	$1.51402e-04$	$3.78481e-05$
$p_{\max}^{M,N}$		2.0045	1.9999	1.9988	2.0001

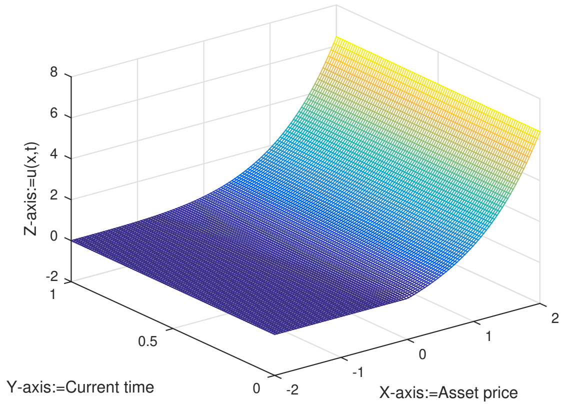

Figure 3. Solution by GTF for European call option for the example 2, $M = N = 100$.

Example 3. *Consider the generalized Black-Scholes equation (2.4) for Binary European call option with* $\hat{\sigma}(x,t) = 0.4(2 + t\sin(\exp(x)))$, $\hat{r}(x,t) = 0.06(1 + (T - t)\exp(-\exp(x)))$, $\hat{D}(x,t) = 0.02\exp(-t - \exp(x)), T = 1$ *and* $E = 1$. *Assume that* $x_{min} = -2$, $x_{max} = 2$ *and* $\epsilon = 10^{-6}$. *The numerics and the GTF solutions are displayed in Table 5 and Figure 4 respectively.*

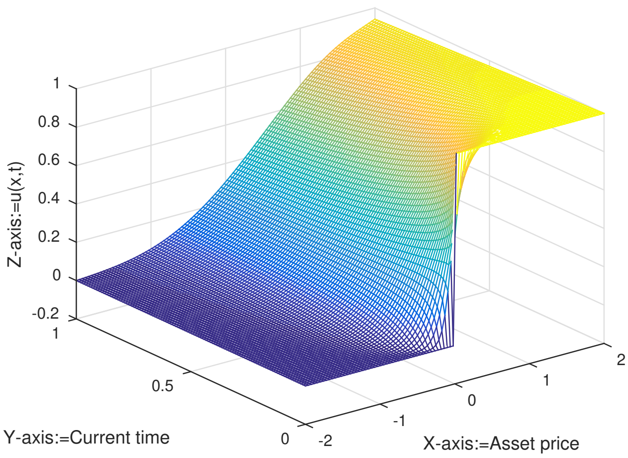

Figure 4. Solution by GTF for Binary European call option for the example 3, $Q = 1, M = 100$, $N = 100$.

Table 5. Example 3: Root mean square error $e_{rms}^{M,N}$,sup norm error $e_{\max}^{M,N}$.

M	20	40	80	160	320
N	20	40	80	160	320
$e_{\max}^{M,N}$	$1.5000e-003$	$3.6512e-004$	$8.9213e-005$	$2.2010e-005$	$5.4679e-006$
$e_{rms}^{M,N}$	$9.4307e-004$	$2.2805e-004$	$5.6190e-005$	$1.3953e-005$	$3.4768e-006$

Here the initial condition and boundary conditions are as follows

$$u(x,0) = \begin{cases} Q: & \text{if } e^x \geq E \\ 0: & \text{if } e^x \geq E \end{cases}$$
$$u(x_{\min}, t) = 0$$
$$u(x_{\max}, t) = Q \exp\left(-\int_0^t \hat{r}(x_{\max}, s)\, ds\right), \quad t \in [0, T]$$

Now we replace the initial condition $u_0(x) = u(x, 0)$ by a smooth function $\phi(x) = \varphi(e^x - E)$ [35] defined as

$$\varphi(y) = \begin{cases} Q & \text{for } y \geq \epsilon \\ c_0 + c_1 y + c_2 y^2 + \ldots + c_9 y^9 & -\epsilon < y < \epsilon \\ 0 & \text{for } y \leq -\epsilon \end{cases}$$

where $\epsilon > 0$ is a small constant and $c_i, i = 0, 1, \ldots, 9$ are the constant coefficients to be determined.

Applying the following ten conditions on the funtion $\varphi(y)$:,

$$\varphi(-\epsilon) = \varphi'(-\epsilon) = \varphi''(-\epsilon) = \varphi'''(-\epsilon) = \varphi^{(4)}(-\epsilon) = 0$$

$$\varphi(\epsilon) = Q,\ \varphi'(\epsilon) = \varphi''(\epsilon) = \varphi'''(\epsilon) = \varphi^{(4)}(\epsilon) = 0$$

we can uniquely determine the unknown coefficients $c_i, i = 0, 1, \ldots, 9$ viz.:

$$c_0 = \frac{Q}{2}, \quad c_1 = \frac{315Q}{256\epsilon}, \quad c_3 = \frac{-105Q}{64\epsilon^3}, \quad c_5 = -\frac{189Q}{128\epsilon^5}$$

$$c_7 = \frac{-45Q}{64\epsilon^7}, \quad c_9 = -\frac{35Q}{256\epsilon^9}, \quad c_2 = c_4 = c_7 = c_8 = 0$$

Figure 5 shows the smoothening procedure of the pay-off in binary European call, wherein the value of ϵ taken as 0.4 for the better view.

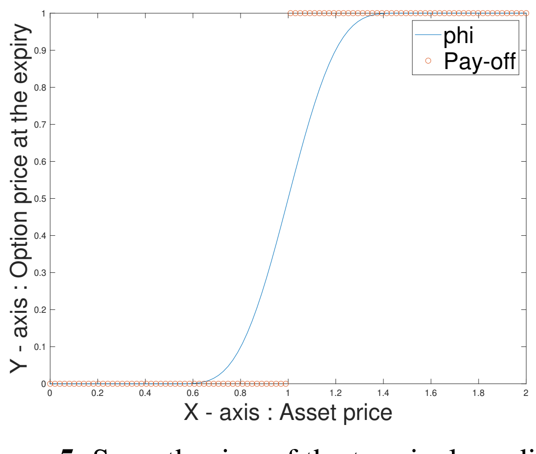

Figure 5. Smoothening of the terminal condition.

Example 4. *Consider the generalized Black-Scholes equation 2.4 for Butter fly spread option with* $\hat{\sigma}(x,t) = 0.4(2+ t\sin(\exp(x)))$, $\hat{r}(x,t) = 0.06(1 + (T - t)\exp(-\exp(x)))$, $\hat{D}(x,t) = 0.02\exp(-t - \exp(x)), T = 1$ *and three singular points* $E_1 = 1,\ E_2 = 2,\ E_3 = 3$. *Assume that* $x_{min} = -2,\ x_{max} = 2$ *and* $\epsilon = 10^{-6}$. *The numerics and the GTF solutions are displayed in Table 6 and Figure 6 respectively.*

Table 6. Example 4: Root mean square error $e^{M,N}_{rms}$,sup norm error $e^{M,N}_{\max}$.

M	20	40	80	160	320
N	20	40	80	160	320
$e^{M,N}_{\max}$	$2.8000e-003$	$4.1934e-004$	$1.3816e-004$	$5.2798e-005$	$4.5299e-006$
$e^{M,N}_{rms}$	$1.4000e-003$	$2.0269e-004$	$6.9207e-005$	$2.7778e-005$	$2.4205e-006$

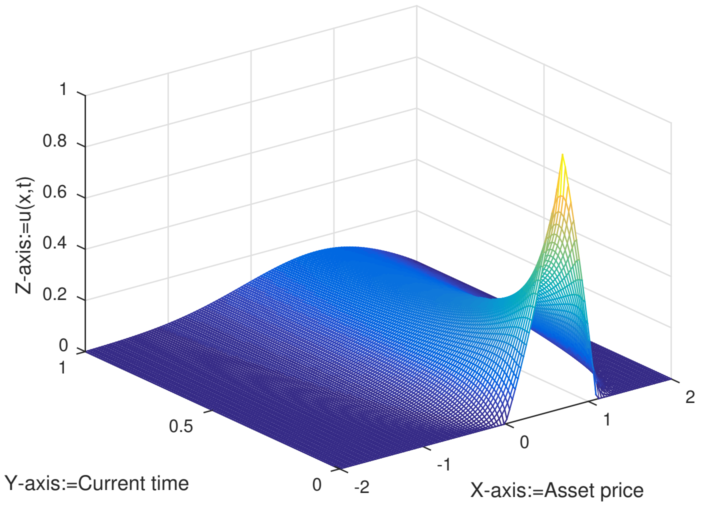

Figure 6. Solution by GTF for Butter fly spread option for the example 4, $M = N = 100$.

Here the initial condition and boundary conditions are as follows

$$\begin{aligned} &u(x,0) = \max(e^x - E_1, 0) - 2\max(e^x - E_2, 0) + \max(e^x - E_3, 0) \\ &u(x_{\min}, t) = 0 \\ &u(x_{\max}, t) = 0 \end{aligned}$$

The payoff for butterfly option has three singularities E_1, E_2 and E_3, So we replace the initial condition $u_0(x) = u(x,0)$ by a smooth function $\phi(x)$ [35] defined as follows:

$$\phi(x) = \begin{cases} 0 & \text{for } e^x \leq E_1 - \epsilon \\ \varphi_1(e^x - E_1) & \text{for } E_1 - \epsilon < e^x < E_1 + \epsilon \\ e^x - E_1 & \text{for } E_1 + \epsilon \leq e^x \leq E_2 - \epsilon \\ \varphi_2(e^x - E_2) & \text{for } E_2 - \epsilon < e^x < E_2 + \epsilon \\ E_3 - e^x & \text{for } E_2 + \epsilon \leq e^x \leq E_3 - \epsilon \\ \varphi_3(e^x - E_3) & \text{for } E_3 - \epsilon < e^x < E_3 + \epsilon \\ 0 & \text{for } e^x \geq E_3 + \epsilon \end{cases}$$

where $\varphi_1(x) = \sum_{i=0}^{9} c_i x^i$ and the coefficients $c_i, i = 0, 1, \ldots, 9$ are computed by solving the following ten conditions:

$$\varphi_1(-\epsilon) = \varphi_1'(-\epsilon) = \varphi_1''(-\epsilon) = \varphi_1'''(-\epsilon) = \varphi_1^{(4)}(-\epsilon) = 0$$
$$\varphi_1(\epsilon) = \epsilon, \varphi_1'(\epsilon) = 1, \varphi_1''(\epsilon) = \varphi_1'''(\epsilon) = \varphi_1^{(4)}(\epsilon) = 0,$$

$\varphi_2(x) = \sum_{1=0}^{9} d_i x^i$ and the coefficients $d_i, i = 0, 1, \ldots, 9$ are computed by solving the following ten conditions:

$$\varphi_2(-\epsilon) = E_2 - E_1 - \epsilon,\ \varphi_2'(-\epsilon) = 1,\ \varphi_2''(-\epsilon) = \varphi_2'''(-\epsilon) = \varphi_2^{(4)}(-\epsilon) = 0$$
$$\varphi_2(\epsilon) = E_2 - E_1 - \epsilon,\ \varphi_2'(\epsilon) = -1,\ \varphi_2''(\epsilon) = \varphi_2'''(\epsilon) = \varphi_2^{(4)}(\epsilon) = 0,$$

and $\varphi_3(x) = \sum_{1=0}^{9} e_i x^i$ and the coefficients $e_i, i = 0, 1, \ldots, 9$ are computed by solving the following ten condi-

$$\varphi_3(-\epsilon) = \epsilon, \varphi_3'(-\epsilon) = -1, \varphi_2''(-\epsilon) = \varphi_2'''(-\epsilon) = \varphi_2^{(4)}(-\epsilon) = 0$$
$$\varphi_3(\epsilon) = \varphi_3'(\epsilon) = \varphi_3''(\epsilon) = \varphi_3'''(\epsilon) = \varphi_3^{(4)}(\epsilon) = 0$$

The coefficients c_i, d_i and $e_i, i = 0, 1, \ldots, 9$ take the following values:

$$c_0 = \frac{35}{256}\epsilon, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{35}{64\epsilon}, \quad c_4 = -\frac{35}{128\epsilon^3}$$

$$c_6 = \frac{7}{64\epsilon^5}, \quad c_8 = -\frac{5}{256\epsilon^7}, \quad c_3 = c_5 = c_7 = c_9 = 0$$

$$d_0 = \frac{64(E_3-E_1)-35\epsilon}{128}, \ d_1 = \frac{315(E_1-2E_2+E_3)}{256\epsilon}, \quad d_2 = \frac{-35}{32\epsilon}$$

$$d_3 = \frac{-(105(E_1-2E_2+E_3))}{64\epsilon^3}, \quad d_4 = \frac{35}{64\epsilon^3}, \quad d_5 = \frac{189(E_1-2E_2+E_3)}{128\epsilon^5}$$

$$d_6 = \frac{-7}{32\epsilon^5}, \quad d_7 = \frac{-(45(E_1-2E_2+E_3))}{64\epsilon^7}, \quad d_8 = \frac{5}{128\epsilon^7}, \quad d_9 = \frac{35(E_1-2E_2+E_3)}{256\epsilon^9}$$

and

$$e_0 = \frac{35}{256}\epsilon, \quad e_1 = \frac{-1}{2}, \quad e_2 = \frac{35}{64\epsilon}, \quad e_4 = -\frac{35}{128\epsilon^3}$$

$$e_6 = \frac{7}{64\epsilon^5}, \quad e_8 = -\frac{5}{256\epsilon^7}, \quad e_3 = e_5 = e_7 = e_9 = 0$$

Figure 7 shows the smoothening procedure of the pay-off in butter fly spread option, wherein the value of ϵ taken as 0.5 for the better view.

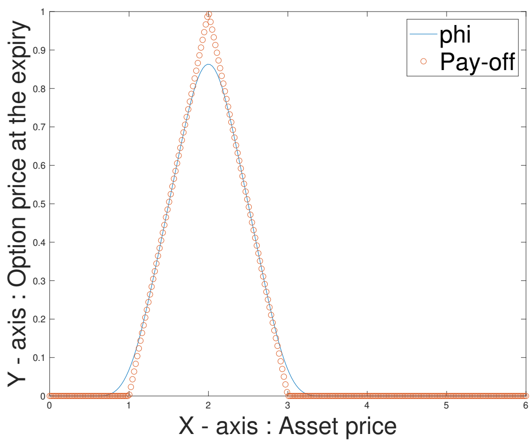

Figure 7. Smoothening of the terminal condition.

6. Conclusions

In this paper, we have applied a computationally convenient time integration scheme for the generalized Black-Scholes equations. The method is based on a central difference spatial discretization on uniform mesh and the generalized trapezoidal formulas (GTF($\frac{1}{3}$)) in time-stepping. The inverse of the matrix associated with the discrete operator is uniformly bounded by the inverse of minimum diagonal dominance and thereby stable for arbitrary volatility and interest rate. The proposed scheme is second-order consistent concerning the spatial variable and third-order in time, and by accepting the uniform bound, we obtain the convergence in the same order as the consistency. Furthermore, it can be seen that GTF ($\frac{1}{3}$) rectifies the singularities of the non-smooth payoff function by approximation with a ninth-degree polynomial. Numerical experiments, including the smoothening of payoffs and comparison with existing literature, are performed to demonstrate the efficiency of the proposed scheme.

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Conflict of interest

The authors declare no conflict of interest.

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