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Research article

On Opial-Wirtinger type inequalities

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Abstract: In the present paper we establish some new Opial-Wirtinger's type inequalities involving Katugampola partial derivatives. These new results in special cases yield Agarwal and Pang's, Traple's and Pachpatte's inequalities and provide new estimates on inequality of this type.

Keywords: Katugampola partial derivatives; Young's inequality; Opial's inequality; Wirtinger's inequality; Hölder's inequality
Mathematics Subject Classification: 26D15

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1. Introduction

In 1960, Opial [12] established the following inequality:

Theorem A *Suppose $f \in C^1[0,h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0,h)$. Then the inequality holds*

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{4}\int_0^h (f'(x))^2 dx, \tag{1.1}$$

where this constant $h/4$ is best possible.

Many generalizations and extensions of Opial's inequality were established [2, 4–11, 15–19]. For an extensive survey on these inequalities, see [13]. Opial's inequality and its generalizations and extensions play a fundamental role in the ordinary and partial differential equations as well as difference equation [2–4, 6–7, 9–11, 17]. In particular, Agarwal and Pang [3] proved the following Opial-Wirtinger's type inequalities.

Theorem B *Let $\lambda \ge 1$ be a given real number, and let $p(t)$ be a nonnegative and continuous function on $[0,a]$. Further, let $x(t)$ be an absolutely continuous function on $[0,a]$, with $x(0) = x(a) = 0$. Then*

$$\int_0^a p(t)|x(t)|^\lambda dt \le \frac{1}{2}\int_0^a [t(a-t)]^{(\lambda-1)/2} p(t)dt \int_0^a |x'(t)|^\lambda\,dt. \tag{1.2}$$

The first aim of the present paper is to establish Opial-Wirtinger's type inequalities involving Katugampola conformable partial derivatives and α-conformable integrals (see Section 2). Our result is given in the following theorem, which is a generalization of (1.2).

Theorem 1.1 *Let $\lambda \geq 1$ be a real number and $\alpha \in (0,1]$, and let $p(s,t)$ be a nonnegative and continuous functions on $[0,a]\times[0,b]$. Further, let $x(s,t)$ be an absolutely continuous function and Katugampola partial derivable on $[0,a]\times[0,b]$, with $x(s,0) = x(0,t) = x(0,0) = 0$ and $x(a,b) = x(a,t) = x(s,b) = 0$. If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ Then*

$$\int_0^a\int_0^b p(s,t)|x(s,t)|^\lambda d_\alpha s d_\alpha t \leq \frac{p+q}{pq}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left(\int_0^a\int_0^b \Gamma_{abpq\lambda\alpha}(s,t)\cdot p(s,t)d_\alpha s d_\alpha t\right) \times \int_0^a\int_0^b \left|\frac{\partial^2}{\partial s\partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t, \tag{1.3}$$

where

$$\Gamma_{abpq\lambda\alpha}(s,t) = \left\{(st)^{1/p}[(a-s)(b-t)]^{1/q}\right\}^{\alpha(\lambda-1)}.$$

Remark 1.1 Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, and $p = q = 2$ and $\alpha = 1$, (1.3) become (1.2).

Theorem C *Let $\lambda \geq 1$ be a given real number, and let $p(t)$ be a nonnegative and continuous function on $[0,a]$. Further, let $x(t)$ be an absolutely continuous function on $[0,a]$, with $x(0) = x(a) = 0$. Then*

$$\int_0^a p(t)|x(t)|^\lambda dt \leq \frac{1}{2}\left(\frac{a}{2}\right)^{\lambda-1}\left(\int_0^a p(t)dt\right)\int_0^a |x'(t)|^\lambda dt. \tag{1.4}$$

Another aim of this paper is to establish the following inequality involving Katugampola conformable partial derivatives and α-conformable integrals. Our result is given in the following theorem.

Theorem 1.2 *Let $j = 1,2$ and $\lambda \geq 1$ be a real number, and let $p_j(s,t)$ be a nonnegative and continuous functions on $[0,a]\times[0,b]$. Further, let $x_j(s,t)$ be an absolutely continuous function and Katugampola partial derivable on $[0,a]\times[0,b]$, with $x_j(s,0) = x_j(0,t) = x_j(0,0) = 0$ and $x_j(a,b) = x_j(a,t) = x_j(s,b) = 0$. Then for $\alpha \in (0,1]$*

$$\begin{aligned}
&\int_0^a\int_0^b \left(p_1(s,t)|x_1(s,t)|^\lambda + p_2(s,t)|x_2(s,t)|^\lambda\right) d_\alpha s d_\alpha t \\
&\leq \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left[\left(\int_0^a\int_0^b (st)^{\alpha(\lambda-1)}p_1(s,t)d_\alpha s d_\alpha t\right)\int_0^a\int_0^b \left|\frac{\partial^2}{\partial s\partial t}(x_1)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right. \\
&\quad \left.+\left(\int_0^a\int_0^b (st)^{\alpha(\lambda-1)}p_2(s,t)d_\alpha s d_\alpha t\right)\int_0^a\int_0^b \left|\frac{\partial^2}{\partial s\partial t}(x_2)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right].
\end{aligned} \tag{1.5}$$

2. Katugampola conformable partial derivatives

Here, let's recall the well-known Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$ and $t \in [0,\infty)$, given by

$$\mathcal{D}_\alpha(f)(t) = \lim_{\varepsilon\to 0}\frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}, \tag{2.1}$$

and

$$\mathcal{D}_\alpha(f)(0) = \lim_{t\to 0} D_\alpha(f)(t), \tag{2.2}$$

provided the limits exist. If f is fully differentiable at t, then

$$\mathcal{D}_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t).$$

A function f is α-differentiable at a point $t \geq 0$, if the limits in (2.1) and (2.2) exist and are finite. Inspired by this, we propose a new concept of α-conformable partial derivative. In the way of (1.4), α-conformable partial derivative is defined in as follows:

Definition 2.1 [20] (α-conformable partial derivative) Let $\alpha \in (0, 1]$ and $s, t \in [0, \infty)$. Suppose $f(s, t)$ is a continuous function and partial derivable, the α-conformable partial derivative at a point $s \geq 0$, denoted by $\frac{\partial}{\partial s}(f)_\alpha(s, t)$, defined by

$$\frac{\partial}{\partial s}(f)_\alpha(s, t) = \lim_{\varepsilon\to 0} \frac{f(se^{\varepsilon s^{-\alpha}}, t) - f(s, t)}{\varepsilon}, \tag{2.3}$$

provided the limits exist, and call α-conformable partial derivable.

Recently, Katugampola conformable partial derivative is defined in as follows:

Definition 2.2 [20] (Katugampola conformable partial derivatives) Let $\alpha \in (0, 1]$ and $s, t \in [0, \infty)$. Suppose $f(s, t)$ and $\frac{\partial}{\partial s}(f)_\alpha(s, t)$ are continuous functions and partial derivable, the Katugampola conformable partial derivative, denoted by $\frac{\partial^2}{\partial s\partial t}(f)_{\alpha^2}(s, t)$, defined by

$$\frac{\partial^2}{\partial s\partial t}(f)_{\alpha^2}(s, t) = \lim_{\varepsilon\to 0} \frac{\frac{\partial}{\partial s}(f)_\alpha(s, te^{\varepsilon t^{-\alpha}}) - \frac{\partial}{\partial s}(f)_\alpha(s, t)}{\varepsilon}, \tag{2.4}$$

provided the limits exist, and call Katugampola conformable partial derivable.

Definition 2.3 [20] (α-conformable integral) Let $\alpha \in (0, 1]$, $0 \leq a < b$ and $0 \leq c < d$. A function $f(x, y) : [a, b] \times [c, d] \to \mathbb{R}$ is α-conformable integrable, if the integral

$$\int_a^b \int_c^d f(x, y) d_\alpha x d_\alpha y := \int_a^b \int_c^d (xy)^{\alpha-1} f(x, y) dx dy \tag{2.5}$$

exists and is finite.

3. Main results

Theorem 3.1 *Let $\lambda \geq 1$ be a real number and $\alpha \in (0, 1]$, and let $p(s, t)$ be a nonnegative and continuous functions on $[0, a] \times [0, b]$. Further, let $x(s, t)$ be an absolutely continuous function and Katugampola partial derivable on $[0, a] \times [0, b]$, with $x(s, 0) = x(0, t) = x(0, 0) = 0$ and $x(a, b) = x(a, t) = x(s, b) = 0$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ Then*

$$\int_0^a \int_0^b p(s, t)|x(s, t)|^\lambda d_\alpha s d_\alpha t \leq \frac{p+q}{pq}\left(\frac{1}{\alpha^2}\right)^{\lambda-1} \left(\int_0^a \int_0^b \Gamma_{abpq\lambda\alpha}(s, t) \cdot p(s, t) d_\alpha s d_\alpha t\right)$$
$$\times \int_0^a \int_0^b \left|\frac{\partial^2}{\partial s\partial t}(x)_{\alpha^2}(s, t)\right|^\lambda d_\alpha s d_\alpha t, \tag{3.1}$$

where

$$\Gamma_{abpq\lambda\alpha}(s,t) = \left\{(st)^{1/p}[(a-s)(b-t)]^{1/q}\right\}^{\alpha(\lambda-1)}.$$

Proof From (2.4) and (2.5), we have

$$x(s,t) = \int_0^s \int_0^t \frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t) d_\alpha s d_\alpha t.$$

By using Hölder's inequality with indices λ and $\lambda/(\lambda-1)$, we have

$$\begin{aligned} |x(s,t)|^{\lambda/p} &\leq \left[\left(\int_0^s \int_0^t \left|\frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t)\right| d_\alpha s d_\alpha t\right)^\lambda\right]^{1/p} \\ &\leq \left(\frac{1}{\alpha^2}(st)^\alpha\right)^{(\lambda-1)/p} \left(\int_0^s \int_0^t \left|\frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right)^{1/p}. \end{aligned} \tag{3.2}$$

Similarly, from

$$x(s,t) = \int_s^a \int_t^b \frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t) d_\alpha s d_\alpha t,$$

we obtain

$$|x(s,t)|^{\lambda/q} \leq \left(\frac{1}{\alpha^2}[(a-s)(b-t)]^\alpha\right)^{(\lambda-1)/q} \left(\int_s^a \int_t^b \left|\frac{\partial^q}{\partial s \partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right)^{1/q}. \tag{3.3}$$

Now a multiplication of (3.2) and (3.3), and by using the well-known Young inequality gives

$$\begin{aligned} |x(s,t)|^\lambda \leq & \left(\frac{1}{\alpha^2}\right)^{\lambda-1} \cdot \Gamma_{abpq\lambda\alpha}(s,t) \cdot \left(\int_0^s \int_0^t \left|\frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right)^{1/p} \\ \times & \left(\int_s^a \int_t^b \left|\frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right)^{1/q} \\ \leq & \left(\frac{1}{\alpha^2}\right)^{\lambda-1} \cdot \Gamma_{abpq\lambda\alpha}(s,t) \cdot \left(\frac{1}{p} \int_0^s \int_0^t \left|\frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t \right. \\ + & \left. \frac{1}{q} \int_s^a \int_t^b \left|\frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right) \\ = & \frac{p+q}{pq} \left(\frac{1}{\alpha^2}\right)^{\lambda-1} \cdot \Gamma_{abpq\lambda\alpha}(s,t) \int_0^a \int_0^b \left|\frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t, \end{aligned} \tag{3.4}$$

where

$$\Gamma_{abpq\lambda\alpha}(s,t) = \left\{(st)^{1/p}[(a-s)(b-t)]^{1/q}\right\}^{\alpha(\lambda-1)}.$$

Multiplying the both sides of (3.4) by $p(s,t)$ and α–conformable integrating both sides over t from 0 to b first and then integrating the resulting inequality over s from 0 to a, we obtain

$$\int_0^a \int_0^b p(s,t)|x(s,t)|^\lambda d_\alpha s d_\alpha t$$

$$\leq \frac{p+q}{pq}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\cdot\int_0^a\int_0^b \Gamma_{abpq\lambda\alpha}(s,t)\cdot p(s,t)\left(\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right)d_\alpha s d_\alpha t$$

$$= \frac{p+q}{pq}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\cdot\left(\int_0^a\int_0^b \Gamma_{abpq\lambda\alpha}(s,t)\cdot p(s,t)d_\alpha s d_\alpha t\right)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t.$$

This completes the proof. ■

Remark 3.1 Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, (3.1) becomes the following result.

$$\int_0^a p(t)|x(t)|^\lambda d_\alpha t \leq \frac{p+q}{pq}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\cdot\int_0^a \Gamma_{apq\lambda\alpha}(t)p(t)d_\alpha t\int_0^a |\mathcal{D}_\alpha(x)(t)|^\lambda d_\alpha t, \tag{3.5}$$

where $\mathcal{D}_\alpha(x)(t)$ is Katugampola derivative (2.1) stated in the introduction, and

$$\Gamma_{apq\lambda\alpha}(t) = \left\{t^{1/p}(a-t)^{1/q}\right\}^{\alpha(\lambda-1)}.$$

Putting $p = q = 2$ and $\alpha = 1$ in (3.5), (3.5) becomes inequality (1.2) established by Agarwal and Pang [3] stated in the introduction.

Taking for $\alpha = 1$, $p = q = 2$ and $p(s,t) =$ constant in (3.1), we have the following interesting result.

$$\int_0^a\int_0^b |x(s,t)|^\lambda dsdt \leq \frac{1}{2}(ab)^\lambda\left[B\left(\frac{\lambda+1}{2},\frac{\lambda+1}{2}\right)\right]^2\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x(s,t)\right|^\lambda dsdt,$$

where B is the Beta function.

Theorem 3.2 *Let $j = 1,2$ and $\lambda \geq 1$ be a real number, and let $p_j(s,t)$ be a nonnegative and continuous functions on $[0,a]\times[0,b]$. Further, let $x_j(s,t)$ be an absolutely continuous function and Katugampola partial derivable on $[0,a]\times[0,b]$, with $x_j(s,0) = x_j(0,t) = x_j(0,0) = 0$ and $x_j(a,b) = x_j(a,t) = x_j(s,b) = 0$. Then for $\alpha \in (0,1]$*

$$\int_0^a\int_0^b\left(p_1(s,t)|x_1(s,t)|^\lambda + p_2(s,t)|x_2(s,t)|^\lambda\right)d_\alpha s d_\alpha t$$

$$\leq \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left[\left(\int_0^a\int_0^b (st)^{\alpha(\lambda-1)}p_1(s,t)d_\alpha s d_\alpha t\right)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x_1)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right.$$

$$\left.+\left(\int_0^a\int_0^b (st)^{\alpha(\lambda-1)}p_2(s,t)d_\alpha s d_\alpha t\right)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x_2)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right]. \tag{3.6}$$

Proof Because

$$x_1(s,t) = \int_0^s\int_0^t \frac{\partial^2}{\partial s\partial t}(x_1)_{\alpha^2}(s,t)d_\alpha s d_\alpha t = \int_s^a\int_t^b \frac{\partial^2}{\partial s\partial t}(x_1)_{\alpha^2}(s,t)d_\alpha s d_\alpha t.$$

Hence

$$|x_1(s,t)| \leq \frac{1}{2}\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x_1)_{\alpha^2}(s,t)\right|d_\alpha s d_\alpha t.$$

By Hölder's inequality with indices λ and $\lambda/(\lambda-1)$, it follows that

$$p_1(s,t)|x_1(s,t)|^\lambda \le \frac{1}{2^\lambda} p_1(s,t)\left(\int_0^a\int_0^b \left|\frac{\partial^2}{\partial s\partial t}(x_1)_{\alpha^2}(s,t)\right| d_\alpha s d_\alpha t\right)^\lambda$$

$$\le \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1} (st)^{\alpha(\lambda-1)} p_1(s,t)\int_0^a\int_0^b \left|\frac{\partial^2}{\partial s\partial t}(x_1)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t, \tag{3.7}$$

Similarly

$$p_2(s,t)|x_2(s,t)|^\lambda \le \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1} (st)^{\alpha(\lambda-1)} p_2(s,t)\int_0^a\int_0^b \left|\frac{\partial^2}{\partial s\partial t}(x_2)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t, \tag{3.8}$$

Taking the sum of (3.7) and (3.8) and α-integrating the resulting inequalities over t from 0 to b first and then over s from 0 to a, we obtain

$$\int_0^a\int_0^b \left(p_1(s,t)|x_1(s,t)|^\lambda + p_2(s,t)|x_2(s,t)|^\lambda\right) d_\alpha s d_\alpha t$$

$$\begin{aligned}
\le\; & \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left\{\int_0^a\int_0^b\left((st)^{\alpha(\lambda-1)}p_1(s,t)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x_1)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right)d_\alpha s d_\alpha t\right.\\
+\; & \left.\int_0^a\int_0^b\left((st)^{\alpha(\lambda-1)}p_2(s,t)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x_2)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right)d_\alpha s d_\alpha t\right\}\\
=\; & \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left[\left(\int_0^a\int_0^b (st)^{\alpha(\lambda-1)}p_1(s,t)d_\alpha s d_\alpha t\right)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x_1)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right.\\
+\; & \left.\left(\int_0^a\int_0^b (st)^{\alpha(\lambda-1)}p_2(s,t)d_\alpha s d_\alpha t\right)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x_2)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t\right].
\end{aligned}$$

Remark 3.2 Taking for $x_1(s,t) = x_2(s,t) = x(s,t)$ and $p_1(s,t) = p_2(s,t) = p(s,t)$ in (3.6), (3.6) changes to the following inequality.

$$\int_0^a\int_0^b p(s,t)|x(s,t)|^\lambda d_\alpha s d_\alpha t \le \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}$$

$$\times\left(\int_0^a\int_0^b (st)^{\alpha(\lambda-1)}p(s,t)d_\alpha s d_\alpha t\right)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t. \tag{3.9}$$

Putting $\alpha = 1$ in (3.9), we have

$$\int_0^a\int_0^b p(s,t)|x(s,t)|^\lambda ds dt \le \frac{1}{2^\lambda}\left(\int_0^a\int_0^b (st)^{\lambda-1}p(s,t)ds dt\right)\int_0^a\int_0^b\left|\frac{\partial^2}{\partial s\partial t}x(s,t)\right|^\lambda ds dt. \tag{3.10}$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, and $\lambda = 1$, (2.10) becomes the following result.

$$\int_0^a p(t)|x(t)|dt \le \frac{1}{2}\left(\int_0^a p(t)dt\right)\int_0^a |x'(t)|\, dt. \tag{3.11}$$

This is just a new inequality established by Agarwal and Pang [4]. For $\lambda = 2$ the inequality (3.11) has appear in the work of Traple [14], Pachpatte [13] proved it for $\lambda = 2m$ ($m \geq 1$ an integer).

Remark 3.3 Let $x_j(s,t)$ reduce to $x_j(t)$ $(j = 1,2)$ and $p_j(s,t)$ reduce to $p_j(t)$ $(j = 1,2)$ with suitable modifications, (3.6) becomes the following interesting result.

$$\int_0^a \left(p_1(t)|x_1(t)|^\lambda + p_2(t)|x_2(t)|^\lambda\right) d_\alpha t \leq \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left[\left(\int_0^a t^{\alpha(\lambda-1)} p_1(t) d_\alpha t\right)\int_0^a \left|\mathcal{D}_\alpha(x_1')(t)\right|^\lambda d_\alpha t + \left(\int_0^a t^{\alpha(\lambda-1)} p_2(t) d_\alpha t\right)\int_0^a \left|\mathcal{D}_\alpha(x_2')(t)\right|^\lambda d_\alpha t\right]. \tag{3.12}$$

Putting $\lambda = 1$ and $\alpha = 1$ in (3.12), we have the following interesting result.

$$\int_0^a (p_1(t)|x_1(t)| + p_2(t)|x_2(t)|)\, dt \leq \frac{1}{2}\left(\int_0^a p_1(t)dt \int_0^a \left|x_1'(t)\right| dt + \int_0^a p_2(t)dt \int_0^a \left|x_2'(t)\right| dt\right).$$

Finally, we give an example to verify the effectiveness of the new inequalities. Estimate the following double integrals:

$$\int_0^1 \int_0^1 \left[st(s-1)(t-1)\right]^\lambda dsdt,$$

where $\lambda \geq 1$.

Let $x_1(s,t) = x_1(s,t) = x(s,t) = st(s-1)(t-1)$, $p_1(s,t) = p_1(s,t) = p(s,t) = (st)^{1-\alpha}$, $a = b = 1$ and $0 < \alpha \leq 1$, and by using Theorem 3.2, we obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 \left[st(s-1)(t-1)\right]^\lambda dsdt \\
= \;& \int_0^1 \int_0^1 p(s,t)\, |x(s,t)|^\lambda\, d_\alpha s d_\alpha t \\
\leq \;& \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left(\int_0^1 \int_0^1 (st)^{\alpha(\lambda-1)} p(s,t) d_\alpha s d_\alpha t\right)\int_0^1 \int_0^1 \left|\frac{\partial^2}{\partial s \partial t}(x)_{\alpha^2}(s,t)\right|^\lambda d_\alpha s d_\alpha t \\
= \;& \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left(\frac{1}{\alpha(\lambda-1)+1}\right)^2 \int_0^1 \int_0^1 \left[(2s-1)(2t-1)\right]^\lambda (st)^{\alpha-1} dsdt \\
= \;& \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left(\frac{1}{\alpha(\lambda-1)+1}\right)^2 \left(\frac{1}{2^{\alpha-1}}\int_{-1}^1 t^\lambda \frac{1}{(t+1)^{1-\alpha}} dt\right)^2 \\
\leq \;& \frac{1}{2^\lambda}\left(\frac{1}{\alpha^2}\right)^{\lambda-1}\left(\frac{1}{\alpha(\lambda-1)+1}\right)^2 \left(\frac{1}{2^{\alpha-1}}\frac{2^\alpha}{\alpha}\right)^2 \\
= \;& \frac{2^{2-\lambda}}{\alpha^{2\lambda}\left(\alpha(\lambda-1)+1\right)^2}.
\end{aligned}$$

4. Conclusions

We have introduced a general version of Opial-Wirtinger's type integral inequality for the Katugampola partial derivatives. The established results are generalization of some existing Opial type integral inequalities in the previous published studies. For further investigations we propose to consider the Opial-Wirtinger's type inequalities for other partial derivatives.

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Conflict of interest

The author declares no conflicts of interest.

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