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Research article

Integral transforms involving the product of Humbert and Bessel functions and its application

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Abstract: In this paper, we develop some integral transforms involving a product of Humbert and Bessel functions with a weight $e^{-\gamma x^2}$. These integral transforms will be evaluated in terms of hypergeometric functions. Various transformation formulae are also evaluated in terms of Appell functions to complete this study. Some special cases of the evaluated integrals yield some infinite series of generalized hypergeometric and Appell functions. As application, one of our main results is investigated to give an expression of the Generalized Humbert-Gaussian beams (GHGBs) propagating through a paraxial ABCD optical system.

Keywords: integral transforms; Bessel functions; Humbert functions; Appell functions; Hypergeometric functions
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1. Introduction

Integral transforms involving special functions have gained considerable attention in the bibliography. In the last decade, many papers are investigated to study the integral transforms involving the product of Bessel and other special functions (see [4, 8, 9]). In view of this, it is worth investigating a general integral transform involving Humbert and Bessel functions with a weight $e^{-\gamma x^2}$. This integral transform is important in evaluating the expression of the generalized Humbert-Gaussian beams propagating in the space. A closed form of the considered integral will be derived. To the best of our knowledge, the results of the present contribution have not been previously published.

The following definitions are essential to recall for the present investigation:

The four two-variables hypergeometric functions, called confluent Humbert functions [7], are defined by

$$\Psi_1(a,b;c,d;x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a)_{r+s}(b)_r}{(c)_r(d)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.1}$$

$$\Phi_1(a,b;c;x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a)_{r+s}(b)_r}{(c)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.2}$$

$$\Xi_1(a,b,c;d;x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a)_r(b)_s(c)_r}{(d)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.3}$$

and

$$\Xi_2(a,b;c;x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a)_r(b)_r}{(c)_{r+s}}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.4}$$

with $|x|<1, |y|<\infty$ and $c,d\neq 0,-1,-2,\ldots$

The hypergeometric function ${}_2F_1$ is defined by (see [1])

$${}_2F_1(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}, \tag{1.5}$$

where $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

with Γ is the gamma function (see [1]).

The Kummer function is defined by the series

$${}_1F_1(\alpha;\beta;z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}\frac{z^k}{k!}. \tag{1.6}$$

In terms of the hypergeometric function (1.5), it's easy to obtain Humbert functions (1.1)–(1.4) (see [13]). Therefore, we can write

$$\Psi_1(a,b;c,d;x,y)=\sum_{s=0}^{\infty}\frac{(a)_s}{(d)_s}\frac{y^s}{s!}\,{}_2F_1(a+s,b;c;x), \tag{1.7}$$

$$\Phi_1(a,b;c;x,y)=\sum_{s=0}^{\infty}\frac{(a)_s}{(c)_s}\frac{y^s}{s!}\,{}_2F_1(a+s,b;c+s;x), \tag{1.8}$$

$$\Xi_1(a,b,c;d;x,y)=\sum_{s=0}^{\infty}\frac{(b)_s}{(d)_s}\frac{y^s}{s!}\,{}_2F_1(a,c;d+s;x), \tag{1.9}$$

and

$$\Xi_2(a,b;c;x,y)=\sum_{s=0}^{\infty}\frac{1}{(c)_s}\frac{y^s}{s!}\,{}_2F_1(a,b;c+s;x), \tag{1.10}$$

with $|x| < 1, |y| < \infty$ and $c, d \neq 0, -1, -2, \ldots$

The four Appell series [2], which are double hypergeometric series, are defined by

$$F_1(a, b, b'; c; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.11}$$

with $\max\{|x|, |y|\} < 1$;

$$F_2(a, b, b'; c, c'; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.12}$$

with $|x| + |y| < 1$;

$$F_3(a, a', b, b'; c; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.13}$$

with $\max\{|x|, |y|\} < 1$;
and

$$F_4(a, b; c, c'; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.14}$$

with $\sqrt{|x|} + \sqrt{|y|} < 1$.

These expressions, with c and c' are neither zero nor a negative integer, can be expressed in terms of ${}_2F_1$ as follows (see [13])

$$F_1(a, b, b'; c; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{x^m}{m!}\, {}_2F_1(a+m, b'; c+m; y), \tag{1.15}$$

$$F_2(a, b, b'; c, c'; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{x^m}{m!}\, {}_2F_1(a+m, b'; c'; y), \tag{1.16}$$

$$F_3(a, a', b, b'; c; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{x^m}{m!}\, {}_2F_1(a', b'; c+m; y), \tag{1.17}$$

and

$$F_4(a, b; c; c'; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{x^m}{m!}\, {}_2F_1(a+m, b+m; c'; y). \tag{1.18}$$

2. Main results

2.1. Evaluation of the integral I_m

In this section, we now evaluate the following integral transform containing Humbert functions $\mathcal{F}$ $(= \Psi_1, \Phi_1, \Xi_1$ or $\Xi_2)$:

$$I_m(\mathcal{F}) = \int_0^{\infty} t^{2m+d} e^{-\gamma t^2} J_{d-1}(\beta t)\mathcal{F}(y, xt^2)dt, \tag{2.1}$$

$$\Re(2m+2d)>0 \text{ and } \Re(\gamma)>0.$$

In (2.1), we replaced x and y with y and xt^2 respectively in (1.1), (1.2), (1.3) and (1.4).

Theorem 2.1. *The following transformations hold true:*

$$I_m(\Psi_1)=\mathcal{A}\sum_{q=0}^{\infty}(a)_q\frac{(m+d)_q}{(c')_q}\frac{(\frac{x}{\gamma})^q}{q!}{}_2F_1\left(a+q,b;d;y\right){}_1F_1\left(m+d+q;d;\frac{-\beta^2}{4\gamma}\right), \tag{2.2}$$

$$I_m(\Phi_1)=\mathcal{A}\sum_{q=0}^{\infty}\frac{(a)_q(m+d)_q}{(d)_q}\frac{(\frac{x}{\gamma})^q}{q!}{}_2F_1\left(a+q,b;d+q;y\right){}_1F_1\left(m+d+q;d;\frac{-\beta^2}{4\gamma}\right), \tag{2.3}$$

$$I_m(\Xi_1)=\mathcal{A}\sum_{q=0}^{\infty}\frac{(b)_q(m+d)_q}{(c')_q}\frac{(\frac{x}{\gamma})^q}{q!}{}_2F_1\left(a,d;c'+q;y\right){}_1F_1\left(m+d+q;d;\frac{-\beta^2}{4\gamma}\right), \tag{2.4}$$

and

$$I_m(\Xi_2)=\mathcal{A}\sum_{q=0}^{\infty}\frac{(m+d)_q}{(d)_q}\frac{(\frac{x}{\gamma})^q}{q!}{}_2F_1\left(a,b;d+q;y\right){}_1F_1\left(m+d+q;d;\frac{-\beta^2}{4\gamma}\right), \tag{2.5}$$

where, $\mathcal{A}=\frac{1}{\beta\gamma^m}\left(\frac{\beta}{2\gamma}\right)^d(d)_m$.

Proof. The proof of integral transform in (2.2) is as follows:

By substituting (1.7) in (2.1), we obtain

$$I_m(\Psi_1)=\mathcal{A}\sum_{q=0}^{\infty}\frac{(a)_q}{(c')_q}\frac{x^q}{q!}{}_2F_1\left(a+q,b;d;y\right)I_q, \tag{2.6}$$

where

$$I_q=\int_0^{\infty}t^{\mu}e^{-\gamma t^2}J_{d-1}(\beta t)dt, \tag{2.7}$$

with

$$\mu=2m+2q+d. \tag{2.8}$$

By means of the identity [6]

$$\int_0^{\infty}t^{\mu}e^{-\gamma t^2}J_{\nu}(\beta t)dt=\frac{1}{2}\left(\frac{\beta}{2}\right)^{\nu}\frac{\Gamma(\frac{\nu+\mu+1}{2})}{\nu!\gamma^{\frac{\nu+\mu+1}{2}}}{}_1F_1\left(\frac{\nu+\mu+1}{2};\nu+1;\frac{-\beta^2}{4\gamma}\right) \tag{2.9}$$

$$(\Re(\mu+\nu)>-1\ ,\ \Re(\gamma)>0),$$

and using (1.6), (2.7) can be written as

$$I_q=\frac{1}{2}\left(\frac{\beta}{2}\right)^{d-1}\frac{1}{(d-1)!}\frac{\Gamma(m+q+d)}{\gamma^{m+q+d}}{}_1F_1\left(m+q+d;d;\frac{-\beta^2}{4\gamma}\right). \tag{2.10}$$

Substituting (2.10) into (2.6), the assertion (2.1) of Theorem 2.1 directly follows.

Note that if $m = 0$ and $c = d$, one finds the undermentioned integral:

$$\begin{aligned} I_{m=0}(\Psi_1) &= \int_0^{\infty} t^c e^{-\gamma t^2} J_{c-1}(\beta t)\Psi_1\left(a, b; c, c'; y, xt^2\right) dt \\ &= \frac{1}{\beta}\left(\frac{\beta}{2\gamma}\right)^c \sum_{q=0}^{\infty} \frac{(a)_q (c)_q}{(c')_q} \frac{(\frac{x}{\gamma})^q}{q!} {}_2F_1\left(a+q, b; c; y\right) {}_1F_1\left(c+q; c; \frac{-\beta^2}{4\gamma}\right). \end{aligned} \tag{2.11}$$

□

Proof. The proof of integral transform (2.3) is as follows:

Considering $\mathcal{F}(y, xt^2) = \Phi_1\left(a, b; d; y, xt^2\right)$ given by (1.8), (2.1) becomes in this case

$$I_m(\Phi_1) = \sum_{q=0}^{\infty} \frac{(a)_q}{(d)_q} \frac{x^q}{q!} {}_2F_1\left(a+q, b; d+q; y\right) I_q, \tag{2.12}$$

where I_q is given by (2.7).

By using the identity (2.9), (2.12) can be written as

$$I_m(\Phi_1) = \frac{1}{\beta}\left(\frac{\beta}{2\gamma}\right)^d \frac{\Gamma(m+q+d)}{\Gamma(d)\gamma^{m+q}} {}_1F_1\left(m+q+d; d; \frac{-\beta^2}{4\gamma}\right). \tag{2.13}$$

Substituting this last equation in the expression of $I_m(\Phi_1)$, yields the assertion (2.3) of Theorem 2.1. □

Note here that for $m = 0$ and $c = d$, (2.12) becomes

$$\begin{aligned} I_{m=0}(\Phi_1) &= \int_0^{\infty} t^c e^{-\gamma t^2} J_{c-1}(\beta t)\, \Phi_1\left(a, b; c; y, xt^2\right) dt \\ &= \frac{1}{\beta}\left(\frac{\beta}{2\gamma}\right)^c \sum_{q=0}^{\infty} (a)_q \frac{(\frac{x}{\gamma})^q}{q!} {}_2F_1\left(a+q, b; c+q; y\right) {}_1F_1\left(c+q; c; \frac{-\beta^2}{4\gamma}\right). \end{aligned} \tag{2.14}$$

Proof. The proof of the integral transform (2.4) is as follows:

Following the same procedure as above, we use the definition of Ξ_1 given by (1.9) and replace $\mathcal{F}(y, xt^2)$ by $\mathcal{F}(y, xt^2) = \Xi_1\left(a, b, c; c'; y, xt^2\right)$, then the considered integral $I_m(\Xi_1)$ can be written as

$$I_m(\Xi_1) = \sum_{q=0}^{\infty} \frac{(b)_q}{(c')_q} \frac{x^q}{q!} {}_2F_1\left(a, d; c'+q; y\right) I_q, \tag{2.15}$$

where I_q is given by (2.7).

Now, by using (2.9) and substituting (2.7) in (2.15), (2.4) is proved. □

By putting $m = 0$ and $c = d$, (2.15) can be written as

$$\begin{aligned} I_{m=0}(\Xi_1) &= \int_0^{\infty} t^c e^{-\gamma t^2} J_{c-1}(\beta t)\, \Xi_1\left(a, b, c; c'; y, xt^2\right) dt \\ &= \frac{1}{\beta}\left(\frac{\beta}{2\gamma}\right)^c \sum_{q=0}^{\infty} \frac{(b)_q (c)_q}{(c')_q} \frac{(\frac{x}{\gamma})^q}{q!} {}_2F_1\left(a, c; c'+q; y\right) {}_1F_1\left(c+q; c; \frac{-\beta^2}{4\gamma}\right). \end{aligned} \tag{2.16}$$

Proof. The proof of the integral transform (2.5) is as follows:
Taking $\mathcal{F}(y, xt^2) = \Xi_2\left(a, b; d; y, xt^2\right)$ given by (1.10), (2.1) becomes

$$I_m(\Xi_2) = \sum_{q=0}^{\infty} \frac{1}{(d)_q} \frac{x^q}{q!}\, {}_2F_1\left(a, b; d+q; y\right) I_q. \tag{2.17}$$

Finally, by introducing the expression of I_q given by (2.10), one finds (2.5). This completes the proof of Theorem 2.1. □

For $m = 0$ and $c = d$, (2.17) becomes

$$I_{m=0}(\Xi_2) = \frac{1}{\beta}\left(\frac{\beta}{2\gamma}\right)^c \sum_{q=0}^{\infty} \frac{(\frac{x}{\gamma})^q}{q!}\, {}_2F_1\left(a, b; c+q; y\right) {}_1F_1\left(c+q; c; \frac{-\beta^2}{4\gamma}\right). \tag{2.18}$$

2.2. *Evaluation of some series*

In this section, we evaluate the series obtained from the integral transforms given by Theorem 2.1 in the case of $m = 0$ and $c = d$. These series are given by (see (2.11), (2.14), (2.16) and (2.18)).

$$\mathcal{S}_1 = \sum_{q=0}^{\infty} \frac{(a)_q (c)_q}{(c')_q} \frac{x^q}{q!} {}_2F_1\left(a+q, b; c; y\right) {}_1F_1\left(c+q; c; z\right), \tag{2.19}$$

$$\mathcal{S}_2 = \sum_{q=0}^{\infty} (a)_q \frac{x^q}{q!} {}_2F_1\left(a+q, b; c+q; y\right) {}_1F_1\left(c+q; c; z\right), \tag{2.20}$$

$$\mathcal{S}_3 = \sum_{q=0}^{\infty} \frac{(b)_q (c)_q}{(c')_q} \frac{x^q}{q!} {}_2F_1\left(a, c; c'+q; y\right) {}_1F_1\left(c+q; c; z\right), \tag{2.21}$$

and

$$\mathcal{S}_4 = \sum_{q=0}^{\infty} \frac{x^q}{q!} {}_2F_1\left(a, b; c+q; y\right) {}_1F_1\left(c+q; c; z\right). \tag{2.22}$$

Theorem 2.2. *The following result holds true:*

$$\mathcal{S}_1 = \frac{e^z}{(1-x)^a} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(\frac{y}{1-x})^n}{n!} \Phi_1\left(a+n, c'-c; c'; \frac{x}{x-1}, \frac{xz}{1-x}\right) \tag{2.23}$$

$$= \mathcal{G} \sum_{n=0}^{\infty} \frac{(a)_n}{(c')_n} \frac{Y^n}{n!} F_2\left(a+n, c'-c, b; c'+n; X, Z\right) \tag{2.24}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} F_2\left(a, c+n, b; c', c; x, y\right). \tag{2.25}$$

We note that in (2.24), $\mathcal{G} = \frac{e^{-\frac{Y}{X}}}{(1-X)^{-a}}$, $X = \frac{x}{x-1}$, $Y = \frac{xz}{1-x}$ *and* $Z = \frac{y}{1-x}$.

Proof of Eq. (2.23): We start from (2.19), which, by using the identities (see [10, 12])

$$(-n)_k = \begin{cases} \frac{(-1)^k (n)!}{(n-k)!}, & 0 \le k \le n \\ 0, & k > n \end{cases}, \tag{2.26}$$

$$(\lambda)_{m+n} = (\lambda)_m (\lambda + m)_n, \tag{2.27}$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n + k), \tag{2.28}$$

can be rearranged as

$$\mathcal{S}_1 = e^z \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{y^n}{n!} \mathcal{G}_n, \tag{2.29}$$

where

$$\mathcal{G}_n = \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \frac{(a+n)_{q+s} (c)_{q+s}}{(c)_s \, (c')_{s+q}} \frac{(xz)^s}{s!} \frac{x^q}{q!}. \tag{2.30}$$

With the help of (1.5) and (2.27), this last expression can be written as

$$\mathcal{G}_n = \sum_{s=0}^{\infty} \frac{(a+n)_s}{(c')_s} \frac{(xz)^s}{s!} \, {}_2F_1 \left(a + n + s, c + s; c' + s; x\right). \tag{2.31}$$

By using the Euler's transformation [11]

$${}_2F_1 \left(a, b; c; z\right) = (1 - z)^{-a} \, {}_2F_1 \left(a, c - b; c; \frac{z}{z-1}\right), \tag{2.32}$$

(2.29) becomes

$$\mathcal{S}_1 = \frac{e^z}{(1-x)^a} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(\frac{y}{1-x})^n}{n!} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a+n)_{s+p} (c'-c)_p}{(c')_{s+p}} \frac{(\frac{x}{x-1})^p}{p!} \frac{(\frac{xz}{1-x})^s}{s!}, \tag{2.33}$$

where, the double summation in this last expression is $\Phi_1 \left(a + n, c' - c; c'; \frac{x}{x-1}, \frac{xz}{1-x}\right)$ given by (1.2). Therefore, the result in (2.23) is proved.

Proof of Eq. (2.24): To prove (2.24), we start from (2.23) by puting

$$X = \frac{x}{x-1}, Y = \frac{xz}{1-x} \text{ and } Z = \frac{y}{1-x}, \tag{2.34}$$

so that, the summation $\mathcal{S}_1$ can be written as

$$\mathcal{S}_1 = \frac{e^{\frac{-Y}{X}}}{(1-X)^{-a}} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(c'-c)_p}{(c')_{m+p}} \sum_{n=0}^{\infty} \frac{(a)_{n+m+p} (b)_n}{(c)_n} \frac{X^p}{p!} \frac{Y^m}{m!} \frac{Z^n}{n!}, \tag{2.35}$$

which, by using the identity

$$(a)_{m+p+n} = (a)_n(a+n)_{m+p} = (a)_{m+p}(a+m+p)_n, \tag{2.36}$$

becomes

$$\mathcal{S}_1 = \frac{e^{\frac{-Y}{X}}}{(1-X)^{-a}} \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{m+p}(c'-c)_p}{(c')_{m+p}} \frac{X^p}{p!}\frac{Y^m}{m!}\, {}_2F_1\left(a+m+p, b; c; Z\right). \tag{2.37}$$

After some simplifications and by using (1.15), and the elementary identities

$$(a+p)_m = \frac{(a+m)_p(a)_m}{(a)_p}, \tag{2.38}$$

and

$$(c'+p)_m = \frac{(c'+m)_p(c')_m}{(c')_p}, \tag{2.39}$$

(2.37) can be expressed in terms of the Appell function F_2 as given in (2.24). This completes the proof.
Proof of Eq. (2.25): We start from the expression of $\mathcal{S}_1$ and with the help of the identities

$$(c)_q(c+q)_n = (c)_{q+n}, \text{ and } (c+q)_n = (c+n)_q\frac{(c)_n}{(c)_q}, \tag{2.40}$$

$\mathcal{S}_1$ can be written in terms of ${}_2F_1$ as follows

$$\mathcal{S}_1 = \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{q=0}^{\infty}\frac{(a)_q(c+n)_q}{(c')_q}\frac{x^q}{q!}\, {}_2F_1\left(a+q, b; c; y\right). \tag{2.41}$$

If one uses (1.15), $\mathcal{S}_1$ can be written in terms of the second Appell function $F_2\left(a, c+n, b; c', c; x, y\right)$. This completes the proof of (2.25) and consequently Theorem 2.2.

Theorem 2.3. *The following result holds true:*

$$\mathcal{S}_2 = \frac{e^z}{(1-x)^a}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{(\frac{y}{1-x})^n}{n!}\Phi_1\left(a+n, n; c+n; \frac{x}{x-1}, \frac{xz}{1-x}\right), \tag{2.42}$$

$$= \frac{e^z}{(1-x)^a}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{(\frac{y}{1-x})^n}{n!}F_2\left(a+n, -, n; c+n; \frac{x}{x-1}, \frac{xz}{1-x}\right), \tag{2.43}$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{n!}F_1\left(a, c+n, b; c', c; x, y\right). \tag{2.44}$$

Proof of Eq. (2.42): To prove the result (2.42), we use the identities given by (2.26), (2.27) and (2.28) and the result [10]

$${}_1F_1\left(c+q; c; z\right) = e^z\, {}_1F_1\left(-q; c; -z\right) = e^z\sum_{s=0}^{q}\frac{(-q)_s}{(c)_s}\frac{(-z)^s}{s!}, \tag{2.45}$$

to rewrite the expression of $\mathcal{S}_2$ given by (2.20). Consequently, it becomes

$$\mathcal{S}_2 = e^z \sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_n (a+n)_{q+s} (b)_n}{(c)_s\,(c+q+s)_n} \frac{x^q}{q!} \frac{(xz)^s}{s!} \frac{y^n}{n!}, \tag{2.46}$$

which can be written as

$$\mathcal{S}_2 = e^z \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{y^n}{n!} \sum_{s=0}^{\infty} \frac{(a+n)_s}{(c+n)_s} \frac{(xz)^s}{s!} {}_2F_1\,(a+n+s, c+s; c+n+s; x)\,. \tag{2.47}$$

The summation in (2.47) can be rearranged, by using the Euler's transformation given by (2.32), as

$$\begin{aligned}\mathcal{S}_2 =& \frac{e^z}{(1-x)^a} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(\frac{y}{1-x})^n}{n!} \\ &\times \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a+n)_s (a+n+s)_p (n)_p}{(c+n)_s\,(c+n+s)_p} \frac{(\frac{xz}{1-x})^s}{s!} \frac{(\frac{x}{x-1})^p}{p!}.\end{aligned} \tag{2.48}$$

Now, by employing the following identities

$$\begin{aligned}(a+n)_{q+s} &= (a+n)_s (a+n+s)_q, \\ (c+n)_{q+s} &= (c+n)_s (c+n+s)_q,\end{aligned} \tag{2.49}$$

The expression of $\mathcal{S}_2$ can be written as

$$\mathcal{S}_2 = \frac{e^z}{(1-x)^a} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(\frac{y}{1-x})^n}{n!} \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a+n)_{s+p} (n)_p}{(c+n)_{s+p}} \frac{(\frac{xz}{1-x})^s}{s!} \frac{(\frac{x}{x-1})^p}{p!}. \tag{2.50}$$

Since, the double summation in (2.50) is the Humbert function given by (1.2), that is, $\Phi_1\left(a+n, n; c+n; \frac{x}{x-1}, \frac{xz}{1-x}\right)$. Therefore, the required proof of (2.42) straightforwardly follows.

Proof of Eq. (2.43): The summation in (2.50) can be rewritten as

$$\begin{aligned}\mathcal{S}_2 =& \frac{e^z}{(1-x)^a} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(\frac{y}{1-x})^n}{n!} \\ &\times \sum_{s=0}^{\infty} \frac{(a+n)_s}{(c+n)_s} \frac{(\frac{xz}{1-x})^s}{s!} \sum_{p=0}^{\infty} \frac{(a+n+s)_p (n)_p}{(c+n+s)_p} \frac{(\frac{x}{x-1})^p}{p!},\end{aligned} \tag{2.51}$$

which after introducing the hypergeometric function, becomes

$$\begin{aligned}\mathcal{S}_2 =& \frac{e^z}{(1-x)^a} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(\frac{y}{1-x})^n}{n!} \\ &\times \sum_{s=0}^{\infty} \frac{(a+n)_s}{(c+n)_s} \frac{(\frac{xz}{1-x})^s}{s!} {}_2F_1\left(a+n+s, n; c+n+s; \frac{x}{x-1}\right).\end{aligned} \tag{2.52}$$

By using the expression of the second Appell function given by (1.15), (2.52) can be rearranged to yield (2.43). This completes the proof.

Proof of Eq.(2.44): By using the elementary identities

$$(\lambda+q)_n = \frac{(\lambda)_n(\lambda+n)_q}{(\lambda)_q}, \tag{2.53}$$

and

$$(\lambda+q)_s = \frac{(\lambda)_{q+s}}{(\lambda)_q}, \tag{2.54}$$

(2.20) can be written as

$$\mathcal{S}_2 = \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty}\frac{(a)_{q+s}(c+n)_q(b)_s}{(c)_{q+s}}\frac{x^q}{q!}\frac{y^s}{s!}. \tag{2.55}$$

With the help of the definition of the first Appell function given by (2.55), we can easily obtain (2.44). This completes the proof of Theorem 2.3.

Theorem 2.4. *The following result holds true:*

$$\mathcal{S}_3 = \frac{e^z}{(1-x)^b}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n}{(c')_n}\frac{y^n}{n!}\,\Phi_1\left(b, c'+n-c; c'+n; \frac{x}{x-1}, \frac{xz}{1-x}\right), \tag{2.56}$$

$$= \frac{e^z}{(1-x)^b}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n}{(c')_n}\frac{y^n}{n!}\,F_1\left(b, -, c'+n-c; c'+n; \frac{x}{x-1}, \frac{xz}{1-x}\right), \tag{2.57}$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{n!}F_3\left(b, a, c+n; c, c'; x, y\right). \tag{2.58}$$

Proof of Eq. (2.56): By the use of (2.28), (2.53) and (2.54), $\mathcal{S}_3$ can be expressed as

$$\mathcal{S}_3 = e^z\sum_{n=0}^{\infty}\frac{(a)_n(c)_n}{(c')_n}\frac{y^n}{n!}\sum_{s=0}^{\infty}\frac{(b)_s}{(c'+n)_s}\frac{(xz)^s}{s!}\sum_{q=0}^{\infty}\frac{(b+s)_q(c+s)_q}{(c'+n+s)_q}\frac{x^q}{q!}. \tag{2.59}$$

By replacing the last summation in (2.59) by the hypergeometric function ${}_2F_1$ and using the Euler's transformation given by (2.32), we find

$$\mathcal{S}_3 = \frac{e^z}{(1-x)^b}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n}{(c')_n}\frac{y^n}{n!}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\frac{(b)_{s+p}(c'+n-c)_p}{(c'+n)_{s+p}}\frac{(\frac{x}{x-1})^p}{p!}\frac{(\frac{xz}{1-x})^s}{s!}. \tag{2.60}$$

The last summation in this equation is none other than the Humbert function $\Phi_1\left(b, c'+n-c; c'+n; \frac{x}{x-1}, \frac{xz}{1-x}\right)$ given by (1.2). From here, the proof of (2.56) straightforwardly follows.

Proof of Eq. (2.57): We start from (2.59) which can be written, with the help of (2.32), as

$$\begin{aligned}\mathcal{S}_3 = &\frac{e^z}{(1-x)^b}\sum_{n=0}^{\infty}\frac{(a)_n(c)_n}{(c')_n}\frac{y^n}{n!}\sum_{s=0}^{\infty}\frac{(b)_s}{(c'+n)_s}\frac{(\frac{xz}{1-x})^s}{s!}\\ &\times{}_2F_1\left(b+s, c'+n-c; c'+n+s; \frac{x}{x-1}\right).\end{aligned} \tag{2.61}$$

Now, by recalling (1.14), $\mathcal{S}_3$ can be rewritten in terms of the first Appell function $F_1(a, b, b'; c; x, y)$ as expressed in (2.57). This completes the proof.

Proof of Eq. (2.58): We write (2.21) of $\mathcal{S}_3$ as

$$\mathcal{S}_3 = \sum_{q=0}^{\infty}\sum_{s=0}^{\infty} \frac{(b)_q(c)_q(c+q)_s}{(c')_q(c)_s} \frac{x^q}{q!} \frac{z^s}{s!} \, {}_2F_1(a, c; c'+q; y), \tag{2.62}$$

which, with the help of the identities given by (2.27), becomes

$$\mathcal{S}_3 = \sum_{s=0}^{\infty} \frac{z^s}{s!} \sum_{q=0}^{\infty} \frac{(b)_q(c+s)_q}{(c')_q} \frac{x^q}{q!} \, {}_2F_1(a, c; c'+q; y). \tag{2.63}$$

Recalling the expression of the third Appell function F_3 given by (1.16) that yields (2.58) in terms of series of this function. This completes the proof of (2.58) and the proof of Theorem 2.4.

Theorem 2.5. *The following result holds true:*

$$\mathcal{S}_4 = \frac{e^z}{(1-x)^a} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{y^n}{n!} \Phi_1\left(a, n, c+n; \frac{x}{x-1}, \frac{xz}{1-x}\right), \tag{2.64}$$

$$= \frac{e^z}{(1-x)^a} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{y^n}{n!} F_1\left(a, -, n, c+n; \frac{x}{x-1}, \frac{xz}{1-x}\right), \tag{2.65}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} F_3(a, a, c+n, b; c; x, y). \tag{2.66}$$

Proof of Eq. (2.64): The use of (2.26), (2.27), (2.28) and (2.45) yields the following expression of $\mathcal{S}_4$

$$\mathcal{S}_4 = e^z \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{y^n}{n!} \sum_{s=0}^{\infty} \frac{(a)_s}{(c)_s} \frac{(xz)^s}{s!} \sum_{q=0}^{\infty} \frac{(a+s)_q(c)_{q+s}}{(c+n)_{q+s}} \frac{x^q}{q!}, \tag{2.67}$$

or in the terms of the hypergeometric function ${}_2F_1$ as

$$\mathcal{S}_4 = e^z \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{y^n}{n!} \sum_{s=0}^{\infty} \frac{(a)_s}{(c+n)_s} \frac{(xz)^s}{s!} \, {}_2F_1(a+s, c+s; c+n+s; x). \tag{2.68}$$

With the help of the Euler's transformation (see (2.32)), (2.68) can be rewritten as

$$\mathcal{S}_4 = \frac{e^z}{(1-x)^a} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{y^n}{n!} \sum_{p=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_{s+p}(n)_p}{(c+n)_{s+p}} \frac{(\frac{x}{x-1})^p}{p!} \frac{(\frac{xz}{1-x})^s}{s!}. \tag{2.69}$$

This last summation in the above equation is the Humbert function $\Phi_1\left(a, n; c+n; \frac{x}{x-1}; \frac{xz}{1-x}\right)$. If one introduces this function, the result (2.64) is proved.

Proof of Eq. (2.65): If we apply the Euler's transformation on the function ${}_2F_1$ which appears in (2.69), we find

$$\mathcal{S}_4 = \frac{e^z}{(1-x)^a}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{y^n}{n!}\sum_{s=0}^{\infty}\frac{(a)_s}{(c+n)_s}\frac{(\frac{xz}{1-x})^s}{s!}{}_2F_1\left(a+s,n;c+n+s;\frac{x}{x-1}\right). \tag{2.70}$$

The last summation in (2.70) is the first Appell function $F_1\left(a,-,n,c+n;\frac{x}{x-1},\frac{xz}{1-x}\right)$ (see (1.14)), which proves the desired result (2.65).

Proof of Eq. (2.66):

The series in (2.22) can be written as

$$\mathcal{S}_4 = \sum_{q=0}^{\infty}(a)_q\frac{x^q}{q!}\sum_{s=0}^{\infty}\frac{(c+q)_s}{(c)_q}\frac{z^s}{s!}\,{}_2F_1\left(a,b;c+q;y\right). \tag{2.71}$$

By using the identity given by (2.48), (2.71) can be written as

$$\mathcal{S}_4 = \sum_{s=0}^{\infty}\frac{z^s}{s!}\sum_{q=0}^{\infty}\frac{(a)_q(c+s)_q}{(c)_q}\frac{x^q}{q!}\,{}_2F_1\left(a;b;c+q;y\right). \tag{2.72}$$

The last summation in (2.72) is the third Appell function $F_3\left(a,a,c+s,b;c;x,y\right)$. This completes the proof of (2.66) and consequently of Theorem 2.5.

3. Applications

In this section, we will apply our main results to express the output field resulting from the propagation of generalized Humbert Gaussian beams through a paraxial ABCD optical system. These beams are obtained by modulating generalized Humbert beams [3] with a Gaussian envelope.

The output field is given by Huygens-Fresnel integral [5]

$$E(\rho,\phi,Z) = \frac{k}{2\pi iB}e^{ik\left(Z+\frac{D}{2B}\rho^2\right)}\int_0^{\infty}\int_0^{2\pi}E(\rho',\phi')e^{\frac{ik}{2B}\left(A\rho'^2-2\rho\rho'cos(\phi-\phi')\right)}\rho' d\rho' d\phi', \tag{3.1}$$

where $k = 2\pi/\lambda$ is the wave number, λ is the optical wavelength and $E(\rho',\phi')$ is the input field of the form

$$E(\rho',\phi') = C_1 e^{\frac{ik\rho'}{2z_0}}e^{i(m+l)\phi'}e^{\frac{-\eta\rho'}{2w_0{}^2}}\rho'^{m+l}\,\Psi_1\left(a,b;c,c';x,y\right), \tag{3.2}$$

where $x = \frac{2i\alpha'}{\alpha+i\alpha'}$, $y = -\frac{k^2\rho'2}{4z_0^2(\alpha+i\alpha')}$, z_0 is a constant, $a = l/2+m+1$, $b = \beta$, $c = m+l+1$ and $c' = m+1$.

In (3.2), C_1 is a constant, w_0 is the waist width of the Gaussian part, m is the beam order and l is the topological charge.

By using the development of Humbert confluent Hypergeometric function in terms of Hypergeometric functions ${}_2F_1\left(a,b;c;x\right)$ [14]

$$\psi_1\left(a,b;c;c';x,y\right) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c')_n}\frac{y^n}{n!}{}_2F_1\left(a+n,b;c;x\right), \tag{3.3}$$

and by substituting (3.2) in (3.1) with the help of the identity given by (2.9) and the well-known relationship [6]

$$\int_0^{2\pi} e^{il\theta_1} e^{-\frac{2r_1 r_2 cos(\theta_1-\theta_2)}{\rho^2}}.d\theta_1 = 2\pi(-i)^l e^{il\theta_2} J_l(\frac{2r_1 r_2}{\rho^2}), \tag{3.4}$$

The output field can be expressed in terms of Hypergeometric functions ${}_1F_1(a;b;x)$ and ${}_2F_1(a,b;c;x)$ as

$$E(\rho,\phi,Z) = C_Z e^{i(m+l)\phi} \sum_{q=0}^{\infty} \frac{(a)_q (c)_q}{(c')_q} \frac{x^q}{q!} \, {}_2F_1(a+q,b;c;y) \, {}_1F_1(c+q;c;z), \tag{3.5}$$

where C_Z is a function of the coordinate Z and $z = \frac{k^2\rho^2}{4\gamma B^2}$.

By using (2.19) of Theorem 2.2, (3.4) can be written in terms of the Humbert function $\Phi_1(a,b;c;x,y)$ as

$$E(\rho,\phi,Z) = C_Z e^{i(m+l)\phi} \frac{e^z}{(1-x)^a} \tag{3.6}$$

$$\times \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(\frac{y}{1-x})^n}{n!} \Phi_1\left(a+n, c'-c; c'; \frac{x}{x-1}, \frac{xz}{1-x}\right), \tag{3.7}$$

or in terms of Appell function $F_2(a,b,b';c,c';x,y)$ as

$$E(\rho,\phi,Z) = C_Z e^{i(m+l)\phi} \sum_{n=0}^{\infty} \frac{z^n}{n!} F_2(a, c+n, b; c', c; x, y). \tag{3.8}$$

In the following, we investigate some numerical simulations of particular cases of GHGBs called as $GHGBs_{sLGBs}$, $GHGBs_{QBGBs}$ and $GHGBs_{WGBs}$ which are derived from standard Laguerre-Gaussian beams, quadratic Bessel-Gaussian beams and Whittaker-Gaussian beams respectively after passing through a paraxial ABCD optical system with a spiral phase plate.

Figure1 illustrates the normalized intensity distribution as a function of the transverse coordinate for particular cases of GHGBs propagating in free space system with two values of the lowest orders beam (m=0 and m=1), where $A = 1$, $B = z$, $C = 0$ and $D = 1$. From the plots of this figure, it's seen that the GHGBs family is characterized by a dark spot at the center with zero intensity. It's observed that when the beam orders increase, the side lobes of $GHGBs_{sLGBs}$ and $GHGBs_{WGBs}$ disappear.

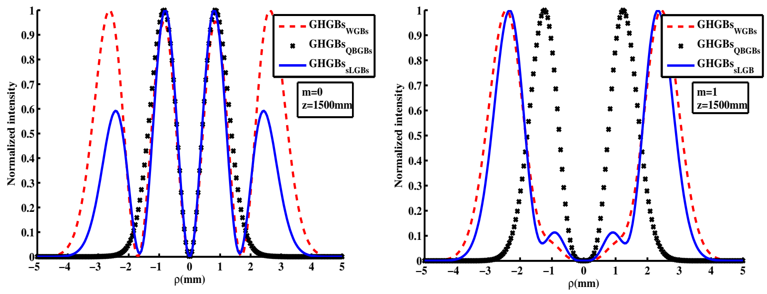

Figure 1. The Normalized intensity distribution versus ρ of $GHGBs_{sLGBs}$, $GHGBs_{WGBs}$ and $GHGBs_{QBGBs}$ as particular cases of GHGBs propagating in free space for two values of the beam orders: $m = 0$ and $m = 1$. The other parameters are given as: $l = 1$, the waist beam $\omega_0 = 0.5mm$, the wavelength $\lambda = 1330nm$.

4. Conclusion

In this note, we have obtained a new general formulae for the integral transforms containing the product of Humbert and Bessel functions. For each confluent Appell function, we have evaluated the corresponding integral transfoms which is an infinite series of generalized hypergeometric and Appell functions. An application of our main results is investigated to evaluate the output generalized Humbert-Gaussian beam propagating through a paraxial ABCD optical system.

Conflict of interest

The authors declare that there is no conflict of interest.

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