AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(2): 1177–1185.
DOI:10.3934/math.2020081
Received: 22 October 2019
Accepted: 07 January 2020
Published: 16 January 2020

Research article

On the Jensen's inequality and its variants

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Abstract: The main purpose of this paper is to discuss operator Jensen inequality for convex functions, without appealing to operator convexity. Several variants of this inequality will be presented, and some applications will be shown too.

Keywords: Jensen's inequality; convex function; self-adjoint operator; positive operator; positive linear map
Mathematics Subject Classification: 26A51, 47A63, 26B25, 39B62, 26D15

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^*–algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. As customary, we reserve m, M for scalars and $\mathbf{1}_{\mathcal{H}}$ for the identity operator on $\mathcal{H}$. A self-adjoint operator A is said to be positive (written $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ holds for all $x \in \mathcal{H}$ also an operator A is said to be strictly positive (denoted by $A > 0$) if A is positive and invertible. If A and B are self-adjoint, we write $B \geq A$ in case $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$–isomorphism between the C^*–algebra $C(\sigma(A))$ of continuous functions on the spectrum $\sigma(A)$ of a selfadjoint operator A and the C^*–algebra generated by A and the identity operator $\mathbf{1}_{\mathcal{H}}$. If $f, g \in C(\sigma(A))$, then $f(t) \geq g(t)$ ($t \in \sigma(A)$) implies that $f(A) \geq g(A)$.

A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It's said to be unital if $\Phi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. A continuous function f defined on the interval J is called an operator convex function if $f((1-v)A + vB) \leq (1-v)f(A) + vf(B)$ for every $0 < v < 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are both in J.

The well-known Jensen inequality for the convex functions states that if f is a convex function on the interval $[m, M]$, then

$$f\left(\sum_{i=1}^{n} w_i a_i\right) \leq \sum_{i=1}^{n} w_i f(a_i) \tag{1.1}$$

for all $a_i \in [m, M]$ and $w_i \in [0, 1]$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^{n} w_i = 1$.

There is an extensive amount of literature devoted to Jensen's inequality concerning different generalizations, refinements, and converse results, see, for example [1, 8, 11].

Mond and Pečarić [10] gave an operator extension of the Jensen inequality as follows: Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $\sigma(A) \subseteq [m, M]$, and let $f(t)$ be a convex function on $[m, M]$, then for any unit vector $x \in \mathcal{H}$,

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle.$$

Choi [2] showed if $f : J \to \mathbb{R}$ is an operator convex function, A is a self-adjoint operator with the spectra in J, and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is unital positive linear mapping, then

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1.2}$$

Though in the case of convex function the inequality (1.2) does not hold in general, we have the following estimate [3, Lemma 2.1]:

$$f(\langle \Phi(A)x, x\rangle) \leq \langle \Phi(f(A))x, x\rangle \tag{1.3}$$

for any unit vector $x \in \mathcal{K}$.

We here cite [4] and [13] as pertinent references to inequalities of types (1.2) and (1.3). For other recent results treating the Jensen operator inequality, we refer the reader to [5, 9, 12].

In the current paper, extensions of Jensen-type inequalities for the continuous function of self-adjoint operators on complex Hilbert spaces are given. Actually, a more generalization of (1.2) is discussed. Of course, this will be at the cost of additional conditions or weaker estimates.

2. Main results

We begin with the following auxiliary result:

Lemma 2.1. *Let $f : J \to \mathbb{R}$ be a convex and differentiable function on $\overset{o}{J}$ (the interior of J) whose derivative f' is continuous on $\overset{o}{J}$, let $A, B \in \mathcal{B}(\mathcal{H})$ be two self-adjoint operators with the spectra in $[m, M] \subset \overset{o}{J}$, and let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a unital positive linear mapping. Then for any unit vector $x \in \mathcal{K}$,*

$$\begin{aligned} & f'(\langle \Phi(B)x, x\rangle)(\langle \Phi(A)x, x\rangle - \langle \Phi(B)x, x\rangle) \\ & \leq \langle \Phi(f(A))x, x\rangle - f(\langle \Phi(B)x, x\rangle) \\ & \leq \langle \Phi(f'(A)A)x, x\rangle - \langle \Phi(B)x, x\rangle \langle \Phi(f'(A))x, x\rangle. \end{aligned}$$

Proof. Since f is convex and differentiable on $\overset{o}{J}$, then we have for any $t, s \in [m, M]$,

$$f'(s)(t - s) \leq f(t) - f(s) \leq f'(t)(t - s). \tag{2.1}$$

it is equivalent to

$$f'(s)t - f'(s)s \leq f(t) - f(s) \leq f'(t)t - f'(t)s. \tag{2.2}$$

If we fix $s \in [m, M]$ and apply the continuous functional calculus for A, we get

$$f'(s) A - f'(s) s\mathbf{1}_{\mathcal{H}} \le f(A) - f(s)\mathbf{1}_{\mathcal{H}} \le f'(A) A - s f'(A).$$

Applying the positive linear mapping Φ, this implies

$$\begin{aligned} f'(s)\Phi(A) - f'(s) s\mathbf{1}_{\mathcal{K}} &\le \Phi(f(A)) - f(s)\mathbf{1}_{\mathcal{K}} \\ &\le \Phi(f'(A) A) - s\Phi(f'(A)). \end{aligned}$$

Therefore, for any unit vector $x \in \mathcal{K}$, we have

$$\begin{aligned} & f'(s)\langle\Phi(A) x, x\rangle - f'(s) s \\ & \le \langle\Phi(f(A)) x, x\rangle - f(s) \\ & \le \langle\Phi(f'(A) A) x, x\rangle - s\langle\Phi(f'(A)) x, x\rangle. \end{aligned}$$

Since Φ is unital, and $\sigma(B) \subseteq [m, M]$, then $\sigma(\Phi(B)) \subseteq [m, M]$. Thus, by substituting $s = \langle\Phi(B) x, x\rangle$, we deduce the desired result. □

Remark 2.1. *By taking $A = B$ in Lemma 2.1, we obtain a counterpart of* (1.3).

We now have all the tools needed to write the proof of the first theorem.

Theorem 2.1. *Let all the assumptions of Lemma 2.1 hold. Then*

$$\Phi(f(A)) \le f(\Phi(A)) + \delta\mathbf{1}_{\mathcal{K}} \tag{2.3}$$

where

$$\delta = \sup\{\langle\Phi(f'(A) A) x, x\rangle - \langle\Phi(A) x, x\rangle\langle\Phi(f'(A)) x, x\rangle : \ x \in \mathcal{K}, \|x\| = 1\}.$$

Proof. One can write,

$$\begin{aligned} 0 &\le \langle\Phi(f(A)) x, x\rangle - f(\langle\Phi(A) x, x\rangle) \\ &\le \langle\Phi(f'(A) A) x, x\rangle - \langle\Phi(A) x, x\rangle\langle\Phi(f'(A)) x, x\rangle \\ &\le \sup\{\langle\Phi(f'(A) A) x, x\rangle - \langle\Phi(A) x, x\rangle\langle\Phi(f'(A)) x, x\rangle : \ x \in \mathcal{K}, \|x\| = 1\} \end{aligned}$$

thanks to Lemma 2.1. Whence,

$$\langle\Phi(f(A)) x, x\rangle \le f(\langle\Phi(A) x, x\rangle) + \delta$$

for any unit vector $x \in \mathcal{K}$.

Now we can write,

$$\begin{aligned} \langle\Phi(f(A)) x, x\rangle &\le f(\langle\Phi(A) x, x\rangle) + \delta \\ &\le \langle f(\Phi(A)) x, x\rangle + \delta \\ &= \langle f(\Phi(A)) x, x\rangle + \langle\delta\mathbf{1}_{\mathcal{K}} x, x\rangle \\ &= \langle f(\Phi(A)) + \delta\mathbf{1}_{\mathcal{K}} x, x\rangle \end{aligned}$$

for any unit vector $x \in \mathcal{K}$.

By replacing x by $\frac{y}{\|y\|}$ where y is any vector in $\mathcal{K}$, we deduce the desired inequality. □

A kind of a converse of Theorem 2.1 can be considered as follows.

Theorem 2.2. *Let all the assumptions of Lemma 2.1 hold. Then*

$$f(\Phi(A)) \le \Phi(f(A)) + \xi \mathbf{1}_{\mathcal{K}} \tag{2.4}$$

where

$$\xi = \sup\{\langle f'(\Phi(A))\Phi(A)x,x\rangle - \langle \Phi(A)x,x\rangle\langle f'(\Phi(A))x,x\rangle : x \in \mathcal{K}, \|x\| = 1\}.$$

Proof. Fix $t \in [m, M]$. Since $[m, M]$ contains the spectra of the A and Φ is unital, so the spectra of $\Phi(A)$ is also contained in $[m, M]$. Then we may replace s in the inequality (2.2) by $\Phi(A)$, via a functional calculus to get

$$f(\Phi(A)) - f(t)\mathbf{1}_{\mathcal{K}} \le f'(\Phi(A))\Phi(A) - tf'(\Phi(A)).$$

This inequality implies, for any $x \in \mathcal{K}$ with $\|x\| = 1$,

$$\langle f(\Phi(A))x,x\rangle - f(t) \le \langle f'(\Phi(A))\Phi(A)x,x\rangle - t\langle f'(\Phi(A))x,x\rangle. \tag{2.5}$$

Substituting t with $\langle \Phi(A)x,x\rangle$ in (2.5). Thus,

$$\begin{aligned} 0 &\le \langle f(\Phi(A))x,x\rangle - f(\langle \Phi(A)x,x\rangle) \\ &\le \langle f'(\Phi(A))\Phi(A)x,x\rangle - \langle \Phi(A)x,x\rangle\langle f'(\Phi(A))x,x\rangle \\ &\le \sup\{\langle f'(\Phi(A))\Phi(A)x,x\rangle - \langle \Phi(A)x,x\rangle\langle f'(\Phi(A))x,x\rangle : x \in \mathcal{K}, \|x\| = 1\} \end{aligned}$$

i.e.,

$$\langle f(\Phi(A))x,x\rangle \le f(\langle \Phi(A)x,x\rangle) + \xi$$

for any $x \in \mathcal{K}$ with $\|x\| = 1$.

On the other hand,

$$\begin{aligned} \langle f(\Phi(A))x,x\rangle &\le f(\langle \Phi(A)x,x\rangle) + \xi \\ &\le \langle \Phi(f(A))x,x\rangle + \xi \end{aligned}$$

where the second inequality follows from (1.3). This completes the proof. □

As we discussed above, inequality (2.1) plays a critical role in our Jensen type inequalities. Now, we intend to improve (2.1).

Proposition 2.1. *Let $f : J \to \mathbb{R}$ be a differentiable and convex, then for any $s, t \in J$*

$$f(s) + f'(s)(t-s) \le f(t) - 2\left(\frac{f(s)+f(t)}{2} - f\left(\frac{s+t}{2}\right)\right).$$

Proof. Since f is convex on the interval J, we have

$$\begin{aligned} f((1-v)s + vt) &= f\left((1-2v)s + 2v\frac{s+t}{2}\right) \\ &\le (1-2v)f(s) + 2vf\left(\frac{s+t}{2}\right) \\ &= (1-v)f(s) + vf(t) - 2r\left(\frac{f(s)+f(t)}{2} - f\left(\frac{s+t}{2}\right)\right) \end{aligned}$$

for any $s, t \in J$ and $r = \min\{v, 1-v\}$. Thus

$$f((1-v)s+vt) \le (1-v)f(s)+vf(t)-2r\left(\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right)\right) \tag{2.6}$$

holds for any $s, t \in J$ and $r = \min\{v, 1-v\}$ with $0 < v < 1$.
From the above inequality one can write

$$f(s+v(t-s))-f(s) \le vf(t)-vf(s)-2r\left(\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right)\right).$$

Dividing by $v > 0$, we get

$$\frac{f(s+v(t-s))-f(s)}{v} \le f(t)-f(s)-2\frac{r}{v}\left(\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right)\right).$$

Now, if $v \to 0$, and by taking into account that for $0 < v \le \frac{1}{2}$, $r = v$ we infer

$$f(s)+f'(s)(t-s) \le f(t)-2\left(\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right)\right)$$

as desired. □

Remark 2.2. *Suppose that all assumptions of Proposition 2.1 hold. The convexity assumption on f guarantees that*

$$\frac{f(s)+f(t)}{2}-f\left(\frac{s+t}{2}\right) \ge 0.$$

Consequently,

$$\begin{aligned} f(s)+f'(s)(t-s) &\le f(t)-2\left(\frac{f(s)+f(t)}{2}-f\left(\frac{t+s}{2}\right)\right) \\ &\le f(t). \end{aligned}$$

Now, from Proposition 2.1, we get the following result.

Theorem 2.3. *Let $f : J \to \mathbb{R}$ be a convex and differentiable function on $\overset{o}{J}$ (the interior of J) whose derivative f' is continuous on $\overset{o}{J}$, let $A \in \mathcal{B}(\mathcal{H})$ self-adjoint operator with the spectra in $[m, M] \subset \overset{o}{J}$, and let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a unital positive linear mapping. Then for any unit vector $x \in \mathcal{K}$,*

$$\begin{aligned} & f(\langle \Phi(A)x, x\rangle) \\ & \le \langle \Phi(f(A))x, x\rangle \\ & \quad -2\left(\frac{f(\langle \Phi(A)x, x\rangle)+\langle \Phi(f(A))x, x\rangle}{2}-\left\langle \Phi\left(f\left(\frac{\langle \Phi(A)x, x\rangle \mathbf{1}_{\mathcal{H}}+A}{2}\right)\right)x, x\right\rangle\right) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} & \langle \Phi(f(A))x, x\rangle + \langle \Phi(A)x, x\rangle \langle \Phi(f'(A))x, x\rangle - \langle \Phi(f'(A)A)x, x\rangle \\ & \quad +2\left(\frac{\langle \Phi(f(A))x, x\rangle + f(\langle \Phi(A)x, x\rangle)}{2}-\left\langle \Phi\left(f\left(\frac{\langle \Phi(A)x, x\rangle \mathbf{1}_{\mathcal{H}}+A}{2}\right)\right)x, x\right\rangle\right) \\ & \le f(\langle \Phi(A)x, x\rangle). \end{aligned} \tag{2.8}$$

Proof. If we fix $s \in [m, M]$ and apply the continuous functional calculus for A, we get from Proposition 2.1,

$$f(s)\mathbf{1}_{\mathcal{H}} + f'(s)(A - s\mathbf{1}_{\mathcal{H}}) \le f(A) - 2\left(\frac{f(s)\mathbf{1}_{\mathcal{H}} + f(A)}{2} - f\left(\frac{s\mathbf{1}_{\mathcal{H}} + A}{2}\right)\right).$$

Applying the positive linear mapping Φ, this implies

$$\begin{aligned}&f(s)\mathbf{1}_{\mathcal{K}} + f'(s)(\Phi(A) - s\mathbf{1}_{\mathcal{K}})\\ &\le \Phi(f(A)) - 2\left(\frac{f(s)\mathbf{1}_{\mathcal{K}} + f(A)}{2} - \Phi\left(f\left(\frac{s\mathbf{1}_{\mathcal{H}} + A}{2}\right)\right)\right).\end{aligned}$$

Therefore, for any unit vector $x \in \mathcal{K}$, we have

$$\begin{aligned}&f(s) + f'(s)(\langle\Phi(A)x, x\rangle - s)\\ &\le \langle\Phi(f(A))x, x\rangle - 2\left(\frac{f(s) + \langle f(A)x, x\rangle}{2} - \left\langle\Phi\left(f\left(\frac{s\mathbf{1}_{\mathcal{H}} + A}{2}\right)\right)x, x\right\rangle\right).\end{aligned}$$

Since Φ is unital, and $\sigma(A) \subseteq [m, M]$, then $\sigma(\Phi(A)) \subseteq [m, M]$. Thus, by substituting $s = \langle\Phi(A)x, x\rangle$, we deduce the inequality (2.7).

On the other hand, if we fix $t \in [m, M]$ and apply the continuous functional calculus for A, then Proposition 2.1 implies,

$$f(A) + f'(A)(t\mathbf{1}_{\mathcal{H}} - A) \le f(t)\mathbf{1}_{\mathcal{H}} - 2\left(\frac{f(A) + f(t)\mathbf{1}_{\mathcal{H}}}{2} - f\left(\frac{A + t\mathbf{1}_{\mathcal{H}}}{2}\right)\right).$$

Applying unital positive linear mapping Φ, we infer

$$\begin{aligned}&\Phi(f(A)) + t\Phi(f'(A)) - \Phi(f'(A)A)\\ &\le f(t)\mathbf{1}_{\mathcal{K}} - 2\left(\frac{\Phi(f(A)) + f(t)\mathbf{1}_{\mathcal{K}}}{2} - \Phi\left(f\left(\frac{A + t\mathbf{1}_{\mathcal{H}}}{2}\right)\right)\right).\end{aligned}$$

Thus, for have any unit vector $x \in \mathcal{K}$,

$$\begin{aligned}&\langle\Phi(f(A))x, x\rangle + t\langle\Phi(f'(A))x, x\rangle - \langle\Phi(f'(A)A)x, x\rangle\\ &\le f(t) - 2\left(\frac{\langle\Phi(f(A))x, x\rangle + f(t)}{2} - \left\langle\Phi\left(f\left(\frac{A + t\mathbf{1}_{\mathcal{H}}}{2}\right)\right)x, x\right\rangle\right).\end{aligned} \tag{2.9}$$

Now, by taking $t = \langle\Phi(A)x, x\rangle$ in (2.9), we get (2.8). □

Remark 2.3. *We emphasize that* (2.7) *provides an improvement of* (1.3)*, and* (2.8) *can be considered as a counterpart of* (1.3)*.*

In the next result we consider a more general case. We remark that this result extends and improves [7, Lemma 2.3]

Theorem 2.4. *Let $f : J \to \mathbb{R}$ be a convex and differentiable function on $\overset{o}{J}$ (the interior of J) whose derivative f' is continuous on $\overset{o}{J}$ with $f(0) \le 0$, let $A \in \mathcal{B}(\mathcal{H})$ self-adjoint operator with the spectra* *in* $[m,M]\subset \overset{o}{J}$, *and let* $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ *be a unital positive linear mapping. Then for any vector* $x\in\mathcal{K}$ *with* $\|x\|\le 1$*,*

$$\begin{aligned}
&f\left(\langle\Phi(A)x,x\rangle\right)\\
&\le \langle\Phi(f(A))x,x\rangle\\
&\quad -2\left(\frac{\|x\|^2 f\left(\frac{1}{\|x\|^2}\langle\Phi(A)x,x\rangle\right)+\langle\Phi(f(A))x,x\rangle}{2}-\left\langle\Phi\left(f\left(\frac{\frac{1}{\|x\|^2}\langle\Phi(A)x,x\rangle\mathbf{1}_{\mathcal{H}}+A}{2}\right)\right)x,x\right\rangle\right).
\end{aligned}$$

Proof. Let $x\in\mathcal{K}$ with $\|x\|=1$. Set $y=x/\|x\|$, so that $\|y\|=1$. We have

$$\begin{aligned}
&f\left(\langle\Phi(A)x,x\rangle\right)\\
&= f\left(\|x\|^2\langle\Phi(A)y,y\rangle+\left(1-\|x\|^2\right)0\right)\\
&\le \|x\|^2 f\left(\langle\Phi(A)y,y\rangle\right)+\left(1-\|x\|^2\right)f(0)\quad\text{(since } f \text{ is convex)}\\
&\le \|x\|^2 f\left(\langle\Phi(A)y,y\rangle\right)\quad\text{(since } f(0)\le 0)\\
&\le \|x\|^2\left[\langle\Phi(f(A))y,y\rangle-2\left(\frac{f\left(\langle\Phi(A)y,y\rangle\right)+\langle\Phi(f(A))y,y\rangle}{2}\right.\right.\\
&\quad\left.\left.-\left\langle\Phi\left(f\left(\frac{\langle\Phi(A)y,y\rangle\mathbf{1}_{\mathcal{H}}+A}{2}\right)\right)y,y\right\rangle\right)\right]\quad\text{(by (2.7))}\\
&= \langle\Phi(f(A))x,x\rangle\\
&\quad -2\left(\frac{\|x\|^2 f\left(\frac{1}{\|x\|^2}\langle\Phi(A)x,x\rangle\right)+\langle\Phi(f(A))x,x\rangle}{2}-\left\langle\Phi\left(f\left(\frac{\frac{1}{\|x\|^2}\langle\Phi(A)x,x\rangle\mathbf{1}_{\mathcal{H}}+A}{2}\right)\right)x,x\right\rangle\right).
\end{aligned}$$

The proof is completed. □

Remark 2.4. *As we can see if* $\|x\|=1$*, then Theorem 2.4 turns out to be* (2.7).

Theorem 2.3 also implies the following result, which presents refinement and reverse of scalar Jensen inequality (1.1).

Corollary 2.1. *Let* $f:J\to\mathbb{R}$ *be a convex and differentiable function, let* $a_1,\dots,a_n\in J$*, and let* $w_1,\dots,w_n$ *be positive scalars such that* $\sum_{i=1}^n w_i=1$*. Then*

$$\begin{aligned}
&f\left(\sum_{i=1}^n w_i a_i\right)\\
&\le \sum_{i=1}^n w_i f(a_i)-2\left(\frac{f\left(\sum_{i=1}^n w_i a_i\right)+\sum_{i=1}^n w_i f(a_i)}{2}-\sum_{i=1}^n w_i f\left(\frac{\sum_{j=1}^n w_j a_j+a_i}{2}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{n} w_i f\left(a_i\right)\\&\quad+\left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} w_i f'\left(a_i\right)\right)-\sum_{i=1}^{n} w_i a_i f'\left(a_i\right)\\&\quad+2\left(\frac{\sum_{i=1}^{n} w_i f\left(a_i\right)+f\left(\sum_{i=1}^{n} w_i a_i\right)}{2}-\sum_{i=1}^{n} w_i f\left(\frac{a_i+\sum_{j=1}^{n} w_j a_j}{2}\right)\right)\\&\leq f\left(\sum_{i=1}^{n} w_i a_i\right).\end{aligned}$$

Proof. The proof follows from Theorem 2.3, by letting

$$\Phi(A)=A,\ A=\begin{bmatrix}a_1 & & 0\\ & \ddots & \\ 0 & & a_n\end{bmatrix}\quad \&\quad x=\begin{bmatrix}\sqrt{w_1}\\ \\ \sqrt{w_n}\end{bmatrix}.$$

□

Remark 2.5. *Closely connected to the Jensen inequality is the Edmundson–Lah–Ribarić inequality [6]. From* (2.6)*, by interchanging* $1-v=\frac{t-m}{M-m}$*,* $v=\frac{M-t}{M-m}$*,* $s=M$*, and* $t=m$*, we get*

$$f(t)\leq\frac{t-m}{M-m}f(M)+\frac{M-t}{M-m}f(m)-2r\left(\frac{f(M)+f(m)}{2}-f\left(\frac{M+m}{2}\right)\right)\tag{2.10}$$

where $r=\min\left\{\frac{t-m}{M-m},\frac{M-t}{M-m}\right\}=\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{M+m}{2}\right|$*. Hence, from* (2.10)*, we get for any unit vector* $x\in\mathcal{K}$

$$\begin{aligned}&\langle\Phi(f(A))x,x\rangle\\&\leq\frac{\langle\Phi(A)x,x\rangle-m}{M-m}f(M)+\frac{M-\langle\Phi(A)x,x\rangle}{M-m}f(m)\\&\quad-2\left(\frac{f(M)+f(m)}{2}-f\left(\frac{M+m}{2}\right)\right)\left(\frac{1}{2}-\frac{1}{M-m}\left\langle\Phi\left(\left|A-\frac{M+m}{2}\mathbf{1}_{\mathcal{H}}\right|\right)x,x\right\rangle\right)\end{aligned}$$

whenever $A\in\mathcal{B}(\mathcal{H})$ *is a self-adjoint operator with the spectra in* $[m,M]$*, and* $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ *is a unital positive linear mapping. Of course, this can be regarded as an extension and improvement of Edmundson–Lah–Ribarić inequality.*

Acknowledgment

The authors wish to thank an anonymous referee for his/her helpful suggestions and comments, improving the readability of the paper.

Conflict of interest

All authors declare no conflicts of interest.

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