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Research article

Some new Chebyshev type inequalities utilizing generalized fractional integral operators

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Abstract: Chebyshev type inequalities for the generalized fractional integral operators are studied based on two synchronous functions in a rather general form. The main results of this paper generalize some previous results obtained by the authors. We also present the special cases of related inequalities for this type of fractional integral is obtained.

Keywords: Chebyshev inequality; fractional integral operators
Mathematics Subject Classification: 26D15, 26A33, 26D10

1. Introduction

In recent years, there has been an increasing interest in the fractional calculus. One of the significant motivations for such deep interest in the subject is its capability to model a number of natural phenomena, see, for example, the papers [9, 12]. On the other hand Chebyshev inequality has broad practicability in statistical problems, numerical quadrature, probability and transform theory, and the bounding of special functions. Its basic appeal develops out of a desire to approximate, for instance, information in the form of a particular measure of the product of functions in terms of the products of the individual function measures. It is, also, of great interest in differential and difference equations [7, 13].

The essential destination of the present study is to prove a Chebyshev type inequality for the generalized fractional integral operators. After some preliminaries and summarization of some previous known results in Section 2, Section 3 deals with general Chebyshev type inequalities for generalized fractional integral operators. Finally, some concluding remarks are given in Section 4.

2. Preliminaries

In this section we recall some basic definitions and previous results which will be used in what follows.
In 1882, Chebyshev [3] proved the following inequality:

Theorem 2.1. *Let f and g be two integrable functions in* $[0, 1]$. *If both functions are simultaneously increasing or decreasing for the same values of x in* $[0, 1]$, *then*

$$\int_0^1 f(x)g(x)dx \geq \int_0^1 f(x)dx \int_0^1 g(x)dx. \tag{2.1}$$

If one function is increasing and the other is decreasing for the same values of x in $[0, 1]$, then (2.1) reverses. In the last years, many papers were devoted to the generalization of the inequalities (2.1), we can mention the works [1, 2, 4–6, 8, 11, 15–19].

2.1. Generalized fractional integral operators

In [14], Raina defined the following results connected with the general class of fractional integral operators.

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathcal{R}), \tag{2.2}$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers and $\mathcal{R}$ is the set of real numbers. With the help of (2.2), in [14], Raina defined the following fractional integral operators, as follows:

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} f(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega (x-t)^{\rho}\right] f(t)dt, \quad x > a, \tag{2.3}$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}$ of order α defined by (see, [10])

$$\left(I^{\alpha}_{a^+} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \quad (x > a; \alpha > 0), \tag{2.4}$$

follow easily by setting

$$\lambda = \alpha, \sigma(0) = 1, \; w = 0, \tag{2.5}$$

in (2.3).

In [19], Usta et. al gave the following Chebychev type inequalities for the generalized fractional integral operators:

Theorem 2.2. *Let f and g be two synchronous functions on* $[0, \infty)$*, that is they are having the same sense of variation on* $[0, \infty)$. *Then for all $t, \rho, \lambda > 0$ and $w \in \mathbb{R}$, we have*

$$\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(1)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(fg)(t) \geq \mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega} f(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega} g(t), \tag{2.6}$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers.

Theorem 2.3. *Let f and g be two synchronous functions on $[0,\infty)$, that is they are having the same sense of variation on $[0,\infty)$. Then for all $t>0$ and $\rho_1,\rho_2,\lambda_1,\lambda_2>0$ and $w_1,w_2\in\mathbb{R}$, we have*

$$t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fg)(t)+t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_2 t^{\rho_1}\right]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fg)(t) \tag{2.7}$$

$$\geq \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(f)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(g)(t)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(g)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(f)(t)$$

where the coefficients $\sigma_1(k),\sigma_2(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ are bounded sequences of positive real numbers.

For more details, one may consult [19].

3. Main findings & Cumulative results

In this section, we will present some fractional integral inequalities for functions defined on the positive real line with the help of fractional integral operators given above. The inequalities to be given in this section are a generalization of the former inequalities.

Theorem 3.1. *Let f and g be two synchronous functions on $[0,\infty)$, that is they are having the same sense of variation on $[0,\infty)$ and $h\geq 0$. Then for all $t>0$ and $\rho_1,\rho_2,\lambda_1,\lambda_2>0$ and $w_1,w_2\in\mathbb{R}$, we have*

$$t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fgh)(t)+t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_2 t^{\rho_1}\right]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fgh)(t)$$

$$\geq \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(hf)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(g)(t)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(hg)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(f)(t)$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(f)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(hg)(t)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(g)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(hf)(t)$$

$$-\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fg)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(h)(t)-\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(h)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fg)(t)$$

where the coefficients $\sigma_1(k),\sigma_2(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ are bounded sequences of positive real numbers.

Proof. Let f, g and h be three functions satisfying the conditions of Theorem 3.1. Then, we have

$$(f(\eta)-f(\xi))(g(\eta)-g(\xi))(h(\eta)+h(\xi))\geq 0.$$

Therefore

$$f(\eta)g(\eta)h(\eta)+f(\xi)g(\xi)h(\xi) \geq h(\eta)f(\eta)g(\xi)+h(\eta)f(\xi)g(\eta)+h(\xi)f(\eta)g(\xi) \tag{3.1}$$

$$+\ h(\xi)f(\xi)g(\eta)-f(\eta)g(\eta)h(\xi)-f(\xi)g(\xi)h(\eta).$$

Now, by multiplying both sides of (3.1) by $(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]$, we obtain:

$$(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]f(\eta)g(\eta)h(\eta) \tag{3.2}$$

$$
\begin{aligned}
& +(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]f(\xi)g(\xi)h(\xi)\\
\geq\ & (t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]h(\eta)f(\eta)g(\xi)\\
& +(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]h(\eta)f(\xi)g(\eta)\\
& +(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]h(\xi)f(\eta)g(\xi)\\
& +(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]h(\xi)f(\xi)g(\eta)\\
& -(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]f(\eta)g(\eta)h(\xi)\\
& -(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]f(\xi)g(\xi)h(\eta).
\end{aligned}
$$

Finally, if we double integrate (3.2) with respect to η and ξ over $(0,t)\times(0,t)$, we get the desired result. □

Corollary 3.2. *Choosing $\lambda_1=\lambda_2=\lambda$, $\sigma_1=\sigma_2=\sigma$, $\rho_1=\rho_2=\rho$ and $w_1=w_2=w$ in Theorem 3.1, we obtain the following inequality*

$$
\begin{aligned}
& t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega t^{\rho}\right]\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(fgh)(t)+\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(h)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(fg)(t)\\
\geq\ & \mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(hf)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(g)(t)+\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(hg)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(f)(t)
\end{aligned}
$$

where the coefficients $\sigma(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Remark 3.3. *If we choose $h=1$ in Theorem 3.1, Theorem 3.1 reduce to Theorem 2.3 which was proved by Usta et al. in [19].*

Theorem 3.4. *Let f, g and h be three monotonic functions defined on $[0,\infty)$, satisfying the following*

$$
(f(\eta)-f(\xi))(g(\eta)-g(\xi))(h(\eta)-h(\xi))\geq 0
$$

for all $\eta,\xi\in[0,t]$, then for all $t>0$ and $\rho_1,\rho_2,\lambda_1,\lambda_2>0$ and $w_1,w_2\in\mathbb{R}$, we have

$$
\begin{aligned}
& t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fgh)(t)-t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_2 t^{\rho_1}\right]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fgh)(t)\\
\geq\ & \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(hf)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(g)(t)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(hg)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(f)(t)\\
& -\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(f)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(hg)(t)-\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(g)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(hf)(t)\\
& +\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fg)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(h)(t)-\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(h)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fg)(t)
\end{aligned}
$$

where the coefficients $\sigma_1(k),\sigma_2(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ are bounded sequences of positive real numbers.

Proof. The proof is similar to previous theorem. □

Theorem 3.5. *Let f, g be two functions on $[0,\infty)$. Then for all $t > 0$ and $\rho_1, \rho_2, \lambda_1, \lambda_2 > 0$ and $w_1, w_2 \in \mathbb{R}$, we have*

$$t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}\left(f^2\right)(t) + t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_2 t^{\rho_1}\right]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}\left(g^2\right)(t)$$

$$\geq\ 2\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(f)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(g)(t),$$

where the coefficients $\sigma_1(k), \sigma_2(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ are bounded sequences of positive real numbers.

Proof. As

$$((f(\eta) - g(\xi))^2 \geq 0$$

we have

$$f^2(\eta) + g^2(\xi) \geq 2f(\eta)g(\xi). \tag{3.3}$$

Now, by multiplying both sides of (3.3) by $(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]$, we get

$$(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]f^2(\eta) \tag{3.4}$$

$$+\,(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]g^2(\xi)$$

$$\geq\ 2(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]f(\eta)g(\xi).$$

Finally by double integration (3.4) over $(0,t)\times(0,t)$, we get the desired result. □

Corollary 3.6. *Choosing $\lambda_1 = \lambda_2 = \lambda$, $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$ and $w_1 = w_2 = w$ in Theorem 3.5, we obtain the following inequality*

$$t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega t^{\rho}\right]\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}\left(f^2\right)(t) + t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega t^{\rho}\right]\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}\left(g^2\right)(t)$$

$$\geq\ 2\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(f)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(g)(t).$$

In particular

$$t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega t^{\rho}\right]\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}\left(f^2\right)(t) \geq \left[\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(f)(t)\right]^2$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers.

Theorem 3.7. *Let f, g be two functions on $[0,\infty)$. Then for all $t > 0$ and $\rho_1, \rho_2, \lambda_1, \lambda_2 > 0$ and $w_1, w_2 \in \mathbb{R}$, we have*

$$\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}\left(f^2\right)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}\left(g^2\right)(t) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}\left(f^2\right)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}\left(g^2\right)(t)$$

$$\geq\ 2\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fg)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fg)(t).$$

where the coefficents $\sigma_1(k), \sigma_2(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ are bounded sequences of positive real numbers.

Proof. As

$$((f(\eta)g(\xi) - f(\xi)g(\eta))^2 \geq 0$$

we have

$$f^2(\eta)g^2(\xi) + f^2(\xi)g^2(\eta) \geq 2f(\eta)g(\eta)g(\xi)f(\xi). \tag{3.5}$$

Then by multiplying both sides of (3.5) by $(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega_1(t-\eta)^{\rho_1}]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}[\omega_2(t-\xi)^{\rho_2}]$ and following the similar steps in Theorem 3.5, we get the desired result. □

Corollary 3.8. *Choosing $\lambda_1 = \lambda_2 = \lambda$, $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$ and $w_1 = w_2 = w$ in Theorem 3.7, we obtain the following inequality*

$$\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}\left(f^2\right)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}\left(g^2\right)(t) \geq \left[\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(fg)(t)\right]^2$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers.

Lemma 3.9. *Let $f : \mathbb{R} \to \mathbb{R}$, and define*

$$\overline{f}(x) = \int_0^x f(t)dt,$$

then for all $t, \rho > 0$, $\lambda > 1$ and $w \in \mathbb{R}$, we have

$$\mathcal{J}^{\sigma}_{\rho,\lambda-1,0+;\omega}\left(\overline{f}\right)(t) = \mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(f)(t)$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers.

Proof.

$$\begin{aligned}
\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}\left(\overline{f}\right)(t) &= \int_0^t (t-\tau)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(x-\tau)^{\rho}]\left(\int_0^{\tau} f(u)du\right)d\tau \\
&= \int_0^t f(u)\int_u^t (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-u)^{\rho}]\,d\tau du \\
&= \int_0^t (t-\tau)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(x-t)^{\rho}]\,f(u)du \\
&= \mathcal{J}^{\sigma}_{\rho,\lambda+1,0+;\omega}(f)(t).
\end{aligned}$$

□

Theorem 3.10. *Let f and g be two functions on $[0,\infty)$, Then for all $t, \rho > 0$, $\lambda_1, \lambda_2 > 1$ and $w_1, w_2 \in \mathbb{R}$, we have*

$$t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}[\omega_2 t^{\rho_2}]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1-1,0+;\omega_1}\left(\overline{fg}\right)(t) + t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}[\omega_2 t^{\rho_1}]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2-1,0+;\omega_2}\left(\overline{fg}\right)(t)$$

$$\geq \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1-1,0+;\omega_1}\left(\overline{f}\right)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2-1,0+;\omega_2}\left(\overline{g}\right)(t)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1-1,0+;\omega_1}\left(\overline{g}\right)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2-1,0+;\omega_2}\left(\overline{f}\right)(t),$$

where the coefficients $\sigma_1(k), \sigma_2(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ are bounded sequences of positive real numbers.

Proof. From Lemma 3.9 and inequality (2.7), we have

$$\begin{aligned}
& t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1-1,0+;\omega_1}\left(\overline{fg}\right)(t)+t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_2 t^{\rho_1}\right]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2-1,0+;\omega_2}\left(\overline{fg}\right)(t) \\
= \; & t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fg)(t)+t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_2 t^{\rho_1}\right]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fg)(t) \\
\geq \; & \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(f)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(g)(t)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(g)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(f)(t) \\
= \; & \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1-1,0+;\omega_1}\left(\overline{f}\right)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2-1,0+;\omega_2}\left(\overline{g}\right)(t)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1-1,0+;\omega_1}\left(\overline{g}\right)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2-1,0+;\omega_2}\left(\overline{f}\right)(t)
\end{aligned}$$

which completes the proof. □

Corollary 3.11. *Choosing $\lambda_1 = \lambda_2 = \lambda$, $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$ and $w_1 = w_2 = w$ in Theorem 3.10, we obtain the following inequality*

$$t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega t^{\rho}\right]\mathcal{J}^{\sigma}_{\rho,\lambda-1,0+;\omega}\left(\overline{fg}\right)(t) \geq \mathcal{J}^{\sigma}_{\rho,\lambda-1,0+;\omega}\left(\overline{f}\right)(t)\mathcal{J}^{\sigma}_{\rho,\lambda-1,0+;\omega}\left(\overline{g}\right)(t)$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers.

Theorem 3.12. *Let f and g be two differentiable function on $[0, \infty)$.Then for all $t, \rho > 0$, $\lambda_1, \lambda_2 > 0$ and $w_1, w_2 \in \mathbb{R}$, we have*

$$\begin{aligned}
& \Big| t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fg)(t)+t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_2 t^{\rho_1}\right]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fg)(t) \\
& -\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(f)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(g)(t)-\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(g)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(f)(t)\Big| \\
\leq \; & \|f'\|_{\infty}\|g'\|_{\infty}\, t^{\lambda_1+\lambda_2+2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 t^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right],
\end{aligned}$$

where

$$\|f'\|_{\infty} = \sup_{x\in[0,\infty)} |f'(x)| < \infty,$$

and the coefficients $\sigma_1(k), \sigma_2(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ are bounded sequences of positive real numbers.

Proof. With basic calculation, we have

$$\begin{aligned}
& t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fg)(t)+t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_2 t^{\rho_1}\right]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fg)(t) \\
& -\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(f)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(g)(t)-\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(g)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(f)(t) \\
= \; & \int_0^t\int_0^t (t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right](f(\eta)-f(\xi))(g(\eta)-g(\xi))\,d\eta d\xi.
\end{aligned}$$

Then taking modulus of the above equality, we find that

$$\left|t^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(fg)(t)+t^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_2 t^{\rho_1}\right]\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(fg)(t)\right.$$

$$\left.-\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(f)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(g)(t)-\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(g)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(f)(t)\right|$$

$$= \left|\int_0^t\int_0^t (t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right](f(\eta)-f(\xi))(g(\eta)-g(\xi))\,d\eta d\xi\right|$$

$$= \left|\int_0^t\int_0^t (t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]\left(\int_\xi^\eta f'(u)du\right)\left(\int_\xi^\eta g'(v)dv\right)d\eta d\xi\right|$$

$$\leq \|f'\|_\infty\|g'\|_\infty\left|\int_0^t\int_0^t (t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right](\eta-\xi)^2\,d\eta d\xi\right|$$

$$\leq \|f'\|_\infty\|g'\|_\infty t^2\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 t^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]$$

which completes the proof. □

Corollary 3.13. *Choosing $\lambda_1=\lambda_2=\lambda$, $\sigma_1=\sigma_2=\sigma$, $\rho_1=\rho_2=\rho$ and $w_1=w_2=w$ in Theorem 3.12, we obtain the following inequality*

$$\left|t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega t^{\rho}\right]\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(fg)(t)-\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(f)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(g)(t)\right|\leq\frac{1}{2}\|f'\|_\infty\|g'\|_\infty t^{2\lambda+2}\left[\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega t^{\rho}\right]\right]^2$$

where the coefficients $\sigma(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Remark 3.14. *If we choose $\lambda_1=\alpha$, $\lambda_2=\beta$, $\sigma_1(0)=\sigma_2(0)=1$, and $w_1=w_2=0$, in Theorem 3.1, Theorem 3.4, Theorem 3.5, Theorem 3.7, Theorem 3.10 and Theorem 3.12, then the inequalities reduces to Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.6 and Theorem 2.7 proved by Sulaiman in [18], respectively.*

Theorem 3.15. *Let f and g be two synchronous functions on $[0,\infty)$, that is they are having the same sense of variation on $[0,\infty)$, and let $v_1, v_2:[0,\infty)\to[0,\infty)$. Then for all $t,\rho>0$, $\lambda_1,\lambda_2>0$ and $w_1,w_2\in\mathbb{R}$, we have*

$$\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1fg)(t)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2fg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1)(t) \tag{3.6}$$

$$\geq \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2g)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1f)(t)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2f)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1g)(t),$$

where the coefficients $\sigma_1(k),\sigma_2(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ are bounded sequences of positive real numbers.

Proof. As f and g be two synchronous functions on $[0,\infty)$, then for all $\eta,\xi\geq 0$ we have

$$(f(\eta)-f(\xi))(g(\eta)-g(\xi))\geq 0.$$

Therefore

$$f(\eta)g(\eta) + f(\xi)g(\xi) \geq f(\eta)g(\xi) + f(\xi)g(\eta). \tag{3.7}$$

Mutlipying both sides of (3.7) by $(t-\eta)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]v_1(\eta)$, $\eta\in(0,t)$, we find that

$$\begin{aligned}&(t-\eta)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]v_1(\eta)f(\eta)g(\eta)+(t-\eta)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]v_1(\eta)f(\xi)g(\xi) \\ \geq\ &(t-\eta)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]v_1(\eta)f(\eta)g(\xi)+(t-\eta)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]v_1(\eta)f(\xi)g(\eta).\end{aligned} \tag{3.8}$$

Integrating (3.7) with repect to η over $(0,t)$, we get

$$\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1fg)(t)+f(\xi)g(\xi)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1)(t)\geq g(\xi)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1f)(t)+f(\xi)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1g)(t). \tag{3.9}$$

Now, similarly, by multiplying both sides of (3.9) by $(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]v_2(\xi)$, $\xi\in(0,t)$ and integrating with respect to ξ over $(0,t)$, we get the desired result. □

Corollary 3.16. *Choosing $\lambda_1=\lambda_2=\lambda$, $\sigma_1=\sigma_2=\sigma$, $\rho_1=\rho_2=\rho$ and $w_1=w_2=w$ in Theorem 3.15, we obtain the following inequality*

$$\begin{aligned}&\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(v_2)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(v_1fg)(t)+\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(v_2fg)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(v_1)(t) \\ \geq\ &\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(v_2g)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(v_1f)(t)+\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(v_2f)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(v_1g)(t),\end{aligned}$$

where the coefficients $\sigma(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Remark 3.17. *If we choose $v_1=v_2=1$ in Theorem 3.15, the inequality (3.14) reduce to inequality (2.7).*

Theorem 3.18. *Let f and g be two synchronous functions on $[0,\infty)$, that is they are having the same sense of variation on $[0,\infty)$, and let $p,q,r:[0,\infty)\to[0,\infty)$. Then for all $t,\rho>0$, $\lambda_1,\lambda_2>0$ and $w_1,w_2\in\mathbb{R}$, we have*

$$\begin{aligned}&\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(q)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pfg)(t) \\ &+2\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qfg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t) \\ &+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(q)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(rfg)(t) \\ &+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(q)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(r)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pfg)(t) \\ &+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(q)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(rfg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t) \\ \geq\ &\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pf)(t)\end{aligned}$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qf)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pg)(t)$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(q)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(rg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pf)(t)$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(q)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(rf)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pg)(t)$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(rf)(t)$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qf)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(rg)(t),$$

where the coefficients $\sigma_1(k), \sigma_2(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ *are bounded sequences of positive real numbers.*

Proof. If we choose $v_1 = p$ and $v_2 = q$ in Theorem 3.15, we can write:

$$\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(q)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pfg)(t) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qfg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t) \tag{3.10}$$

$$\geq \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pf)(t) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qf)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pg)(t).$$

Multiplying both sides of (3.10) by $\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)$, we get

$$\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(q)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pfg)(t) \tag{3.11}$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qfg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)$$

$$\geq \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pf)(t)$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qf)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pg)(t).$$

If we choose $v_1 = r$ and $v_2 = q$ in Theorem 3.15 and multiplying by $\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)$, then we find that

$$\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(q)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(rfg)(t) \tag{3.12}$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qfg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(r)(t)$$

$$\geq \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(rf)(t)$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(qf)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(rg)(t).$$

Similarly, if we choose $v_1 = p$ and $v_2 = r$ in Theorem 3.15 and multiplying by $\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(q)(t)$, then we find that

$$\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(q)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(r)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pfg)(t) \tag{3.13}$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(q)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(rfg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(p)(t)$$

$$\geq \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(q)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(rg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pf)(t)$$

$$+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(q)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(rf)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(pg)(t).$$

Then by adding the inequalities of (3.11)-(3.13), the desired inequality has been obtained. □

Corollary 3.19. *Choosing $\lambda_1 = \lambda_2 = \lambda$, $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$ and $w_1 = w_2 = w$ in Theorem 3.18, we obtain the following inequality*

$$2\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(r)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(q)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(pfg)(t)$$

$$+2\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(r)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(p)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(qfg)(t)$$

$$+2\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(p)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(q)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(rfg)(t)$$

$$\geq \mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(r)(t)\left[\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(qg)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(pf)(t)+\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(qf)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(pg)(t)\right]$$

$$+\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(q)(t)\left[\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(rg)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(pf)(t)+\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(rf)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(pg)(t)\right]$$

$$+\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(p)(t)\left[\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(qg)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(rf)(t)+\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(qf)(t)\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(rg)(t)\right]$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers.

Remark 3.20. *If we choose $\lambda_1 = \alpha$, $\lambda_2 = \beta$, $\sigma_1(0) = \sigma_2(0) = 1$, and $w_1 = w_2 = 0$, in Theorem 3.15 and Theorem 3.18 then the inequalities reduces to Lemma 3 and Theorem 4 proved by Dahmani in [5], respectively.*

Theorem 3.21. *Let v_1 and v_2 be two positive functions on $[0,\infty)$ and let f and g be two differentiable functions on $[0,\infty)$.If $f' \in L_p([0,\infty))$, $g' \in L_q([0,\infty))$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then for all $t > 0$ and $\rho_1, \rho_2, \lambda_1, \lambda_2 > 0$ and $w_1, w_2 \in \mathbb{R}$, we have*

$$T(f,g;v_1,v_2) \leq t\,\|f'\|_p\,\|g'\|_q\,\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2)(t),$$

where

$$T(f,g;v_1,v_2) := \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1fg)(t)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2fg)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1)(t)$$

$$- \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2g)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1f)(t)-\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2f)(t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1g)(t)$$

and the coefficients $\sigma_1(k), \sigma_2(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ are bounded sequences of positive real numbers.

Proof. Following the similar steps of proof of Theorem 8, we can write

$$T(f,g;v_1,v_2) := \int_0^t \int_0^t H(\eta,\xi)(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]v_1(\eta)v_2(\xi)d\eta d\xi \tag{3.14}$$

where

$$H(\eta,\xi) = (f(\eta)-f(\xi))(g(\eta)-g(\xi)) = \int_\xi^\eta \int_\xi^\eta f'(x)g'(y)dxdy.$$

Using the well-known Hölder inequality for double integral, we find that

$$|H(\eta,\xi)| \le \left|\int_\xi^\eta \int_\xi^\eta |f'(x)|^p\, dxdy\right|^{\frac{1}{p}} \left|\int_\xi^\eta \int_\xi^\eta |g'(y)|^q\, dxdy\right|^{\frac{1}{q}} = |\eta-\xi| \left|\int_\xi^\eta |f'(x)|^p\, dx\right|^{\frac{1}{p}} \left|\int_\xi^\eta |g'(y)|^q\, dy\right|^{\frac{1}{q}}. \tag{3.15}$$

Substituting (3.15) into (3.14), we have

$$\begin{aligned} |T(f,g;v_1,v_2)| \le & \int_0^t \int_0^t |\eta-\xi|(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]v_1(\eta)v_2(\xi) \\ & \times \left|\int_\xi^\eta |f'(x)|^p\, dx\right|^{\frac{1}{p}} \left|\int_\xi^\eta |g'(y)|^q\, dy\right|^{\frac{1}{q}} d\eta d\xi. \end{aligned} \tag{3.16}$$

Appying again Hölder inequality to the right hand side of (3.16), we find that

$$\begin{aligned} & |T(f,g;v_1,v_2)| \\ \le & \left(\int_0^t \int_0^t |\eta-\xi|(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]v_1(\eta)v_2(\xi)\left|\int_\xi^\eta |f'(x)|^p\, dx\right| d\eta d\xi\right)^{\frac{1}{p}} \\ & \times\left(\int_0^t \int_0^t |\eta-\xi|(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]v_1(\eta)v_2(\xi)\left|\int_\xi^\eta |g'(x)|^q\, dx\right| d\eta d\xi\right)^{\frac{1}{q}} \\ \le & \|f'\|_p \|g'\|_q \\ & \times\left(\int_0^t \int_0^t |\eta-\xi|(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(t-\xi)^{\rho_2}\right]v_1(\eta)v_2(\xi)d\eta d\xi\right)^{\frac{1}{p}} \end{aligned}$$

$$\times\left(\int_0^t\int_0^t |\eta-\xi|\,(t-\eta)^{\lambda_1-1}\,(t-\xi)^{\lambda_2-1}\,\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1\,(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2\,(t-\xi)^{\rho_2}\right]v_1(\eta)v_2(\xi)\right)^{\frac{1}{q}}.$$

Now using the fact that $|\eta-\xi|\le t$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}
&|T(f,g;v_1,v_2)|\\
\le\ & t\,\|f'\|_p\,\|g'\|_q\left(\int_0^t\int_0^t (t-\eta)^{\lambda_1-1}\,(t-\xi)^{\lambda_2-1}\,\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1\,(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2\,(t-\xi)^{\rho_2}\right]v_1(\eta)v_2(\xi)d\eta d\xi\right)^{\frac{1}{p}}\\
&\times\left(\int_0^t\int_0^t (t-\eta)^{\lambda_1-1}\,(t-\xi)^{\lambda_2-1}\,\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1\,(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2\,(t-\xi)^{\rho_2}\right]v_1(\eta)v_2(\xi)\right)^{\frac{1}{q}}\\
=\ & t\,\|f'\|_p\,\|g'\|_q\left[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1)(t)\right]^{\frac{1}{p}}\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2)(t)\right]^{\frac{1}{p}}\left[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1)(t)\right]^{\frac{1}{q}}\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2)(t)\right]^{\frac{1}{q}}\\
=\ & t\,\|f'\|_p\,\|g'\|_q\,\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v_1)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v_2)(t).
\end{aligned}$$

Thus the proof is completed. □

Corollary 3.22. *If we choose $\lambda_1=\lambda_2=\lambda$, $\sigma_1=\sigma_2=\sigma$, $\rho_1=\rho_2=\rho$, $w_1=w_2=w$ and $v_1=v_2=v$ in Theorem 3.21, we have the following inequality*

$$T(f,g;v,v)\le t\,\|f'\|_p\,\|g'\|_q\left[\mathcal{J}^{\sigma}_{\rho,\lambda,0+;\omega}(v)(t)\right]^2$$

where the coefficients $\sigma(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Remark 3.23. *In particular, putting $\lambda_1=\lambda_2=\alpha$, $\sigma_1(0)=\sigma_2(0)=1$, and $w_1=w_2=0$, then Corollary 3.22 reduce to Theorem 3.1 proved by Dahmani et. al in [6].*

Corollary 3.24. *If we choose $v_1=v_2=v$ in Theorem 3.21, we have the following inequality*

$$T(f,g;v,v)\le t\,\|f'\|_p\,\|g'\|_q\,\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,0+;\omega_1}(v)(t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,0+;\omega_2}(v)(t)$$

where the coefficients $\sigma_1(k),\sigma_2(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ are bounded sequences of positive real numbers.

Remark 3.25. *In particular, putting $\lambda_1=\alpha$, $\lambda_2=\beta$, $\sigma_1(0)=\sigma_2(0)=1$, and $w_1=w_2=0$, then Corollary 3.24 reduce to Theorem 3.2 proved by Dahmani et. al in [6].*

Corollary 3.26. *If we choose $v_1=v_2=1$ in Theorem 3.21, we have the following inequality*

$$T(f,g)\le t^{\lambda_1+\lambda_2+1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 t^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 t^{\rho_2}\right]\|f'\|_p\,\|g'\|_q\,.$$

where the coefficients $\sigma_1(k),\sigma_2(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ are bounded sequences of positive real numbers.

Remark 3.27. *In particular, putting $\lambda_1=\alpha$, $\lambda_2=\beta$, $\sigma_1(0)=\sigma_2(0)=1$, and $w_1=w_2=0$, then Corollary 3.26 reduce to Corollary 3.4 given by Dahmani et. al in [6].*

4. Concluding remarks

We have introduced a general version of Chebyshev type integral inequality for the generalised fractional integral operators based on two synchronous functions. The established results are generalization of some existing Chebychev type integral inequalities in the previous published studies. For further investigations we propose to consider the Chebyshev type inequalities for other fractional integral operators.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. S. Belarbi, Z. Dahmani, *On Some new fractional integral inequalities*, J. Inequal. Pure Appl. Math., **10** (2009), 1–12.
2. K. Boukerrioua, A. G. Lakoud, *On generalization of Čebyšev type inequalities*, J. Inequal. Pure Appl. Math., **8** (2007), 1–9.
3. P. L. Čebyšev, *Sur less expressions approximatives des integrales definies par les autres prises entre les memes limites,* Proc. Math. Soc. Charkov, **2** (1882), 93–98.
4. J. Choi, P. Agarwal, *Certain fractional integral inequalities involving hypergeometric operators*, East Asian Mathematical Journal, **30** (2014), 283–291.
5. Z. Dahmani, *New inequalities in fractional integrals*, Int. J. Nonlinear Sci., **9** (2010), 493–497.
6. Z. Dahmani, O. Mechouar, S. Brahami, *Certain inequalities related to the Chebyshev functional involving a type Riemann-Liouville operator*, Bull. Math. Anal. Appl., **3** (2011), 38–44.
7. S. S. Dragomir, *A generalization of Grüss inequality in inner product spaces and applications*, J. Math. Annal. Appl., **237** (1999), 74–82.
8. A. G. Lakoud, F. Aissaouinew, *New Čebyšev type inequalities for double integrals*, J. Math. Inequal., **5** (2011), 453–462.
9. T. Hayat, S. Nadeem, S. Asghar, *Periodic unidirectional fows of a viscoelastic fluid with the fractional Maxwell model*, Appl. Math. Comput., **151** (2004), 153–161.
10. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
11. S. M. Malamud, *Some complements to the Jensen and Chebyshev inequalities and a problem of W. Walter,* Proc. Amer. Math. Soc., **129** (2001), 2671–2678.
12. S. Momani, Z. Odiba, *Numerical approach to differential equations of fractional order*, J. Comput. Appl. Math., **207**(2007), 96–110.
13. D. S. Mitrinovic, J. E. Pecaric, A. M. Fink, *Classical and new inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

14. R. K. Raina, *On generalized Wright's hypergeometric functions and fractional calculus operators*, East Asian mathematical journal, **21** (2005), 191–203.

15. B. G. Pachpatte, *On Čebyšev-Grüss type inequalities via Pecaric's extention of the Montgomery identity,* J. Inequal. Pure Appl. Math., **7** (2006), 1–10.

16. J. E. Pecaric, *On the Čebyšev inequality,*Bul. Sti. Tehn. Inst. Politehn "Tralan Vuia" Timişora(Romania), **25** (1980), 5–9.

17. M. Z. Sarıkaya, N. Aktan, H. Yıldırım, *Weighted Čebyšev-Grüss type inequalities on time scales*, J. Math. Inequal., **2** (2008), 185–195.

18. W. T. Sulaiman, *Some new fractional integral inequalities*, J. Math. Anal., **2** (2011), 23–28.

19. F. Usta, H. Budak and M. Z. Sarıkaya, *On Chebyshev type inequalities for fractional integral operators*, AIP Conf. Proc., **1833** (2017), 1–4.

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