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Research article

Lyapunov-type inequalities for Hadamard type fractional boundary value problems

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Abstract: In this paper, we present few Lyapunov-type inequalities for Hadamard fractional boundary value problems associated with different sets of boundary conditions. Further, we discuss two applications of the established results.

Keywords: Hadamard fractional derivative; boundary value problem; Green's function; Lyapunov-type inequality; eigenvalue; lower bound; disconjugacy; disfocality
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1. Introduction

In the last few decades, fractional differential equations have gained more importance due to its applications in various sciences such as physics, mechanics, chemistry, engineering, etc. For a detailed introduction on this topic, we refer the monographs of Podlubny [23], Miller & Ross [16], Kibas et al. [12] and the references therein. Many mathematicians and scientists have contributed to the development of the theory of fractional differential equations. In this process, several types of fractional derivatives were introduced including the Hadamard fractional derivative, which is the focus of this article.

There has been a rigorous development in the theory and applications of fractional boundary value problems. However, most of the results are concerned with the Riemann-Liouville or the Caputo fractional derivatives. Recently, much attention has been paid to the study of two-point boundary value problems for fractional differential equations involving Hadamard fractional derivatives. In [2], Ahmad and Ntouyas studied a coupled system of Hadamard fractional differential equations together with integral boundary conditions. Also, they developed Hadamard fractional integro-differential boundary value problems in [3]. Wang et al. [26] investigated a non-local Hadamard fractional boundary value problem with Hadamard integral and discrete boundary condition on half line. Recently, Mao et al.

[19] and Ardjouni [1] established sufficient conditions on positive solutions for Hadamard fractional boundary value problems. More recently, Wang et al. [27, 28] analysed the stability properties of nonlinear Hadamard fractional differential system.

On the other hand, Lyapunov [14] proved a necessary condition for the existence of a nontrivial solution of Hill's equation associated with Dirichlet boundary conditions.

Theorem 1.1. *[14] If the boundary value problem*

$$\begin{cases} y''(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.1}$$

has a nontrivial solution, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{1.2}$$

The Lyapunov inequality (1.2) has several applications in various problems related to differential equations. Due to its importance, it has been generalized in many forms. Many researchers have derived Lyapunov-type inequalities for various classes of fractional boundary value problems in the recent years. For the first time, in 2013, Ferreira [8] generalized Theorem 1.1 to the case where the classical second-order derivative in (1.1) is replaced by the α^{th}-order ($1 < \alpha \leq 2$) Riemann–Liouville fractional derivative. Further, in 2014, Ferreira [9] developed a Lyapunov-type inequality for the Caputo fractional derivative. In 2018, Ntouyas et al. [21] presented a survey of results on Lyapunov-type inequalities for fractional differential equations associated with a variety of boundary conditions. For more details on Lyapunov-type inequalities and their applications, we refer [4, 6, 10, 11, 22, 24, 25, 29, 30] and the references therein.

In particular, Ma et al. [17] developed a Lyapunov-type inequality for the Hadamard fractional boundary value problem in 2017.

Theorem 1.2. *If the Hadamard fractional boundary value problem*

$$\begin{cases} {}^HD^\alpha y(t) - q(t)y(t) = 0, \quad 1 < t < e, \quad 1 < \alpha \leq 2, \\ y(1) = y(e) = 0, \end{cases} \tag{1.3}$$

has a non-trivial solution, where $q : [1, e] \to \mathbb{R}$ is a continuous function, then

$$\int_1^e |q(s)|ds > \Gamma(\alpha)(1-\lambda)^{1-\alpha}\lambda^{1-\alpha}e^\lambda. \tag{1.4}$$

Here $\lambda = \frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}$ and ${}^HD^\alpha$ denotes the α^{th}-order Hadamard fractional differential operator with $1 < \alpha \leq 2$.

Recently, Dhar [7] and Laadjal et al. [15] generalized the Lyapunov-type inequality in Theorem 1.2 by replacing the interval $[1, e]$ with $[a, b]$.

Theorem 1.3. *If the Hadamard fractional boundary value problem*

$$\begin{cases} {}^{H}D^{\alpha}y(t) + q(t)y(t) = 0, \quad 0 < a < t < b, \quad 1 < \alpha \le 2, \\ y(a) = y(b) = 0, \end{cases} \tag{1.5}$$

has a non-trivial solution and $y(t) \neq 0$ *on* (a, b)*, where* $q : [a, b] \to \mathbb{R}$ *is a continuous function, then*

$$\int_a^b q_+(s)ds > \frac{4^{\alpha-1}\Gamma(\alpha)}{(\log \frac{b}{a})^{\alpha-1}}. \tag{1.6}$$

Here $q_+(t) = \max\{q(t), 0\}$.

Theorem 1.4. *If the Hadamard fractional boundary value problem*

$$\begin{cases} {}^{H}D^{\alpha}y(t) + q(t)y(t) = 0, \quad 1 \le a < t < b, \quad 1 < \alpha \le 2, \\ y(a) = y(b) = 0, \end{cases} \tag{1.7}$$

has a non-trivial solution, where $q : [a, b] \to \mathbb{R}$ *is a continuous function, then*

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\xi_1 \left(\frac{\log \frac{\xi_1}{a} \log \frac{b}{\xi_1}}{\log \frac{b}{a}} \right)^{1-\alpha}. \tag{1.8}$$

where

$$\xi_1 = \exp\left(\frac{1}{2} \left[[(2\alpha - 2) + \log ba] - \sqrt{(2\alpha - 2)^2 + \log^2 \frac{b}{a}} \right] \right).$$

Motivated by the works in [7, 15, 17], in this article, we establish a few Lyapunov-type inequalities for Hadamard fractional boundary value problems associated with a variety of boundary conditions.

2. Preliminaries

Throughout, we shall use the following notations, definitions and some lemmas from the theory of fractional calculus in the sense of Hadamard. These results can be found in the monographs [6, 12]. Denote the set of all real numbers and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively.

Definition 2.1. [12] Let $\alpha > 0$ and $a \in \mathbb{R}$. The α^{th}-order Hadamard fractional integral of a function $y : [a, b] \to \mathbb{R}$ is defined by

$$\left({}^{H}I^{\alpha}y\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s}, \quad a < t < b, \tag{2.1}$$

provided the right-hand side exists. Here $\Gamma(.)$ denotes the Euler's Gamma function.

Definition 2.2. [12] The α^{th}-order Hadamard fractional derivative of a function $y : [a, b] \to \mathbb{R}$ is defined by

$$\left({}^{H}D^{\alpha}y\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} y(s)\frac{ds}{s}, \quad a < t < b, \tag{2.2}$$

where $n = [\alpha] + 1$.

Definition 2.3. [12] $C[a,b]$ be the space of all continuous functions $y:[a,b]\to\mathbb{R}$ with the norm

$$\|y\|_C=\max_{t\in[a,b]}|y(t)|.$$

Definition 2.4. [12] Let $0\le\gamma<1$, $y:(a,b]\to\mathbb{R}$ and define

$$y_{\gamma,\log}(t)=\left(\log\frac{t}{a}\right)^{\gamma}y(t),\quad t\in[a,b].$$

$C_{\gamma,log}[a,b]$ be the weighted space of functions y such that $y_{\gamma,\log}\in C[a,b]$, and

$$\|y\|_{C_{\gamma,\log}}=\max_{t\in[a,b]}\left|\left(\log\frac{t}{a}\right)^{\gamma}y(t)\right|.$$

Lemma 2.1. *[12] If $\alpha>0,\beta>0$ and $0<a<b<\infty$, then*

$$\left({}^{H}I^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1},$$
$$\left({}^{H}D^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}.$$

Lemma 2.2. *[6] Let $\alpha>0$, $n=[\alpha]+1$, and $0<a<b<\infty$. Assume $y\in C(a,b)$. The equality*

$$({}^{H}D^{\alpha}y)(t)=0$$

is valid if, and only if,

$$y(t)=C_1\left(\log\frac{t}{a}\right)^{\alpha-1}+C_2\left(\log\frac{t}{a}\right)^{\alpha-2}+\cdots+C_n\left(\log\frac{t}{a}\right)^{\alpha-n},$$

where C_1, C_2, $\cdots$, C_n are arbitrary constants.

Lemma 2.3. *[6] Let $\alpha>0$, $n=[\alpha]+1$, and $0<a<b<\infty$. Assume $y\in C(a,b)$. Then,*

$$\left({}^{H}D^{\alpha}\left({}^{H}I^{\alpha}y\right)\right)(t)=y(t),$$

and

$$\left({}^{H}I^{\alpha}\left({}^{H}D^{\alpha}y\right)\right)(t)=y(t)+\sum_{i=1}^{n}C_i\left(\log\frac{t}{a}\right)^{\alpha-i},$$

where C_1, C_2, $\cdots$, C_n are arbitrary constants.

3. Main results

In this section, we obtain Lyapunov-type inequalities for Hadamard fractional boundary value problems associated with mixed, Sturm-Liouville, Robin and general boundary conditions, using the properties of the corresponding Green's functions.

Theorem 3.1. *Let* $1 < \alpha \leq 2$ *and* $h : [a,b] \to \mathbb{R}$*. The fractional boundary value problem*

$$\begin{cases} ({}^HD_a^\alpha y)(t) + h(t) = 0, \quad a < t < b, \\ l({}^HI_a^{2-\alpha} y)(a) - m({}^HD_a^{\alpha-1} y)(a) = 0, \\ ny(b) + p({}^HD_a^{\alpha-1} y)(b) = 0, \end{cases} \tag{3.1}$$

has the unique solution

$$y(t) = \int_a^b G(t,s)h(s)ds, \tag{3.2}$$

where $G(t,s)$ *is given by*

$$G(t,s) = \begin{cases} G_1(t,s), \quad a < s \leq t \leq b, \\ G_2(t,s), \quad a < t \leq s \leq b, \end{cases} \tag{3.3}$$

$$G_1(t,s) = G_2(t,s) - \frac{(\log \frac{t}{s})^{\alpha-1}}{s\Gamma(\alpha)}, \tag{3.4}$$

and

$$G_2(t,s) = \left[\frac{l(\log \frac{t}{a})^{\alpha-1} + m(\alpha-1)(\log \frac{t}{a})^{\alpha-2}}{sA}\right]\left[\frac{n(\log \frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)} + p\right]. \tag{3.5}$$

Here $l,\ p \geq 0$*;* $m,\ n > 0$ *and*

$$A = ln\left(\log \frac{b}{a}\right)^{\alpha-1} + mn(\alpha-1)\left(\log \frac{b}{a}\right)^{\alpha-2} + lp\Gamma(\alpha) > 0.$$

Proof. Applying ${}^HI_a^\alpha$ (the α^{th}-order Hadamard fractional integral operator) on both sides of (3.1) and using Lemma 2.2, we have

$$y(t) = -({}^HI_a^\alpha h)(t) + C_1\left(\log \frac{t}{a}\right)^{\alpha-1} + C_2\left(\log \frac{t}{a}\right)^{\alpha-2}, \tag{3.6}$$

for some $C_1, C_2 \in \mathbb{R}$. Applying ${}^HI_a^{2-\alpha}$ on both sides of (3.6) and using Lemmas 2.1, we get

$$({}^HI_a^{2-\alpha} y)(t) = -({}^HI_a^2 h)(t) + C_1\Gamma(\alpha)\left(\log \frac{t}{a}\right) + C_2\Gamma(\alpha-1). \tag{3.7}$$

Applying ${}^HD_a^{\alpha-1}$ (the $(\alpha-1)^{\text{th}}$-order Hadamard fractional differential operator) on both sides of (3.6) and using Lemmas 2.1, we obtain

$$({}^HD_a^{\alpha-1} y)(t) = -({}^HI_a^1 h)(t) + C_1\Gamma(\alpha). \tag{3.8}$$

From the first boundary condition, we have

$$-mC_1(\alpha-1) + lC_2 = 0. \tag{3.9}$$

The second boundary condition yields

$$C_1\left[n\left(\log \frac{b}{a}\right)^{\alpha-1} + p\Gamma(\alpha)\right] + nC_2\left(\log \frac{b}{a}\right)^{\alpha-2} = n({}^HI_a^\alpha h)(b) + p({}^HI_a^1 h)(b). \tag{3.10}$$

Solving (3.9) and (3.10) for C_1 and C_2, we have

$$C_1 = \frac{l}{A}\int_a^b \left(\frac{n(\log\frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)} + p\right)h(s)\frac{ds}{s},$$

and

$$C_2 = \frac{m(\alpha-1)}{A}\int_a^b \left(\frac{n(\log\frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)} + p\right)h(s)\frac{ds}{s}.$$

Substituting C_1 and C_2 in (3.6), it follows that

$$\begin{aligned} y(t) &= \frac{l(\log\frac{t}{a})^{\alpha-1}}{A}\int_a^b \left(\frac{n(\log\frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)} + p\right)h(s)\frac{ds}{s} \\ &+ \frac{m(\alpha-1)(\log\frac{t}{a})^{\alpha-2}}{A}\int_a^b \left(\frac{n(\log\frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)} + p\right)h(s)\frac{ds}{s} \\ &- \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{a}\right)^{\alpha-1}h(s)\frac{ds}{s} \\ &= \int_a^b G(t,s)h(s)ds. \end{aligned}$$

The proof is complete. □

Corollary 1. *Let* $1 < \alpha \le 2$ *and* $h : [a,b] \to \mathbb{R}$*. The fractional boundary value problem*

$$\begin{cases} ({}^H D_a^\alpha y)(t) + h(t) = 0, \quad a < t < b, \\ y(a) = 0,\ ny(b) + p({}^H D_a^{\alpha-1} y)(b) = 0, \end{cases} \tag{3.11}$$

has the unique solution

$$y(t) = \int_a^b \bar{G}(t,s)h(s)ds, \tag{3.12}$$

where $\bar{G}(t,s)$ *is given by*

$$\bar{G}(t,s) = \begin{cases} \bar{G}_1(t,s), \quad a \le s \le t \le b, \\ \bar{G}_2(t,s), \quad a \le t \le s \le b, \end{cases} \tag{3.13}$$

$$\bar{G}_1(t,s) = \bar{G}_2(t,s) - \frac{\left(\log\frac{t}{s}\right)^{\alpha-1}}{s\Gamma(\alpha)}, \tag{3.14}$$

and

$$\bar{G}_2(t,s) = \frac{\left(\log\frac{t}{a}\right)^{\alpha-1}}{s\bar{A}}\left(\frac{n(\log\frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)} + p\right). \tag{3.15}$$

Here $n \ge 0$*,* $p > 0$ *and* $\bar{A} = n\left(\log\frac{b}{a}\right)^{\alpha-1} + p\Gamma(\alpha) > 0$.

Proof. The proof is similar to Theorem 3.1. □

We define $H(t,s) = sG(t,s)$ and $\bar{H}(t,s) = s\bar{G}(t,s)$. Now, we prove that these Green's functions are positive and obtain upper bounds for both the Green's functions and their integrals.

Theorem 3.2. *The Green's function $H(t,s)$ for* (3.1) *satisfies $H(t,s) > 0$ for $(t,s) \in (a,b] \times (a,b]$.*

Proof. Clearly, for $a < t \le s \le b$,

$$H(t,s) = \left[\frac{l(\log\frac{t}{a})^{\alpha-1} + m(\alpha-1)(\log\frac{t}{a})^{\alpha-2}}{A}\right]\left[\frac{n(\log\frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)} + p\right] > 0.$$

Now, suppose $a < s \le t \le b$. Consider

$$\begin{aligned}
H(t,s) = &\left[\frac{l(\log\frac{t}{a})^{\alpha-1} + m(\alpha-1)(\log\frac{t}{a})^{\alpha-2}}{A}\right]\left[\frac{n(\log\frac{b}{s})^{\alpha-1}}{\Gamma(\alpha)} + p\right]\\
&- \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}\\
= &\frac{ln}{A\Gamma(\alpha)}\left\{\left(\log\frac{t}{a}\right)^{\alpha-1}\left(\log\frac{b}{s}\right)^{\alpha-1} - \left(\log\frac{b}{a}\right)^{\alpha-1}\left(\log\frac{t}{s}\right)^{\alpha-1}\right\}\\
&+ \frac{mn(\alpha-1)}{A\Gamma(\alpha)}\left\{\left(\log\frac{t}{a}\right)^{\alpha-2}\left(\log\frac{b}{s}\right)^{\alpha-1} - \left(\log\frac{b}{a}\right)^{\alpha-2}\left(\log\frac{t}{s}\right)^{\alpha-1}\right\}\\
&+ \frac{lp}{A}\left\{\left(\log\frac{t}{a}\right)^{\alpha-1} - \left(\log\frac{t}{s}\right)^{\alpha-1}\right\} + \frac{mp(\alpha-1)}{A}\left(\log\frac{t}{a}\right)^{\alpha-2}\\
= &\frac{1}{A\Gamma(\alpha)}[X_1 + X_2 + X_3 + X_4]. \qquad (3.16)
\end{aligned}$$

Clearly, $A\Gamma(\alpha) > 0$. Consider

$$\begin{aligned}
&\left(\log\frac{t}{a}\right)\left(\log\frac{b}{s}\right) - \left(\log\frac{b}{a}\right)\left(\log\frac{t}{s}\right)\\
=&(\log t - \log a)(\log b - \log s) - (\log b - \log a)(\log t - \log s)\\
=&(\log s - \log a)(\log b - \log t) \ge 0,
\end{aligned}$$

implying that

$$X_1 = ln\left\{\left(\log\frac{t}{a}\right)^{\alpha-1}\left(\log\frac{b}{s}\right)^{\alpha-1} - \left(\log\frac{b}{a}\right)^{\alpha-1}\left(\log\frac{t}{s}\right)^{\alpha-1}\right\} \ge 0. \qquad (3.17)$$

Since

$$a < s \le t \le b,$$

we have

$$\left(\log\frac{t}{a}\right)^{\alpha-2} \ge \left(\log\frac{b}{a}\right)^{\alpha-2}, \quad \left(\log\frac{b}{s}\right)^{\alpha-1} \ge \left(\log\frac{t}{s}\right)^{\alpha-1}$$

and

$$\left(\log\frac{t}{a}\right)^{\alpha-1} > \left(\log\frac{t}{s}\right)^{\alpha-1},$$

implying that

$$X_2 = mn(\alpha-1)\left\{\left(\log\frac{t}{a}\right)^{\alpha-2}\left(\log\frac{b}{s}\right)^{\alpha-1} - \left(\log\frac{b}{a}\right)^{\alpha-2}\left(\log\frac{t}{s}\right)^{\alpha-1}\right\}$$

$$\geq mn(\alpha-1)\left(\log\frac{b}{a}\right)^{\alpha-2}\left\{\left(\log\frac{b}{s}\right)^{\alpha-1}-\left(\log\frac{t}{s}\right)^{\alpha-1}\right\}\geq 0, \tag{3.18}$$

and

$$X_3 = lp\Gamma(\alpha)\left[\left(\log\frac{t}{a}\right)^{\alpha-1}-\left(\log\frac{t}{s}\right)^{\alpha-1}\right]>0. \tag{3.19}$$

Clearly,

$$X_4 = mp(\alpha-1)\Gamma(\alpha)\left(\log\frac{t}{a}\right)^{\alpha-2}>0. \tag{3.20}$$

Using (3.17) - (3.20) in (3.16), we have $H(t,s)>0$. The proof is complete. □

Corollary 2. *The Green's function $\bar{H}(t,s)$ for* (3.11) *satisfies $\bar{H}(t,s)\geq 0$ for $(t,s)\in[a,b]\times[a,b]$.*

Proof. The proof is similar to Theorem 3.2. □

Theorem 3.3. *For the Green's function $H(t,s)$ defined in* (3.3)*,*

$$\max_{s\in(a,b]} H(t,s) = H(t,t), \quad t\in(a,b],$$

and

$$\left(\log\frac{t}{a}\right)^{2-\alpha}H(t,t) < \left[\frac{l\left(\log\frac{b}{a}\right)+m(\alpha-1)}{A}\right]\left[\frac{n\left(\log\frac{b}{a}\right)^{\alpha-1}}{\Gamma(\alpha)}+p\right], \quad t\in[a,b].$$

Proof. For the first part, we show that for any fixed $t\in(a,b]$, $H(t,s)$ increases with respect to s from a to t, and then decreases with respect s from t to b. Let $a<t\leq s\leq b$. Consider

$$\frac{\partial}{\partial s}H(t,s) = \left[\frac{-n(\alpha-1)\left(\log\frac{b}{s}\right)^{\alpha-2}}{s\Gamma(\alpha)}\right]\left[\frac{l\left(\log\frac{t}{a}\right)^{\alpha-1}+m(\alpha-1)\left(\log\frac{t}{a}\right)^{\alpha-2}}{A}\right]<0,$$

implying that $H(t,s)$ is a decreasing function of s. Now, suppose $a<s\leq t\leq b$. Consider

$$\begin{aligned}
\frac{\partial}{\partial s}H(t,s) &= \left[\frac{-n(\alpha-1)\left(\log\frac{b}{s}\right)^{\alpha-2}}{s\Gamma(\alpha)}\right]\left[\frac{l\left(\log\frac{t}{a}\right)^{\alpha-1}+m(\alpha-1)\left(\log\frac{t}{a}\right)^{\alpha-2}}{A}\right]\\
&\quad+\frac{(\alpha-1)\left(\log\frac{t}{s}\right)^{\alpha-2}}{s\Gamma(\alpha)}\\
&= \frac{ln(\alpha-1)}{A\Gamma(\alpha)}\left\{-\left(\log\frac{t}{a}\right)^{\alpha-1}\left(\log\frac{b}{s}\right)^{\alpha-2}+\left(\log\frac{b}{a}\right)^{\alpha-1}\left(\log\frac{t}{s}\right)^{\alpha-2}\right\}\\
&\quad+\frac{mn(\alpha-1)^2}{A\Gamma(\alpha)}\left\{-\left(\log\frac{t}{a}\right)^{\alpha-2}\left(\log\frac{b}{s}\right)^{\alpha-2}+\left(\log\frac{b}{a}\right)^{\alpha-2}\left(\log\frac{t}{s}\right)^{\alpha-2}\right\}\\
&\quad+\frac{lp(\alpha-1)}{A}\left(\log\frac{t}{s}\right)^{\alpha-2}\\
&= \frac{(\alpha-1)}{A\Gamma(\alpha)}[X_5+X_6+X_7].
\end{aligned} \tag{3.21}$$

Clearly, $\frac{(\alpha-1)}{A\Gamma(\alpha)} > 0$. Since $a < s < t \le b$, we have

$$\left(\log \frac{t}{s}\right)^{\alpha-2} \ge \left(\log \frac{b}{s}\right)^{\alpha-2} \text{ and } \left(\log \frac{b}{a}\right)^{\alpha-1} \ge \left(\log \frac{t}{a}\right)^{\alpha-1},$$

implying that

$$\begin{aligned} X_5 &= ln\left[-\left(\log \frac{t}{a}\right)^{\alpha-1}\left(\log \frac{b}{s}\right)^{\alpha-2} + \left(\log \frac{b}{a}\right)^{\alpha-1}\left(\log \frac{t}{s}\right)^{\alpha-2}\right] \\ &\ge ln\left(\log \frac{t}{s}\right)^{\alpha-2}\left[-\left(\log \frac{t}{a}\right)^{\alpha-1} + \left(\log \frac{b}{a}\right)^{\alpha-1}\right] \ge 0. \end{aligned} \tag{3.22}$$

Since

$$\begin{aligned} &\left(\log \frac{t}{a}\right)\left(\log \frac{b}{s}\right) - \left(\log \frac{b}{a}\right)\left(\log \frac{t}{s}\right) \\ =&(\log t - \log a)(\log b - \log s) - (\log b - \log a)(\log t - \log s) \\ =&(\log s - \log a)(\log b - \log t) \ge 0, \end{aligned}$$

we have that

$$X_6 = mn(\alpha-1)^2\left\{-\left(\log \frac{t}{a}\right)^{\alpha-2}\left(\log \frac{b}{s}\right)^{\alpha-2} + \left(\log \frac{b}{a}\right)^{\alpha-2}\left(\log \frac{t}{s}\right)^{\alpha-2}\right\} \ge 0. \tag{3.23}$$

Clearly,

$$X_7 = lp\Gamma(\alpha)\left(\log \frac{t}{s}\right)^{\alpha-2} > 0. \tag{3.24}$$

Using (3.22) - (3.24) in (3.21), we have $\frac{\partial}{\partial s}H(t,s) > 0$, implying that $H(t,s)$ is an increasing function of s. Then, it follows that

$$\max_{s\in(a,b]} H(t,s) = H(t,t), \quad t \in (a,b].$$

To prove the second part, for $t \in [a,b]$, consider

$$\begin{aligned} \left(\log \frac{t}{a}\right)^{2-\alpha} H(t,t) &= \left[\frac{l\left(\log \frac{t}{a}\right) + m(\alpha-1)}{A}\right]\left[\frac{n\left(\log \frac{b}{t}\right)^{\alpha-1}}{\Gamma(\alpha)} + p\right] \\ &< \left[\frac{l\left(\log \frac{b}{a}\right) + m(\alpha-1)}{A}\right]\left[\frac{n\left(\log \frac{b}{a}\right)^{\alpha-1}}{\Gamma(\alpha)} + p\right]. \end{aligned}$$

The proof is complete. □

Corollary 3. *For the Green's function $\bar{H}(t,s)$ defined in* (3.13)*,*

$$\max_{s\in[a,b]} \bar{H}(t,s) = \bar{H}(t,t), \quad t \in [a,b],$$

and

$$\bar{H}(t,t) < \left[\frac{\left(\log \frac{b}{a}\right)^{\alpha-1}}{\bar{A}}\right]\left[\frac{n\left(\log \frac{b}{a}\right)^{\alpha-1}}{\Gamma(\alpha)} + p\right], \quad t \in [a,b].$$

Proof. The first part of the proof is similar to the proof of Theorem 3.3. To prove the second part, for $t \in [a, b]$, consider

$$\begin{aligned}\bar{H}(t,t) &= \left[\frac{\left(\log\frac{t}{a}\right)^{\alpha-1}}{\bar{A}}\right]\left[\frac{n\left(\log\frac{b}{t}\right)^{\alpha-1}}{\Gamma(\alpha)}+p\right]\\ &< \left[\frac{\left(\log\frac{b}{a}\right)^{\alpha-1}}{\bar{A}}\right]\left[\frac{n\left(\log\frac{b}{a}\right)^{\alpha-1}}{\Gamma(\alpha)}+p\right].\end{aligned}$$

The proof is complete. □

We are now able to formulate Lyapunov-type inequalities for the fractional boundary value problems (3.1) and (3.11).

Theorem 3.4. *If the following fractional boundary value problem*

$$\begin{cases}({}^H D^{\alpha} y)(t) + p(t)y(t) = 0, \quad 0 < a < t < b,\\ l({}^H I^{2-\alpha} y)(a) - m({}^H D^{\alpha-1} y)(a) = 0,\\ ny(b) + p({}^H D^{\alpha-1} y)(b) = 0,\end{cases} \tag{3.25}$$

has a nontrivial solution, then

$$\int_a^b \left(\log\frac{s}{a}\right)^{\alpha-2}|p(s)|ds > \frac{A\Gamma(\alpha)}{\left[n\left(\log\frac{b}{a}\right)^{\alpha-1}+p\Gamma(\alpha)\right]\left[l\left(\log\frac{b}{a}\right)+m(\alpha-1)\right]}. \tag{3.26}$$

Proof. Let $\mathfrak{B} = C_{\gamma,\log}[a, b]$ be the Banach space of functions y endowed with norm

$$\|y\|_{C_{\gamma,\log}} = \max_{t\in[a,b]}\left|\left(\log\frac{t}{a}\right)^{\gamma} y(t)\right|.$$

It follows from Theorem 3.1 that a solution to (3.25) satisfies the equation

$$y(t) = \int_a^b H(t,s)p(s)y(s)ds = \int_a^b sG(t,s)p(s)y(s)ds.$$

Hence,

$$\begin{aligned}\|y\|_{C_{2-\alpha,\log}} &= \max_{t\in[a,b]}\left|\left(\log\frac{t}{a}\right)^{2-\alpha}\int_a^b sG(t,s)p(s)y(s)ds\right|\\ &\le \max_{t\in[a,b]}\left[\int_a^b \left|\left(\log\frac{t}{a}\right)^{2-\alpha} sG(t,s)\right||p(s)||y(s)|ds\right]\\ &\le \|y\|_{C_{2-\alpha,\log}}\left[\max_{t\in[a,b]}\int_a^b \left|\left(\log\frac{t}{a}\right)^{2-\alpha} H(t,s)\right|\left(\log\frac{s}{a}\right)^{\alpha-2}|p(s)|ds\right]\\ &\le \|y\|_{C_{2-\alpha,\log}}\left[\max_{t\in[a,b]}\left|\left(\log\frac{t}{a}\right)^{2-\alpha} H(t,s)\right|\right]\int_a^b \left(\log\frac{s}{a}\right)^{\alpha-2}|p(s)|ds,\end{aligned}$$

or, equivalently,

$$1 < \left[\left|\left(\log\frac{t}{a}\right)^{2-\alpha} H(t,t)\right|\right]\int_a^b \left(\log\frac{s}{a}\right)^{\alpha-2}|p(s)|ds.$$

An application of Theorem 3.3 yields the result. □

Corollary 4. *If the following fractional boundary value problem*

$$\begin{cases}({}^H D_a^{\alpha} y)(t) + q(t)y(t) = 0, \quad a < t < b,\\ y(a) = 0,\ ny(b) + p({}^H D_a^{\alpha-1} y)(b) = 0,\end{cases} \tag{3.27}$$

has a nontrivial solution, then

$$\int_a^b \left|q(s)\right| ds > \frac{\bar{A}\Gamma(\alpha)}{\left[n\left(\log \frac{b}{a}\right)^{2\alpha-2} + p\left(\log \frac{b}{a}\right)^{\alpha-1} \Gamma(\alpha)\right]}. \tag{3.28}$$

Proof. Let $\mathfrak{B} = C[a, b]$ be the Banach space of functions y endowed with norm

$$\|y\| = \max_{t\in[a,b]} |y(t)|.$$

It follows from Corollary 1 that a solution to (3.27) satisfies the equation

$$y(t) = \int_a^b \bar{H}(t, s)q(s)y(s)ds = \int_a^b s\bar{H}(t, s)q(s)y(s)ds.$$

Hence,

$$\begin{aligned}\|y\| = \max_{t\in[a,b]} \left| \int_a^b \bar{H}(t, s)q(s)y(s)ds\right| &\le \max_{t\in[a,b]} \left[\int_a^b \bar{H}(t, s)\left|q(s)\right|\left|y(s)\right|ds\right]\\ &\le \|y\| \left[\max_{t\in[a,b]} \int_a^b \bar{H}(t, s)\left|q(s)\right|ds\right]\\ &\le \|y\| \left[\max_{t\in[a,b]} \bar{H}(t, t)\right] \int_a^b \left|q(s)\right|ds,\end{aligned}$$

or, equivalently,

$$1 < \left[\max_{t\in[a,b]} \bar{H}(t, t)\right] \int_a^b \left|q(s)\right|ds.$$

An application of Corollary 3 yields the result. □

Take $l = p = 0$ in Theorem 3.4. Then, we obtain the following Lyapunov-type inequality for the left-focal fractional boundary value problem.

Corollary 5. *If the following fractional boundary value problem*

$$\begin{cases}({}^H D_a^{\alpha} y)(t) + q(t)y(t) = 0, \quad a < t < b,\\ ({}^H D_a^{\alpha-1} y)(a) = 0,\ y(b) = 0,\end{cases} \tag{3.29}$$

has a nontrivial solution, then

$$\int_a^b \left(\log \frac{s}{a}\right)^{\alpha-2} \left|q(s)\right|ds > \frac{\Gamma(\alpha)}{(\log \frac{b}{a})}. \tag{3.30}$$

Take $n = 0$ in Corollary 4. Then, we obtain the following Lyapunov-type inequality for the right-focal fractional boundary value problem.

Corollary 6. *If the following fractional boundary value problem*

$$\begin{cases}({}^H D_a^{\alpha} y)(t) + q(t)y(t) = 0, \quad a < t < b,\\ y(a) = 0,\ ({}^H D_a^{\alpha-1} y)(b) = 0,\end{cases} \tag{3.31}$$

has a nontrivial solution, then

$$\int_a^b \left|q(s)\right| ds > \frac{\Gamma(\alpha)}{\left(\log \frac{b}{a}\right)^{\alpha-1}}. \tag{3.32}$$

Take $l = m = n = p = 1$ in Theorem 3.4. Then, we obtain the following Lyapunov-type inequality for the fractional boundary value problem with Robin boundary conditions.

Corollary 7. *If the following fractional boundary value problem*

$$\begin{cases}({}^H D_a^{\alpha} y)(t) + q(t)y(t) = 0, \quad a < t < b,\\ ({}^H I_a^{2-\alpha} y)(a) - ({}^H D_a^{\alpha-1} y)(a) = 0,\\ y(b) + ({}^H D_a^{\alpha-1} y)(b) = 0,\end{cases} \tag{3.33}$$

has a nontrivial solution, then

$$\int_a^b \left(\log \frac{s}{a}\right)^{\alpha-2} \left|q(s)\right| ds > \frac{\Gamma(\alpha)\left[\left(\log \frac{b}{a}\right)^{\alpha-1} + (\alpha-1)\left(\log \frac{b}{a}\right)^{\alpha-2} + \Gamma(\alpha)\right]}{\left[\left(\log \frac{b}{a}\right)^{\alpha-1} + \Gamma(\alpha)\right]\left[\log \frac{b}{a} + \alpha - 1\right]}. \tag{3.34}$$

Take $l > 0$ and $p = 0$ in Theorem 3.4. Then, we obtain the following Lyapunov-type inequality for the fractional boundary value problem with Sturm-Liouville boundary conditions.

Corollary 8. *If the following fractional boundary value problem*

$$\begin{cases}({}^H D_a^{\alpha} y)(t) + q(t)y(t) = 0, \quad a < t < b,\\ l({}^H I_a^{2-\alpha} y)(a) - m({}^H D_a^{\alpha-1} y)(a) = 0,\ y(b) = 0,\end{cases} \tag{3.35}$$

has a nontrivial solution, then

$$\int_a^b \left(\log \frac{s}{a}\right)^{\alpha-2} \left|q(s)\right| ds > \frac{\Gamma(\alpha)}{\log \frac{b}{a}}. \tag{3.36}$$

4. Anti-periodic boundary condition

In this section, we obtain a Lyapunov-type inequality for an anti-periodic fractional boundary value problem using the properties of the corresponding Green's function.

Theorem 4.1. *Let $1 < \alpha \le 2$ and $h : [1, T] \to \mathbb{R}$. The fractional boundary value problem*

$$\begin{cases}{}^H D^{\alpha} y(t) + h(t) = 0, \quad 1 < t < T,\\ ({}^H I^{2-\alpha} y)(1) + ({}^H I^{2-\alpha} y)(T) = 0,\\ ({}^H D^{\alpha-1} y)(1) + ({}^H D^{\alpha-1} y)(T) = 0,\end{cases} \tag{4.1}$$

has the unique solution

$$y(t) = \int_1^T \tilde{G}(t, s)h(s)ds,\ 1 < t \le T, \tag{4.2}$$

where

$$\tilde{G}(t, s) = \begin{cases} \frac{(\log t)^{\alpha-1}}{2s\Gamma(\alpha)} + \frac{(\log t)^{\alpha-2}(\log T - 2\log s)}{4s\Gamma(\alpha-1)} - \frac{(\log\frac{t}{s})^{\alpha-1}}{s\Gamma(\alpha)}, & 1 \le s \le t \le T, \\ \frac{(\log t)^{\alpha-1}}{2s\Gamma(\alpha)} + \frac{(\log t)^{\alpha-2}(\log T - 2\log s)}{4s\Gamma(\alpha-1)}, & 1 \le t \le s \le T. \end{cases} \tag{4.3}$$

Proof. Applying ${}^H I^\alpha$ on both sides of (4.1) and using Lemma 2.2, we have

$$y(t) = C_1(\log t)^{\alpha-1} + C_2(\log t)^{\alpha-2} - \int_1^t (\log t - \log s)^{\alpha-1} h(s)\frac{ds}{s}, \tag{4.4}$$

for some $C_1, C_2 \in \mathbb{R}$. Applying ${}^H I^{2-\alpha}$ on both sides of (4.4) and using Lemma (2.1), we get

$$\left({}^H I^{2-\alpha} y\right)(t) = -\int_1^t h(s)\frac{ds}{s} + C_1\Gamma(\alpha)(\log t) + C_2\Gamma(\alpha - 1). \tag{4.5}$$

Applying ${}^H D^{\alpha-1}$ on both sides of (4.4) and using Lemma (2.1), we obtain

$$\left({}^H D^{\alpha-1} y\right)(t) = -\int_1^t h(s)\frac{ds}{s} + C_1\Gamma(\alpha). \tag{4.6}$$

From the first boundary condition, we get

$$C_1\Gamma(\alpha)(\log T) + 2C_2\Gamma(\alpha - 1) = \int_1^T \left(\log\frac{T}{s}\right)h(s)\frac{ds}{s}. \tag{4.7}$$

The second boundary condition yields

$$\int_1^T h(s)\frac{ds}{s} = 2C_1\Gamma(\alpha). \tag{4.8}$$

Solving (4.7) and (4.8) for C_1 and C_2, we have

$$C_1 = \frac{1}{2\Gamma(\alpha)}\int_1^T h(s)\frac{ds}{s},$$

and

$$C_2 = \frac{1}{4\Gamma(\alpha - 1)}\int_1^T (\log T - 2\log s)h(s)\frac{ds}{s}.$$

Substituting C_1 and C_2 in (3.6), we obtain the unique solution of (4.1) as

$$\begin{aligned} y(t) &= \frac{(\log t)^{\alpha-1}}{2\Gamma(\alpha)}\int_1^T h(s)\frac{ds}{s} + \frac{(\log t)^{\alpha-2}}{4\Gamma(\alpha - 1)}\int_1^T (\log T - 2\log s)h(s)\frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_1^t (\log t - \log s)^{\alpha-1}h(s)\frac{ds}{s} \\ &= \int_1^t \left[\frac{(\log t)^{\alpha-1}}{2s\Gamma(\alpha)} + \frac{(\log t)^{\alpha-2}(\log T - 2\log s)}{4s\Gamma(\alpha - 1)} - \frac{(\log\frac{t}{s})^{\alpha-1}}{s\Gamma(\alpha)}\right]h(s)ds \end{aligned}$$

$$+\int_t^T \left[\frac{(\log t)^{\alpha-1}}{2s\Gamma(\alpha)} + \frac{(\log t)^{\alpha-2}(\log T - 2\log s)}{4s\Gamma(\alpha-1)}\right] h(s)ds$$
$$= \int_1^T \tilde{G}(t,s)h(s)ds.$$

The proof is complete. □

Let us define $\tilde{H}(t,s) = s(\log t)^{2-\alpha}\tilde{G}(t,s)$ Now, we obtain an upper bound for the Green's function $\tilde{H}(t,s)$.

Theorem 4.2. *For the Green's function $\tilde{H}(t,s)$ defined in* (4.3)*, we observe that*

$$\left|\tilde{H}(t,s)\right| < \frac{(3-\alpha)\log T}{4\Gamma(\alpha)}, \ \forall\ (t,s) \in [1,T]\times[1,T].$$

Proof. Consider

$$\tilde{H}(t,s) = \begin{cases} \frac{\log t}{2\Gamma(\alpha)} + \frac{(\log T - 2\log s)}{4\Gamma(\alpha-1)} - \frac{(\log t)^{2-\alpha}(\log t - \log s)^{\alpha-1}}{\Gamma(\alpha)}, & 1 \le s \le t \le T, \\ \frac{\log t}{2\Gamma(\alpha)} + \frac{(\log T - 2\log s)}{4\Gamma(\alpha-1)}, & 1 \le t \le s \le T. \end{cases} \tag{4.9}$$

Denote by

$$\tilde{H}_1(t,s) = \frac{\log t}{2\Gamma(\alpha)} + \frac{(\log T - 2\log s)}{4\Gamma(\alpha-1)} - \frac{(\log t)^{2-\alpha}(\log t - \log s)^{\alpha-1}}{\Gamma(\alpha)}$$

and

$$\tilde{H}_2(t,s) = \frac{\log t}{2\Gamma(\alpha)} + \frac{(\log T - 2\log s)}{4\Gamma(\alpha-1)}.$$

For a fixed $t \in [1,T]$,

$$\frac{d}{ds}\tilde{H}_1(t,s) = \frac{(\alpha-1)(\log t - \log s)^{\alpha-2}(\log t)^{2-\alpha}}{s\Gamma(\alpha)} - \frac{2}{4s\Gamma(\alpha-1)}$$
$$= \frac{-(\log t - \log s)^{2-\alpha} + (\log t)^{2-\alpha}}{2s\Gamma(\alpha-1)(\log t - \log s)^{2-\alpha}} \ge 0,$$

for all $s \in [1,t]$. So, $\tilde{H}_1(t,s)$ is an increasing function of s. Thus,

$$\max_{s\in[1,t]} \left|\tilde{H}_1(t,s)\right| = \max\left\{\left|\tilde{H}_1(t,1)\right|, \left|\tilde{H}_1(t,t)\right|\right\}.$$

We observe that $\tilde{H}_1(t,t)$ is an increasing function of s, since

$$\frac{d}{dt}\tilde{H}_1(t,t) = \frac{2-\alpha}{2t\Gamma(\alpha)} > 0.$$

Therefore, we have

$$\max_{s\in[1,T]} \tilde{H}_1(t,t) = \max\left\{\left|\tilde{H}_1(1,1)\right|, \left|\tilde{H}_1(T,T)\right|\right\}$$
$$= \max\left\{\frac{\log T}{4\Gamma(\alpha-1)}, \frac{(3-\alpha)\log T}{4\Gamma(\alpha)}\right\}$$

$$= \frac{(3-\alpha)\log T}{4\Gamma(\alpha)}.$$

Now, consider

$$\frac{d}{dt}\tilde{H}_1(t,1) = \frac{-1}{2t\Gamma(\alpha)} < 0,$$

implying that $H_1(t,1)$ is a decreasing function of t. So, we have

$$\begin{aligned}\max_{t\in[1,T]}\left|\tilde{H}_1(t,1)\right| &= \max\left\{\left|\tilde{H}_1(1,1)\right|, \left|\tilde{H}_1(T,1)\right|\right\}\\ &= \max\left\{\frac{\log T}{4\Gamma(\alpha-1)}, \frac{(3-\alpha)\log T}{4\Gamma(\alpha)}\right\}\\ &= \frac{(3-\alpha)\log T}{4\Gamma(\alpha)}.\end{aligned}$$

Therefore,

$$\max_{s\in[1,t],t\in[1,T]}\left|\tilde{H}_1(t,s)\right| = \frac{(3-\alpha)\log T}{4\Gamma(\alpha)}. \tag{4.10}$$

For a fixed $s \in [1,T]$,

$$\frac{d}{dt}\tilde{H}_2(t,s) = \frac{1}{2t\Gamma(\alpha)} > 0,$$

implying that $\tilde{H}_2(t,s)$ is an increasing function of t. So,

$$\max_{t\in[1,s]}\left|\tilde{H}_2(t,s)\right| = \max\left\{\left|\tilde{H}_2(1,s)\right|, \left|\tilde{H}_2(s,s)\right|\right\}.$$

Since $\left|\tilde{H}_2(s,s)\right| = \left|\tilde{H}_1(s,s)\right|$ for $s \in [1,T]$, we only consider $\tilde{H}_2(1,s)$. Since

$$\frac{d}{ds}\tilde{H}_2(1,s) = \frac{-1}{2s\Gamma(\alpha-1)} < 0,$$

$\tilde{H}_2(1,s)$ is a decreasing function of s. Thus, we have

$$\begin{aligned}\max_{s\in[1,T]}|\tilde{H}_2(1,s)| &= \max\{|\tilde{H}_2(1,1)|, |\tilde{H}_2(1,T)|\}\\ &= \max\left\{\frac{\log T}{4\Gamma(\alpha-1)}, \frac{\log T}{4\Gamma(\alpha-1)}\right\}\\ &= \frac{\log T}{4\Gamma(\alpha-1)}.\end{aligned}$$

Hence, we have

$$\max_{t\in[1,s],s\in[1,T]}|\tilde{H}_2(t,s)| = \frac{(3-\alpha)\log T}{4\Gamma(\alpha)}. \tag{4.11}$$

The final result follows from (4.10) and (4.11). □

We are now able to formulate a Lyapunov-type inequality for the Hadamard type fractional boundary value problem with anti-periodic boundary condition.

Theorem 4.3. *If the following fractional boundary value problem*

$$\begin{cases} \left({}^{H}D^{\alpha}y\right)\right)(t) + p(t)y(t) = 0, \quad 1 < t < T, \\ ({}^{H}I^{2-\alpha}y)(0) + ({}^{H}I^{2-\alpha}y)(T) = 0, \\ ({}^{H}D^{\alpha-1}y)(0) + ({}^{H}D^{\alpha-1}y)(T) = 0, \end{cases} \tag{4.12}$$

has a nontrivial solution, then

$$\int_1^T (\log s)^{\alpha-2}|p(s)|ds > \frac{4\Gamma(\alpha)}{(3-\alpha)\log T}. \tag{4.13}$$

Proof. Let $\mathfrak{B} = C_{\gamma,\log}[1,T]$ be the Banach space of functions y endowed with norm

$$||y||_{C_{\gamma,\log}} = \max_{t\in[1,T]} \left|(\log t)^{\gamma} y(t)\right|.$$

It follows from Theorem 4.1 that a solution to (4.12) satisfies the equation

$$y(t) = \int_1^T \tilde{H}(t,s)p(s)y(s)ds = \int_1^T (\log t)^{2-\alpha} s\tilde{G}(t,s)p(s)y(s)ds.$$

Hence,

$$\begin{aligned} ||y||_{C_{2-\alpha,\log}} &= \max_{t\in[1,T]} \left|(\log t)^{2-\alpha} \int_1^T s\tilde{G}(t,s)p(s)y(s)ds\right| \\ &\leq \max_{t\in[1,T]} \left[\int_1^T \left|(\log t)^{2-\alpha} s\tilde{G}(t,s)\right||p(s)||y(s)|ds\right] \\ &\leq ||y||_{C_{2-\alpha,\log}} \left[\max_{t\in[1,T]} \int_1^T \left|(\log t)^{2-\alpha} s\tilde{G}(t,s)\right|(\log s)^{\alpha-2}|p(s)|ds\right] \\ &\leq ||y||_{C_{2-\alpha,\log}} \left[\max_{t\in[1,T]} \left|(\log t)^{2-\alpha} s\tilde{G}(t,s)\right|\right] \int_1^T (\log s)^{\alpha-2}|p(s)|ds, \end{aligned}$$

or, equivalently,

$$1 < \left[\max_{t\in[1,T]} \left|\tilde{H}(t,s)\right|\right] \int_1^T (\log s)^{\alpha-2}|p(s)|ds.$$

An application of Theorem 4.2 yields the result. □

5. Applications

In this section, we discuss two applications of the results established in previous sections. First, we estimate lower bounds on the smallest eigenvalues of the eigenvalue problems corresponding to (3.25), (3.27) and (3.25).

Theorem 5.1. *Assume y is a nontrivial solution of the Hadamard fractional eigenvalue problem*

$$\begin{cases} ({}^{H}D_a^{\alpha}y)(t) + p(t)y(t) = 0, \quad a < t < b, \\ l({}^{H}I_a^{2-\alpha}y)(a) - m({}^{H}D_a^{\alpha-1}y)(a) = 0, \\ ny(b) + p({}^{H}D_a^{\alpha-1}y)(b) = 0, \end{cases}$$

where $y(t) \neq 0$ *for each* $t \in (a, b)$. *Then,*

$$|\lambda| > \frac{(-1)^{\alpha} A\Gamma(\alpha)\Big[\Gamma(\alpha-1) - \Gamma(\alpha-1, -\log\frac{b}{a})\Big]^{-1}}{a\Big[n\left(\log\frac{b}{a}\right)^{\alpha-1} + p\Gamma(\alpha)\Big]\Big[l\left(\log\frac{b}{a}\right) + m(\alpha-1)\Big]}.$$

Corollary 9. *Assume y is a nontrivial solution of the Hadamard fractional eigenvalue problem*

$$\begin{cases} ({}^H D_a^{\alpha} y)(t) + p(t)y(t) = 0, \quad a < t < b, \\ y(a) = 0, \ ny(b) + p({}^H D_a^{\alpha-1} y)(b) = 0, \end{cases}$$

where $y(t) \neq 0$ *for each* $t \in (a, b)$. *Then,*

$$|\lambda| > \frac{\bar{A}\Gamma(\alpha)}{\left(\log\frac{b}{a}\right)^{\alpha}\left[n\left(\log\frac{b}{a}\right)^{\alpha-1} + p\Gamma(\alpha)\right]}.$$

Theorem 5.2. *Assume y is a nontrivial solution of the Hadamard fractional eigenvalue problem*

$$\begin{cases} ({}^H D^{\alpha} y)(t) + \lambda y(t) = 0, \quad 1 < t < T, \\ ({}^H I^{2-\alpha} y)(0) + ({}^H I^{2-\alpha} y)(T) = 0, \\ ({}^H D^{\alpha-1} y)(0) + ({}^H D^{\alpha-1} y)(T) = 0, \end{cases} \tag{5.1}$$

where $y(t) \neq 0$ *for each* $t \in (1, T)$. *Then,*

$$|\lambda| > \frac{4\Gamma(\alpha)(\alpha-1)}{[\Gamma(\alpha-1) - \Gamma(\alpha-1, -\log T)]\log T(3-\alpha)}.$$

Proof. From (4.13), we obtain

$$\int_1^T (\log s)^{\alpha-2}|\lambda| ds > \frac{4\Gamma(\alpha)}{(3-\alpha)\log T},$$

or, equivalently,

$$|\lambda| > \frac{4\Gamma(\alpha)}{[\Gamma(\alpha-1) - \Gamma(\alpha-1, -\log T)]\log T(3-\alpha)}.$$

This proves the result (5.2). The proof is complete. □

Now we will discuss the disconjugacy and disfocality for Hadamard fractional boundary value problems (1.5), (3.29) and (3.31).

Definition 5.1. The Hadamard fractional boundary value problem (1.5) is disconjugate on $[a, b]$ if and only if each nontrivial solution has less than $[\alpha] + 1$ zeros on $[a, b]$.

Definition 5.2. The Hadamard fractional boundary value problem (3.29) is left disfocal on $[a, b]$ if and only if each nontrivial solution has less than $[\alpha]$ zeros on $[a, b]$.

Definition 5.3. The Hadamard fractional boundary value problem (3.31) is right disfocal on $[a, b]$ if and only if each nontrivial solution has less than $[\alpha]$ zeros on $[a, b]$.

Using these definitions, we introduce non-existence criteria for non-trivial solutions as follows:

Theorem 5.3. *The Hadamard fractional boundary value problem* (1.5) *is disconjugate if*

$$\int_a^b \left|q(s)\right| ds \leq \frac{\Gamma(\alpha)(4)^{\alpha-1}}{\left(\log \frac{b}{a}\right)^{\alpha-1}}. \tag{5.2}$$

Theorem 5.4. *Assume that the assumptions of Theorem 5.3 are satisfied. Then, the Hadamard fractional boundary value problem* (1.5) *has no non-trivial solution on* $[a, b]$.

Theorem 5.5. *The Hadamard fractional boundary value problem* (3.29) *is left disfocal if*

$$\int_a^b \left(\log \frac{s}{a}\right)^{\alpha-2} \left|q(s)\right| ds \leq \frac{\Gamma(\alpha)}{\log \frac{b}{a}}. \tag{5.3}$$

Theorem 5.6. *Assume that the assumptions of Theorem 5.5 are satisfied. Then, the Hadamard fractional boundary value problem* (3.29) *has no non-trivial solution on* $[a, b]$.

Theorem 5.7. *The Hadamard fractional boundary value problem* (3.31) *is right disfocal if*

$$\int_a^b \left|q(s)\right| ds \leq \frac{\Gamma(\alpha)}{\left(\log \frac{b}{a}\right)^{\alpha-1}}. \tag{5.4}$$

Theorem 5.8. *Assume that the assumptions of Theorem 5.7 are satisfied. Then, the Hadamard fractional boundary value problem* (3.31) *has no non-trivia l solution on* $[a, b]$.

6. Conclusions

In this article, we considered Hadamard fractional boundary value problems associated with two different types of boundary conditions-general and anti-periodic, and established Lyapunov-type inequalities for the same. In this process, we derived a few important properties of the corresponding Green's functions. In the later part of the article, we illustrated the applicability of established results through two applications. As the first application, we obtained lower bounds on the smallest eigenvalues for the corresponding Hadamard fractional eigenvalue problems. For the second application, we introduced the concepts of disconjugacy and disfocality and using which we obtained non-existence criteria for non-trivial solutions of Hadamard fractional boundary value problems.

Conflict of interest

All the authors declare no conflicts of interest.

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