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Research article

Periodic mild solutions of impulsive fractional evolution equations

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Abstract: This paper studies the periodic mild solutions of impulsive fractional evolution equations. Firstly, the existence and stability of periodic solutions of impulsive fractional differential equations with varying lower limits for general impulses and small shifted impulses are considered. Secondly, the existence of periodic solutions of impulsive fractional differential equations with fixed lower limits is proved. Lastly, an example is given to demonstrate the result.

Keywords: impulsive fractional differential equations; periodic mild solution; existence; stability
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1. Introduction

Fractional differential equations rise in many fields, such as biology, physics and engineering. There are many results about the existence of solutions and control problems (see [1–6]).

It is well known that the nonexistence of nonconstant periodic solutions of fractional differential equations was shown in [7, 8, 11] and the existence of asymptotically periodic solutions was derived in [8–11]. Thus it gives rise to study the periodic solutions of fractional differential equations with periodic impulses.

Recently, Fečkan and Wang [12] studied the existence of periodic solutions of fractional ordinary differential equations with impulses periodic condition and obtained many existence and asymptotic stability results for the Caputo's fractional derivative with fixed and varying lower limits. In this paper, we study the Caputo's fractional evolution equations with varying lower limits and we prove the existence of periodic mild solutions to this problem with the case of general periodic impulses as well as small equidistant and shifted impulses. We also study the Caputo's fractional evolution equations with fixed lower limits and small nonlinearities and derive the existence of its periodic mild solutions. The current results extend some results in [12].

2. Caputo derivatives with varying lower limits

Set $\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\varpi_q(\theta^{-\frac{1}{q}}) \geq 0$, $\varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q)$, $\theta \in (0,\infty)$. Note that $\xi_q(\theta)$ is a probability density function defined on $(0,\infty)$, namely $\xi_q(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^{\infty}\xi_q(\theta)d\theta = 1$.

Define $\mathscr{T} : X \to X$ and $\mathscr{S} : X \to X$ given by

$$\mathscr{T}(t) = \int_0^{\infty}\xi_q(\theta)S(t^q\theta)d\theta, \quad \mathscr{S}(t) = q\int_0^{\infty}\theta\xi_q(\theta)S(t^q\theta)d\theta.$$

Lemma 2.1. *([13, Lemmas 3.2, 3.3]) The operators $\mathscr{T}(t)$ and $\mathscr{S}(t), t \geq 0$ have following properties:*

(1) Suppose that $\sup_{t\geq 0}\|S(t)\| \leq M$. For any fixed $t \geq 0$, $\mathscr{T}(\cdot)$ and $\mathscr{S}(\cdot)$ are linear and bounded operators, i.e., for any $u \in X$,

$$\|\mathscr{T}(t)u\| \leq M\|u\| \text{ and } \|\mathscr{S}(t)u\| \leq \frac{M}{\Gamma(q)}\|u\|.$$

(2) $\{\mathscr{T}(t), t \geq 0\}$ and $\{\mathscr{S}(t), t \geq 0\}$ are strongly continuous.

(3) $\{\mathscr{T}(t), t > 0\}$ and $\{\mathscr{S}(t), t > 0\}$ are compact, if $\{S(t), t > 0\}$ is compact.

2.1. General impulses

Let $\mathbb{N}_0 = \{0, 1, \cdots, \infty\}$. We consider the following impulsive fractional equations

$$\begin{cases} {}^cD^q_{t_k,t}u(t) = Au(t) + f(t,u(t)),\ q \in (0,1),\ t \in (t_k, t_{k+1}),\ k \in \mathbb{N}_0, \\ u(t_k^+) = u(t_k^-) + \Delta_k(u(t_k^-)),\ k \in \mathbb{N}, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where ${}^cD^q_{t_k,t}$ denotes the Caputo's fractional time derivative of order q with the lower limit at t_k, $A : D(A) \subseteq X \to X$ is the generator of a C_0-semigroup $\{S(t), t \geq 0\}$ on a Banach space X, $f : \mathbb{R}\times X \to X$ satisfies some assumptions. We suppose the following conditions:

(I) f is continuous and T-periodic in t.

(II) There exist constants $a > 0$, $b_k > 0$ such that

$$\begin{cases} \|f(t,u) - f(t,v)\| \leq a\|u - v\|, \forall\, t \in \mathbb{R},\ u, v \in X, \\ \|u - v + \Delta_k(u) - \Delta_k(v)\| \leq b_k\|u - v\|, \forall\, k \in \mathbb{N},\ u, v \in X. \end{cases}$$

(III) There exists $N \in \mathbb{N}$ such that $T = t_{N+1}, t_{k+N+1} = t_k + T$ and $\Delta_{k+N+1} = \Delta_k$ for any $k \in \mathbb{N}$.

It is well known [3] that (2.1) has a unique solution on $\mathbb{R}_+$ if the conditions (I) and (II) hold. So we can consider the Poincaré mapping

$$P(u_0) = u(T^-) + \Delta_{N+1}(u(T^-)).$$

By [14, Lemma 2.2] we know that the fixed points of P determine T-periodic mild solutions of (2.1).

Theorem 2.2. *Assume that (I)-(III) hold. Let* $\Xi := \prod_{k=0}^{N} Mb_k E_q(Ma(t_{k+1}-t_k)^q)$*, where* E_q *is the Mittag-Leffler function (see [3, p.40]), then there holds*

$$\|P(u) - P(v)\| \leq \Xi\|u - v\|, \ \forall u, v \in X. \tag{2.2}$$

If $\Xi < 1$*, then* (2.1) *has a unique* T*-periodic mild solution, which is also asymptotically stable.*

Proof. By the mild solution of (2.1), we mean that $u \in C((t_k, t_{k+1}), X)$ satisfying

$$u(t) = \mathscr{T}(t - t_k)u(t_k^+) + \int_{t_k}^{t} \mathscr{S}(t - s)f(s, u(s))ds. \tag{2.3}$$

Let u and v be two solutions of (2.3) with $u(0) = u_0$ and $v(0) = v_0$, respectively. By (2.3) and (II), we can derive

$$\begin{aligned} &\|u(t) - v(t)\| \\ \leq\ &\|\mathscr{T}(t - t_k)(u(t_k^+) - v(t_k^+))\| + \int_{t_k}^{t} (t - s)^{q-1}\|\mathscr{S}(t - s)(f(s, u(s) - f(s, v(s))\|ds \\ \leq\ &M\|u(t_k^+) - v(t_k^+)\| + \frac{Ma}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1}\|f(s, u(s) - f(s, v(s))\|ds. \end{aligned} \tag{2.4}$$

Applying Gronwall inequality [15, Corollary 2] to (2.4), we derive

$$\|u(t) - v(t)\| \leq M\|u(t_k^+) - v(t_k^+)\|E_q(Ma(t - t_k)^q), \ t \in (t_k, t_{k+1}), \tag{2.5}$$

which implies

$$\|u(t_{k+1}^-) - v(t_{k+1}^-)\| \leq ME_q(Ma(t_{k+1} - t_k)^q)\|u(t_k^+) - v(t_k^+)\|, k = 0, 1, \cdots, N. \tag{2.6}$$

By (2.6) and (II), we derive

$$\begin{aligned} &\|P(u_0) - P(v_0)\| \\ =\ &\|u(t_{N+1}^-) - v(t_{N+1}^-) + \Delta_{N+1}(u(t_{N+1}^-)) - \Delta_{N+1}(v(t_{N+1}^-))\| \\ \leq\ &b_{N+1}\|u(t_{N+1}^-) - v(t_{N+1}^-)\| \\ \leq\ &\Big(\prod_{k=0}^{N} Mb_k E_q(Ma(t_{k+1} - t_k)^q)\Big)\|u_0 - v_0\| \\ =\ &\Xi\|u_0 - v_0\|, \end{aligned} \tag{2.7}$$

which implies that (2.2) is satisfied. Thus $P : X \to X$ is a contraction if $\Xi < 1$. Using Banach fixed point theorem, we obtain that P has a unique fixed point u_0 if $\Xi < 1$. In addition, since

$$\|P^n(u_0) - P^n(v_0)\| \leq \Xi^n\|u_0 - v_0\|, \ \forall v_0 \in X,$$

we get that the corresponding periodic mild solution is asymptotically stable. □

2.2. Small equidistant and shifted impulses

We study

$$\begin{cases} {}^cD^q_{kh}u(t) = Au(t) + f(u(t)),\ q \in (0,1),\ t \in (kh,(k+1)h),\ k \in \mathbb{N}_0, \\ u(kh^+) = u(kh^-) + \bar{\Delta}h^q,\ k \in \mathbb{N}, \\ u(0) = u_0, \end{cases} \tag{2.8}$$

where $h > 0$, $\bar{\Delta} \in X$, and $f : X \to X$ is Lipschitz. We know [3] that under above assumptions, (2.8) has a unique mild solution $u(u_0, t)$ on $\mathbb{R}_+$, which is continuous in $u_0 \in X$, $t \in \mathbb{R}_+ \setminus \{kh|k \in \mathbb{N}\}$ and left continuous in t ant impulsive points $\{kh|k \in \mathbb{N}\}$. We can consider the Poincaré mapping

$$P_h(u_0) = u(u_0, h^+).$$

Theorem 2.3. *Let $w(t)$ be a solution of following equations*

$$\begin{cases} w'(t) = \bar{\Delta} + \dfrac{1}{\Gamma(q+1)} f(w(t)),\ t \in [0,T], \\ w(0) = u_0. \end{cases} \tag{2.9}$$

Then there exists a mild solution $u(u_0, t)$ of (2.8) *on* $[0,T]$*, satisfying*

$$u(u_0, t) = w(tq^{q-1}) + O(h^q).$$

If $w(t)$ is a stable periodic solution, then there exists a stable invariant curve of Poincaré mapping of (2.8) *in a neighborhood of $w(t)$. Note that h is sufficiently small.*

Proof. For any $t \in (kh,(k+1)h), k \in \mathbb{N}_0$, the mild solution of (2.8) is equivalent to

$$\begin{aligned} u(u_0, t) &= \mathscr{T}(t - kh)u(kh^+) + \int_{kh}^{t} (t-s)^{q-1}\mathscr{S}(t-s) f(u(u_0, s))ds \\ &= \mathscr{T}(t - kh)u(kh^+) + \int_0^{t-kh} (t - kh - s)^{q-1}\mathscr{S}(t - kh - s) f(u(u(kh^+), s))ds. \end{aligned} \tag{2.10}$$

So

$$u((k+1)h^+) = \mathscr{T}(h)u(kh^+) + \bar{\Delta}h^q + \int_0^h (h-s)^{q-1}\mathscr{S}(h-s) f(u(u(kh^+), s))ds = P_h(u(kh^+)), \tag{2.11}$$

and

$$P_h(u_0) = u(u_0, h^+) = \mathscr{T}(h)u_0 + \bar{\Delta}h^q + \int_0^h (h-s)^{q-1}\mathscr{S}(h-s) f(u(u_0, s))ds. \tag{2.12}$$

Inserting

$$u(u_0, t) = \mathscr{T}(t)u_0 + h^q v(u_0, t),\ t \in [0,h],$$

into(2.10), we obtain

$$v(u_0, t) = \frac{1}{h^q}\int_0^t (t-s)^{q-1}\mathscr{S}(t-s) f(\mathscr{T}(t)u_0 + h^q v(u_0, t))ds$$

$$
\begin{aligned}
&= \frac{1}{h^q}\int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f(\mathscr{T}(t)u_0)ds \\
&\quad + \frac{1}{h^q}\int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\big(f(\mathscr{T}(t)u_0+h^q v(u_0,t))-f(\mathscr{T}(t)u_0)\big)ds \\
&= \frac{1}{h^q}\int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f(\mathscr{T}(t)u_0)ds + O(h^q),
\end{aligned}
$$

since

$$
\begin{aligned}
&\left\|\int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\big(f(\mathscr{T}(t)u_0+h^q v(u_0,t))-f(\mathscr{T}(t)u_0)\big)ds\right\| \\
\leq\ & \int_0^t (t-s)^{q-1}\|\mathscr{S}(t-s)\|\|f(\mathscr{T}(t)u_0+h^q v(u_0,t))-f(\mathscr{T}(t)u_0)\|ds \\
\leq\ & \frac{ML_{loc}h^q t^q}{\Gamma(q+1)}\max_{t\in[0,h]}\{\|v(u_0,t)\|\} \\
\leq\ & h^{2q}\frac{ML_{loc}}{\Gamma(q+1)}\max_{t\in[0,h]}\{\|v(u_0,t)\|\},
\end{aligned}
$$

where L_{loc} is a local Lipschitz constant of f. Thus we get

$$
u(u_0,t) = \mathscr{T}(t)u_0 + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f(\mathscr{T}(t)u_0)ds + O(h^{2q}),\ t\in[0,h], \tag{2.13}
$$

and (2.12) gives

$$
P_h(u_0) = \mathscr{T}(h)u_0 + \bar{\Delta}h^q + \int_0^h (h-s)^{q-1}\mathscr{S}(h-s)f(\mathscr{T}(h)u_0)ds + O(h^{2q}).
$$

So (2.11) becomes

$$
\begin{aligned}
& u((k+1)h^+) \\
=\ & \mathscr{T}(h)u(kh^+) + \bar{\Delta}h^q + \int_{kh}^{(k+1)h} ((k+1)h-s)^{q-1}\mathscr{S}((k+1)h-s)f(\mathscr{T}(h)u(kh^+))ds + O(h^{2q}).
\end{aligned} \tag{2.14}
$$

Since $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are strongly continuous,

$$
\lim_{t\to 0}\mathscr{T}(t) = I \text{ and } \lim_{t\to 0}\mathscr{S}(t) = \frac{1}{\Gamma(q)}I. \tag{2.15}
$$

Thus (2.14) leads to its approximation

$$
w((k+1)h^+) = w(kh^+) + \bar{\Delta}h^q + \frac{h^q}{\Gamma(q+1)}f(w(kh^+)),
$$

which is the Euler numerical approximation of

$$
w'(t) = \bar{\Delta} + \frac{1}{\Gamma(q+1)}f(w(t)).
$$

Note that (2.10) implies

$$\|u(u_0,t) - \mathscr{T}(t-kh)u(kh^+)\| = O(h^q),\ \forall t \in [kh,(k+1)h]. \tag{2.16}$$

Applying (2.15), (2.16) and the already known results about Euler approximation method in [16], we obtain the result of Theorem 2.3.

□

Corollary 2.4. *We can extend* (2.8) *for periodic impulses of following form*

$$\begin{cases} {}^cD^q_{kh}u(t) = Au(t) + f(u(t)),\ t \in (kh,(k+1)h),\ k \in \mathbb{N}_0, \\ u(kh^+) = u(kh^-) + \bar{\Delta}_k h^q,\ k \in \mathbb{N}, \\ u(0) = u_0, \end{cases} \tag{2.17}$$

where $\bar{\Delta}_k \in X$ *satisfy* $\bar{\Delta}_{k+N+1} = \bar{\Delta}_k$ *for any* $k \in \mathbb{N}$. *Then Theorem 2.3 can directly extend to* (2.17) *with*

$$\begin{cases} w'(t) = \dfrac{\sum_{k=1}^{N+1}\bar{\Delta}_k}{N+1} + \dfrac{1}{\Gamma(q+1)}f(w(t)),\ t \in [0,T],\ k \in \mathbb{N}, \\ w(0) = u_0 \end{cases} \tag{2.18}$$

instead of (2.9).

Proof. We can consider the Poincaré mapping

$$P_h(u_0) = u(u_0,(N+1)h^+),$$

with a form of

$$P_h = P_{N+1,h} \circ \cdots \circ P_{1,h}$$

where

$$P_{k,h}(u_0) = \bar{\Delta}_k h^q + u(u_0,h).$$

By (2.13), we can derive

$$P_{k,h}(u_0) = \bar{\Delta}_k h^q + u(u_0,h) = \mathscr{T}(h)u_0 + \bar{\Delta}_k h^q + \int_0^h (h-s)^{q-1}\mathscr{S}(h-s)f(\mathscr{T}(h)u_0)ds + O(h^{2q}).$$

Then we get

$$P_h(u_0) = \mathscr{T}(h)u_0 + \sum_{k=1}^{N+1}\bar{\Delta}_k h^q + (N+1)\int_0^h (h-s)^{q-1}\mathscr{S}(h-s)f(\mathscr{T}(h)u_0)ds + O(h^{2q}).$$

By (2.15), we obtain that $P_h(u_0)$ leads to its approximation

$$u_0 + \sum_{k=1}^{N+1}\bar{\Delta}_k h^q + \frac{(N+1)h^q}{\Gamma(q+1)}f(u_0). \tag{2.19}$$

Moreover, equations

$$w'(t) = \frac{\sum_{k=1}^{N+1}\bar{\Delta}_k}{N+1} + \frac{1}{\Gamma(q+1)}f(w(t))$$

has the Euler numerical approximation

$$u_0 + h^q\Big(\frac{\sum_{k=1}^{N+1}\bar{\Delta}_k}{N+1} + \frac{1}{\Gamma(q+1)}f(u_0)\Big)$$

with the step size h^q, and its approximation of $N+1$ iteration is (2.19), the approximation of P_h. Thus Theorem 2.3 can directly extend to (2.17) with (2.18). □

3. Caputo derivatives with fixed lower limits and weak nonlinearities

Now we consider following equations with small nonlinearities of the form

$$\begin{cases} {}^cD_0^q u(t) = Au(t) + \epsilon f(t, u(t)),\ q \in (0,1),\ t \in (t_k, t_{k+1}),\ k \in \mathbb{N}_0, \\ u(t_k^+) = u(t_k^-) + \epsilon\Delta_k(u(t_k^-)),\ k \in \mathbb{N}, \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where ϵ is a small parameter, ${}^cD_0^q$ is the generalized Caputo fractional derivative with lower limit at 0. Then (3.1) has a unique mild solution $u(\epsilon, t)$. Give the Poincaré mapping

$$P(\epsilon, u_0) = u(\epsilon, T^-) + \epsilon\Delta_{N+1}(u(\epsilon, T^-)).$$

Assume that

(H1) f and Δ_k are C^2-smooth.

Then $P(\epsilon, u_0)$ is also C^2-smooth. In addition, we have

$$u(\epsilon, t) = \mathscr{T}(t)u_0 + \epsilon\omega(t) + O(\epsilon^2),$$

where $\omega(t)$ satisfies

$$\begin{cases} {}^cD_0^q \omega(t) = A\omega(t) + f(t, \mathscr{T}(t)u_0),\ t \in (t_k, t_{k+1}),\ k = 0, 1, \cdots, N, \\ \omega(t_k^+) = \omega(t_k^-) + \Delta_k(\mathscr{T}(t_k)u_0),\ k = 1, 2, \cdots, N+1, \\ \omega(0) = 0, \end{cases}$$

and

$$\omega(T^-) = \sum_{k=1}^{N} \mathscr{T}(T - t_k)\Delta_k(\mathscr{T}(t_k)u_0) + \int_0^T (T-s)^{q-1}\mathscr{S}(T-s)f(s, \mathscr{T}(s)u_0)ds.$$

Thus we derive

$$\begin{cases} P(\epsilon, u_0) = u_0 + M(\epsilon, u_0) + O(\epsilon^2) \\ M(\epsilon, u_0) = (\mathscr{T}(T) - I)u_0 + \epsilon\omega(T^-) + \epsilon\Delta_{N+1}(\mathscr{T}(T)u_0). \end{cases} \tag{3.2}$$

Theorem 3.1. *Suppose that (I), (III) and (H1) hold.*

1). If $(\mathscr{T}(T) - I)$ has a continuous inverse, i.e. $(\mathscr{T}(T) - I)^{-1}$ exists and continuous, then (3.1) *has a unique T-periodic mild solution located near* 0 *for any $\epsilon \neq 0$ small.*

2). If $(\mathscr{T}(T) - I)$ is not invertible, we suppose that $\ker(\mathscr{T}(T) - I) = [u_1, \cdots, u_k]$ *and $X = im(\mathscr{T}(T) - I) \oplus X_1$ for a closed subspace X_1 with* $\dim X_1 = k$. *If there is $v_0 \in [u_1, \cdots, u_k]$ such that $B(0, v_0) = 0$ (see* (3.7)*) and the $k \times k$-matrix $DB(0, v_0)$ is invertible, then* (3.1) *has a unique T-periodic mild solution located near $\mathscr{T}(t)v_0$ for any $\epsilon \neq 0$ small.*

3). If $r_\sigma(D_{u_0}M(\epsilon, u_0)) < 0$, then the T-periodic mild solution is asymptotically stable. If $r_\sigma(D_{u_0}M(\epsilon, u_0)) \cap (0, +\infty) \neq \emptyset$, then the T-periodic mild solution is unstable.

Proof. The fixed point u_0 of $P(\epsilon, x_0)$ determines the T-periodic mild solution of (3.1), which is equivalent to

$$M(\epsilon, u_0) + O(\epsilon^2) = 0. \tag{3.3}$$

Note that $M(0, u_0) = (\mathscr{T}(T) - I)u_0$. If $(\mathscr{T}(T) - I)$ has a continuous inverse, then (3.3) can be solved by the implicit function theorem to get its solution $u_0(\epsilon)$ with $u_0(0) = 0$.

If $(\mathscr{T}(T) - I)$ is not invertible, then we take a decomposition $u_0 = v + w$, $v \in [u_1, \cdots, u_k]$, take bounded projections $Q_1 : X \to im(\mathscr{T}(T) - I)$, $Q_2 : X \to X_1$, $I = Q_1 + Q_2$ and decompose (3.3) to

$$Q_1 M(\epsilon, v + w) + Q_1 O(\epsilon^2) = 0, \tag{3.4}$$

and

$$Q_2 M(\epsilon, v + w) + Q_2 O(\epsilon^2) = 0. \tag{3.5}$$

Now $Q_1 M(0, v+w) = (\mathscr{T}(T)-I)w$, so we can solve by implicit function theorem from (3.4), $w = w(\epsilon, v)$ with $w(0, v) = 0$. Inserting this solution into (3.5), we get

$$B(\epsilon, v) = \frac{1}{\epsilon}(Q_2 M(\epsilon, v + w) + Q_2 O(\epsilon^2)) = Q_2 \omega(T^-) + Q_2 \Delta_{N+1}(\mathscr{T}(t)v + w(\epsilon, v)) + O(\epsilon). \tag{3.6}$$

So

$$B(0, v) = \sum_{k=1}^{N} Q_2 \mathscr{T}(T - t_k)\Delta_k(\mathscr{T}(t_k)v) + Q_2 \int_0^T (T - s)^{q-1} \mathscr{S}(T - s) f(s, \mathscr{T}(s)v) ds. \tag{3.7}$$

Consequently we get, if there is $v_0 \in [u_1, \cdots, u_k]$ such that $B(0, v_0) = 0$ and the $k \times k$-matrix $DB(0, v_0)$ is invertible, then (3.1) has a unique T-periodic mild solution located near $\mathscr{T}(t)v_0$ for any $\epsilon \neq 0$ small.

In addition, $D_{u_0} P(\epsilon, u_0(\epsilon)) = I + D_{u_0} M(\epsilon, u_0) + O(\epsilon^2)$. Thus we can directly derive the stability and instability results by the arguments in [17].

□

4. An example

In this section, we give an example to demonstrate Theorem 2.2.

Example 4.1. *Consider the following impulsive fractional partial differential equation:*

$$\begin{cases} {}^c D_{t_k,t}^{\frac{1}{2}} u(t, y) = \dfrac{\partial^2}{\partial y^2} u(t, y) + \sin u(t, y) + \cos 2\pi t, & t \in (t_k, t_{k+1}),\ k \in \mathbb{N}_0,\ \ y \in [0, \pi], \\ \Delta_k(u(t_k^-, y)) = u(t_k^+, y) - u(t_k^-, y) = \xi u(t_k^-, y), & k \in \mathbb{N},\ \ y \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, & t \in (t_k, t_{k+1}),\ \ k \in \mathbb{N}_0, \\ u(0, y) = u_0(y), & y \in [0, \pi], \end{cases} \tag{4.1}$$

for $\xi \in \mathbb{R}$, $t_k = \frac{k}{3}$. *Let* $X = L^2[0, \pi]$. *Define the operator* $A : D(A) \subseteq X \to X$ *by* $Au = \frac{d^2 u}{dy^2}$ *with the domain*

$$D(A) = \{u \in X \mid \frac{du}{dy}, \frac{d^2 u}{dy^2} \in X,\ u(0) = u(\pi) = 0\}.$$

Then A is the infinitesimal generator of a C_0-semigroup $\{S(t), t \geq 0\}$ on X and $\|S(t)\| \leq M = 1$ for any $t \geq 0$. Denote $u(\cdot, y) = u(\cdot)(y)$ and define $f : [0, \infty) \times X \to X$ by

$$f(t, u)(y) = \sin u(y) + \cos 2\pi t.$$

Set $T = t_3 = 1$, $t_{k+3} = t_k + 1$, $\Delta_{k+3} = \Delta_k$, $a = 1$, $b_k = |1 + \xi|$. Obviously, conditions (I)-(III) hold. Note that

$$\Xi = \prod_{k=0}^{2} |1 + \xi| E_{\frac{1}{2}}(\frac{1}{\sqrt{3}}) = |1 + \xi|^3 (E_{\frac{1}{2}}(\frac{1}{\sqrt{3}}))^3.$$

Letting $\Xi < 1$, we get $-E_{\frac{1}{2}}(\frac{1}{\sqrt{3}}) - 1 < \xi < E_{\frac{1}{2}}(\frac{1}{\sqrt{3}}) - 1$. Now all assumptions of Theorem 2.2 hold. Hence, if $-E_{\frac{1}{2}}(\frac{1}{\sqrt{3}}) - 1 < \xi < E_{\frac{1}{2}}(\frac{1}{\sqrt{3}}) - 1$, (4.1) *has a unique* 1*-periodic mild solution, which is also asymptotically stable.*

5. Conclusion

This paper deals with the existence and stability of periodic solutions of impulsive fractional evolution equations with the case of varying lower limits and fixed lower limits. Although, Fečkan and Wang [12] prove the existence of periodic solutions of impulsive fractional ordinary differential equations in finite dimensional Euclidean space, we extend some results to impulsive fractional evolution equation on Banach space by involving operator semigroup theory. Our results can be applied to some impulsive fractional partial differential equations and the proposed approach can be extended to study the similar problem for periodic impulsive fractional evolution inclusions.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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