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Research article

New reproducing kernel functions in the reproducing kernel Sobolev spaces

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Abstract: In this paper we construct some new reproducing kernel functions in the reproducing kernel Sobolev space. These functions are new in the literature. We can solve many problems by these functions in the reproducing kernel Sobolev spaces.

Keywords: reproducing kernel functions; reproducing kernel Sobolev spaces
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1. Introduction

The reproducing-kernel Hilbert space construction associates a positive definite kernel with a Hilbert space of functions often referred to as the native space of the kernel. This construction can be used to deal with the problem of reconstructing an unknown function which lies in the reproducing kernel Hilbert space from a given multivariate data sample in an “optimal” way [1].

One of the most investigated spaces in mathematical analysis are Sobolev spaces. The knowledge of the reproducing kernels is very useful in the analysis of many computational problems. It is usually enough to analyse reproducing kernels instead of the corresponding Hilbert spaces. It is therefore somehow surprising that it is difficult to find in the literature explicit formulas for the reproducing kernels of the Sobolev spaces [2].

Reproducing kernel space is a special Hilbert space. There are many works on the solution of the nonlinear problems with reproducing kernel method [3]. The concept of reproducing kernel can be traced back to the paper of Zaremba [4] in 1908. It was given for investigating the boundary value problems of the harmonic functions. In the early development stage of the reproducing kernel theory, most of the works were implemented by Bergman [5]. Bergman has investigated the corresponding kernels of the harmonic functions with one or several variables, and the corresponding kernel of the analytic function in squared metric, and implemented them in the research of the boundary value problem of the elliptic partial differential equation.

Zorzi et al. [6] have investigated the harmonic analysis of kernel functions and the sparse plus low rank network identification [7]. The empirical Bayesian learning in AR graphical models has been studied in [8].

The reproducing kernel Sobolev space method (RKSSM), which computes the numerical solution, is of great attention to many branches of applied sciences. Many studies have been dedicated to the application of the RKSSM [1,9]. For more details see [9–15].

We organize the paper as: We present the main definitions in Section 2. We construct some important reproducing kernel Sobolev spaces in Section 3. We obtain very reproducing kernel functions in this section. We give the conclusion in Section 4.

2. Main definitions

Definition 1. We define the reproducing kernel Sobolev space $A_2^m[a,b]$ as [3]:

$$A_2^m[a,b] = \left\{ f \mid f^{(m-1)} \text{is absolutely continuous function}, f^{(m)} \in L^2[a,b], x \in [a,b] \right\},$$

We define the inner product and norm for this space as:

$$\langle f, g \rangle_{A_2^m} = \int_a^b \left(\sum_{i=0}^{m} f^{(i)}(x) g^{(i)}(x) \right) dx$$

and

$$\|f\|_{A_2^m} = \sqrt{\langle f, f \rangle_{W_2^m}}.$$

Definition 2. We have the reproducing property as [3]:

$$\langle f, R_y \rangle_{A_2^m} = f(y).$$

3. New reproducing kernel functions

We obtain very useful reproducing kernel functions in the reproducing kernel Sobolev spaces in this section. These kernel functions are new in the literature.

3.1. m=1

We have the inner product as:

$$\langle u, R_y \rangle_{S_1^1[0,1]} = \int_0^1 \left(u(x) R_y(x) + u'(x) R_y'(x) \right) dx$$

We use integration by parts and obtain:

$$\langle u, R_y \rangle_{S_1^1[0,1]} = \int_0^1 u(x) R_y(x) dx + u(1) R_y'(1) - u(0) R_y'(0) - \int_0^1 u(x) R_y''(x) dx$$

$$= \quad u(1)R'_y(1) - u(0)R'_y(0) - \int_0^1 u(x)(R''_y(x) - R_y(x))dx.$$

If we choose

$$1)R'_y(0) = 0,$$
$$2)R'_y(1) = 0,$$

then, we obtain

$$R''_y(x) - R_y(x) = -\delta(x - y).$$

When $x \neq y$, we get

$$R''_y(x) - R_y(x) = 0.$$

$$\lambda^2 - 1 = 0 \Longrightarrow \quad \lambda = \mp 1.$$

Thus, we reach

$$R_y(x) = \begin{cases} c_1e^x + c_2e^{-x}, x \leq y, \\ d_1e^x + d_2e^{-x}, x > y. \end{cases}$$

By the property of the Dirac-Delta function, we get

$$3)R_{y^+}(y) = R_{y^-}(y),$$
$$4)R'_{y^+}(y) - R'_{y^-}(y) = -1.$$

We have

$$R'_y(x) = \begin{cases} c_1e^x - c_2e^{-x}, \quad x < y \\ d_1e^x - d_2e^{-x}, \quad x > y. \end{cases}$$

$$R'_y(0) = c_1 - c_2 = 0 \Longrightarrow c_1 = c_2$$

$$R'_y(1) = d_1e - \frac{d_2}{e} = 0 \Longrightarrow d_2 = e^2d_1$$

$$R_{y^+}(y) = R_{y^-}(y)$$
$$d_1e^y + d_2e^{-y} = c_1e^y + c_2e^{-y}$$

$$R'_{y^+}(y) - R'_{y^-}(y) = -1$$

$$d_1e^y - d_2e^{-y} - c_1e^y + c_2e^{-y} = -1$$

when we solve these equations, we get:

$$c_1 = \frac{1}{2}\frac{e^{1-y} + e^{y-1}}{(e - e^{-1})},$$

$$c_2 = \frac{1}{2}\frac{e^{1-y} + e^{y-1}}{(e - e^{-1})},$$

$$d_1 = \frac{1}{2}\frac{e^{y-1} + e^{-y-1}}{(e - e^{-1})},$$

$$d_2 = \frac{1}{2}\frac{e^{1+y} + e^{1-y}}{(e - e^{-1})}.$$

Thus, we obtain the reproducing kernel function as:

$$R_y(x) = \begin{cases} \frac{1}{2}\frac{(e^{1-y}+e^{-1+y})(e^{1+x}+e^{1-x})}{e^2-1}, & x \leq y \\ \frac{1}{2}\frac{(e^{1+y}+e^{1-y})(e^{-1+x}+e^{1-x})}{e^2-1}, & x > y. \end{cases}$$

For simplify, we define

$$a(x) = \frac{1}{2}\frac{(e^{1-y} + e^{-1+y})(e^{1+x} + e^{1-x})}{e^2 - 1},$$

$$b(x) = \frac{1}{2}\frac{(e^{1+y} + e^{1-y})(e^{-1+x} + e^{1-x})}{e^2 - 1}.$$

We need to show

$$\langle u, R_y \rangle_{S_2^1} = u(y).$$

Therefore, we get

$$\langle u, R_y \rangle_{S_2^1} = \int_0^y u(x)R_y(x)dx + \int_y^1 u(x)R_y(x)dx + \int_0^y u'(x)R_y'(x)dx + \int_y^1 u'(x)R_y'(x)dx.$$

$$\langle u, R_y \rangle_{S_2^1} = \int_0^y u(x)a(x)dx + \int_y^1 u(x)b(x)dx + \int_0^y u'(x)a'(x)dx + \int_y^1 u'(x)b'(x)dx$$

$$
\begin{aligned}
&= \int_0^y u(x)\frac{1}{2}\frac{(e^{1-y}+e^{-1+y})}{e^2-1}[e^{1+x}+e^{1-x}]dx \\
&+ \int_y^1 u(x)\frac{1}{2}\frac{(e^{1+y}+e^{1-y})}{e^2-1}[e^{-1+x}+e^{1-x}]dx \\
&+ \int_0^y \frac{u'(x)}{2}\frac{(e^{1-y}+e^{-1+y})}{e^2-1}[e^{1+x}-e^{1-x}]dx \\
&+ \int_y^1 \frac{u'(x)}{2}\frac{(e^{1+y}+e^{1-y})}{e^2-1}[e^{-1+x}-e^{1-x}]dx.
\end{aligned}
$$

$$
\begin{aligned}
\langle u, R_y\rangle_{S_2^1} &= \frac{e^{1+y}+e^{1-y}}{2(e^2-1)}[\int_y^1 (u(x)(e^{x-1}+e^{1-x})+u'(x)(e^{1+x}-e^{1-x}))dx] \\
&+\frac{1}{2}\frac{e^{1-y}+e^{y-1}}{(e^2-1)}[\int_0^y (u(x)(e^{x+1}+e^{1-x})+u'(x)(e^{1+x}-e^{1-x}))dx].
\end{aligned}
$$

We use integration by parts and obtain

$$
\begin{aligned}
\langle u, R_y\rangle_{S_2^1} = \frac{1}{2}\frac{e^{1+y}+e^{1-y}}{2(e^2-1)}\Bigg[\int_y^1 u(x)(e^{x-1}+e^{1-x})dx \\
+(e^0-e^0)u(1)-(e^{y-1}-e^{1-y})u(y)-\int_y^1 u(x)(e^{x-1}+e^{1-x})dx\Bigg] \\
+\frac{1}{2}\frac{(e^{1-y}+e^{-1+y})}{(e^2-1)}\Bigg[\int_0^y u(x)(e^{x+1}+e^{1-x})dx \\
+u(y)(e^{1+y}-e^{1-y})-u(0)(e^1-e^1)-\int_0^y u(x)(e^{x+1}+e^{1-x})dx\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\langle u, R_y\rangle_{S_2^1} &= \frac{1}{2}\frac{e^{1+y}+e^{1-y}}{(e^2-1)}u(y)(e^{1-y}-e^{y-1})+\frac{1}{2}\frac{e^{1-y}+e^{y-1}}{(e^2-1)}u(y)(e^{1+y}-e^{1-y}) \\
&= \frac{1}{2}\frac{u(y)}{(e^2-1)}[(e^{1-y}-e^{y-1})(e^{1+y}+e^{1-y})+(e^{1+y}-e^{1-y})(e^{1-y}+e^{y-1})]
\end{aligned}
$$

$$\begin{aligned}\langle u,R_y\rangle_{S_2^1} &= \frac{1}{2(e^2-1)}u(y)[e^2+e^{2-2y}-e^{2y}-1+e^2+e^{2y}-e^{2-2y}-1]\\ &= \frac{u(y)}{2(e^2-1)}(2e^2-2)\\ &= u(y).\end{aligned}$$

This completes the proof.

3.2. m=2

We have

$$\begin{aligned}\langle u,R_y\rangle_{S_2^2} &= \int_0^1 [u(x)R_y(x)+u'(x)R_y'(x)+u''(x)R_y''(x)]dx\\ &= \int_0^1 u(x)R_y(x)dx+\int_0^1 u'(x)R_y'(x)dx+\int_0^1 u''(x)R_y''(x)dx\end{aligned}$$

After integration by parts, we get

$$\begin{aligned}\langle u,R_y\rangle_{S_2^2} = \int_0^1 u(x)R_y(x)dx+u(1)R_y'(1)-u(0)R_y'(0)\\ -\int_0^1 u(x)R_y''(x)dx+u'(1)R_y''(1)-u'(0)R_y''(0)\\ -u(1)R_y'''(1)+u(0)R_y'''(0)+\int_0^1 u(x)R_y^{(4)}(x)dx.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\langle u,R_y\rangle_{S_2^2} &= u(1)R_y'(1)-u(0)R_y'(0)+u'(1)R_y''(1)-u'(0)R_y''(0)\\ &\quad -u(1)R_y'''(1)+u(0)R_y'''(0)+\int_0^1 u(x)[R_y(x)-R_y''(x)+R_y^{(4)}(x)]dx.\end{aligned}$$

We have

$$\begin{aligned}&1) R_y'(1)-R_y'''(1)=0,\\ &2) -R_y'(0)+R_y'''(0)=0,\end{aligned}$$

$$3)R''_y(1) = 0,$$
$$4)R''_y(0) = 0.$$

Therefore, we get

$$\langle u, R_y\rangle_{S_2^2} = \int_0^1 u(x)[R_y(x) - R''_y(x) + R_y^{(4)}(x)]dx.$$

By reproducing property, we have

$$R_y(x) - R''_y(x) + R_y^{(4)}(x) = \delta(x - y).$$

When $x \neq y$, we get

$$R_y(x) - R''_y(x) + R_y^{(4)}(x) = 0.$$

Then, we have

$$1 - \lambda^2 + \lambda^4 = 0.$$

$$\lambda_1 = \frac{\sqrt{3}}{2} - \frac{1}{2}I,$$
$$\lambda_2 = \frac{-\sqrt{3}}{2} + \frac{1}{2}I,$$
$$\lambda_3 = \frac{\sqrt{3}}{2} + \frac{1}{2}I,$$
$$\lambda_4 = \frac{-\sqrt{3}}{2} - \frac{1}{2}I.$$

$$R_y(x) = \begin{cases} c_1 e^{\frac{\sqrt{3}x}{2}} \cos(\frac{x}{2}) + c_2 e^{\frac{\sqrt{3}x}{2}} \sin(\frac{x}{2}) \\ +c_3 e^{\frac{-\sqrt{3}x}{2}} \cos(\frac{x}{2}) + c_4 e^{\frac{-\sqrt{3}x}{2}} \sin(\frac{x}{2}), & x \leq y \\ \\ d_1 e^{\frac{\sqrt{3}x}{2}} \cos(\frac{x}{2}) + d_2 e^{\frac{\sqrt{3}x}{2}} \sin(\frac{x}{2}) \\ +d_3 e^{\frac{-\sqrt{3}x}{2}} \cos(\frac{x}{2}) + d_4 e^{\frac{-\sqrt{3}x}{2}} \sin(\frac{x}{2}), & x > y \end{cases}$$

By properties of the Dirac-Delta function, we have

$$5)R_{y^+}(y) = R_{y^-}(y),$$
$$6)R'_{y^+}(y) = R'_{y^-}(y),$$
$$7)R''_{y^+}(y) = R''_{y^-}(y),$$
$$8)R'''_{y^+}(y) - R'''_{y^-}(y) = 1.$$

Solving these eight equations give,

$$c_1 = -\frac{1}{6}\frac{\left(-4\sin(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\sqrt{3} - 8\cos(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}} - 4\cos(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\right)\sqrt{3}}{\left(4\sin(\tfrac{1}{2}y)^2e^{\frac{1}{2}\sqrt{3}} - 4\sin(\tfrac{1}{2}y)^2e^{-\frac{1}{2}\sqrt{3}} + 4\cos(\tfrac{1}{2}y)^2e^{\frac{1}{2}\sqrt{3}} - 4\cos(\tfrac{1}{2}y)^2e^{-\frac{1}{2}\sqrt{3}}\right)}$$

$$c_2 = \frac{1}{6}\frac{\left(-4\cos(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\sqrt{3} + 8\sin(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}} + 4\sin(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\right)\sqrt{3}}{\left(\sin(\tfrac{1}{2}y)^2 + \cos(\tfrac{1}{2}y)^2\right)\left(4e^{\frac{1}{2}\sqrt{3}} - 4e^{-\frac{1}{2}\sqrt{3}}\right)}$$

$$c_3 = -\frac{1}{6}\frac{\left(4\sin(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}}\sqrt{3} - 4\cos(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}} - 8\cos(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\right)\sqrt{3}}{\left(\sin(\tfrac{1}{2}y)^2 + \cos(\tfrac{1}{2}y)^2\right)\left(4e^{\frac{1}{2}\sqrt{3}} - 4e^{-\frac{1}{2}\sqrt{3}}\right)}$$

$$c_4 = \frac{1}{6}\frac{\left(4\cos(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}}\sqrt{3} + 4\sin(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}} + 8\sin(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\right)\sqrt{3}}{\left(\sin(\tfrac{1}{2}y)^2 + \cos(\tfrac{1}{2}y)^2\right)\left(4e^{\frac{1}{2}\sqrt{3}} - 4e^{-\frac{1}{2}\sqrt{3}}\right)}$$

$$d_1 = -\frac{1}{6}\frac{\left(-4\sin(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}}\sqrt{3} - 8\cos(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}} - 4\cos(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}}\right)\sqrt{3}}{\left(\sin(\tfrac{1}{2}y)^2 + \cos(\tfrac{1}{2}y)^2\right)\left(4e^{\frac{1}{2}\sqrt{3}} - 4e^{-\frac{1}{2}\sqrt{3}}\right)}$$

$$d_2 = \frac{1}{6}\frac{\left(-4\cos(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}}\sqrt{3} + 8\sin(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}} + 4\sin(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{-\frac{1}{2}\sqrt{3}}\right)\sqrt{3}}{\left(\sin(\tfrac{1}{2}y)^2 + \cos(\tfrac{1}{2}y)^2\right)\left(4e^{\frac{1}{2}\sqrt{3}} - 4e^{-\frac{1}{2}\sqrt{3}}\right)}$$

$$d_3 = -\frac{1}{6}\frac{\left(4\sin(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\sqrt{3} - 4\cos(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}} - 8\cos(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\right)\sqrt{3}}{\left(\sin(\tfrac{1}{2}y)^2 + \cos(\tfrac{1}{2}y)^2\right)\left(4e^{\frac{1}{2}\sqrt{3}} - 4e^{-\frac{1}{2}\sqrt{3}}\right)}$$

$$d_4 = \frac{1}{6}\frac{\left(4\cos(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\sqrt{3} + 4\sin(\tfrac{1}{2}y)e^{\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}} + 8\sin(\tfrac{1}{2}y)e^{-\frac{1}{2}\sqrt{3}y}e^{\frac{1}{2}\sqrt{3}}\right)\sqrt{3}}{\left(\sin(\tfrac{1}{2}y)^2 + \cos(\tfrac{1}{2}y)^2\right)\left(4e^{\frac{1}{2}\sqrt{3}} - 4e^{-\frac{1}{2}\sqrt{3}}\right)}$$

Then we find the reproducing kernel function for $x \le y$ as:

$$\begin{aligned}
R_y(x) = \frac{1}{6}\frac{1}{e^{\sqrt{3}}-1}e^{\frac{1}{2}\sqrt{3}}\Bigg(&\cos(\frac{1}{2}y)e^{\frac{1}{2}(x-y+1)\sqrt{3}}\cos(\frac{1}{2}x)\sqrt{3} + 2\cos(\frac{1}{2}y)e^{-\frac{1}{2}(x+y-1)\sqrt{3}}\cos(\frac{1}{2}x)\sqrt{3}\\
&+2\cos(\frac{1}{2}y)e^{\frac{1}{2}(x+y-1)\sqrt{3}}\cos(\frac{1}{2}x)\sqrt{3} + \cos(\frac{1}{2}y)e^{-\frac{1}{2}(x-y+1)\sqrt{3}}\cos(\frac{1}{2}x)\sqrt{3}\\
&+\sin(\frac{1}{2}y)e^{\frac{1}{2}(x-y+1)\sqrt{3}}\sin(\frac{1}{2}x)\sqrt{3} + 2\sin(\frac{1}{2}y)e^{-\frac{1}{2}(x+y-1)\sqrt{3}}\sin(\frac{1}{2}x)\sqrt{3}\\
&+2\sin(\frac{1}{2}y)e^{\frac{1}{2}(x+y-1)\sqrt{3}}\sin(\frac{1}{2}x)\sqrt{3} + \sin(\frac{1}{2}y)e^{-\frac{1}{2}(x-y+1)\sqrt{3}}\sin(\frac{1}{2}x)\sqrt{3}\\
&-3\cos(\frac{1}{2}y)e^{\frac{1}{2}(x-y+1)\sqrt{3}}\sin(\frac{1}{2}x) + 3\cos(\frac{1}{2}y)e^{-\frac{1}{2}(x-y+1)\sqrt{3}}\sin(\frac{1}{2}x)
\end{aligned}$$

$$+3\sin(\frac{1}{2}y)e^{\frac{1}{2}(x-y+1)\sqrt{3}}\cos(\frac{1}{2}x)-3\sin(\frac{1}{2}y)e^{-\frac{1}{2}(x-y+1)\sqrt{3}}\cos(\frac{1}{2}x)\Big).$$

The reproducing kernel function for $x > y$ is obtained as:

$$\begin{aligned}
R_y(x) = \frac{1}{6}\frac{1}{e^{\sqrt{3}}-1}e^{\frac{1}{2}\sqrt{3}}\Big(&\cos(\frac{1}{2}y)e^{\frac{1}{2}(x-y-1)\sqrt{3}}\cos(\frac{1}{2}x)\sqrt{3}+2\cos(\frac{1}{2}y)e^{-\frac{1}{2}(x+y-1)\sqrt{3}}\cos(\frac{1}{2}x)\sqrt{3}\\
&+2\cos(\frac{1}{2}y)e^{\frac{1}{2}(x+y-1)\sqrt{3}}\cos(\frac{1}{2}x)\sqrt{3}+\cos(\frac{1}{2}y)e^{-\frac{1}{2}(x-y-1)\sqrt{3}}\cos(\frac{1}{2}x)\sqrt{3}\\
&+\sin(\frac{1}{2}y)e^{\frac{1}{2}(x-y-1)\sqrt{3}}\sin(\frac{1}{2}x)\sqrt{3}+2\sin(\frac{1}{2}y)e^{-\frac{1}{2}(x+y-1)\sqrt{3}}\sin(\frac{1}{2}x)\sqrt{3}\\
&+2\sin(\frac{1}{2}y)e^{\frac{1}{2}(x+y-1)\sqrt{3}}\sin(\frac{1}{2}x)\sqrt{3}+\sin(\frac{1}{2}y)e^{-\frac{1}{2}(x-y-1)\sqrt{3}}\sin(\frac{1}{2}x)\sqrt{3}\\
&-3\cos(\frac{1}{2}y)e^{\frac{1}{2}(x-y-1)\sqrt{3}}\sin(\frac{1}{2}x)+3\cos(\frac{1}{2}y)e^{-\frac{1}{2}(x-y-1)\sqrt{3}}\sin(\frac{1}{2}x)\\
&+3\sin(\frac{1}{2}y)e^{\frac{1}{2}(x-y-1)\sqrt{3}}\cos(\frac{1}{2}x)-3\sin(\frac{1}{2}y)e^{-\frac{1}{2}(x-y-1)\sqrt{3}}\cos(\frac{1}{2}x)\Big).
\end{aligned}$$

3.3. m=3

We have

$$\begin{aligned}
\langle u, B_y\rangle_{S_2^3[0,1]} &= \int_0^1 u(x)B_y(x)dx+\int_0^1 u'(x)B_y'(x)dx\\
&+\int_0^1 u''(x)B_y''(x)dx+\int_0^1 u'''(x)B_y'''(x)dx.
\end{aligned}$$

After using integration by parts, we obtain

$$\begin{aligned}
\langle u, B_y\rangle_{S_2^3[0,1]} &= \int_0^1 u(x)B_y(x)dx+u(1)B_y'(1)-u(0)B_y'(0)-\int_0^1 u(x)B_y''(x)dx\\
&+ u'(1)B_y''(1)-u'(0)B_y''(0)-u(1)B_y'''(1)+u(0)B_y'''(0)+\int_0^1 u(x)B_y^{(4)}(x)dx\\
&+ u''(1)B_y'''(1)-u''(0)B_y'''(0)-u'(1)B_y^{(4)}(1)+u'(0)B_y^{(4)}(0)\\
&+ u(1)B_y^{(5)}(1)-u(0)B_y^{(5)}(0)-\int_0^1 u(x)B_y^{(6)}(x)dx.
\end{aligned}$$

Then, we get

$$\langle u, B_y \rangle_{S_2^3[0,1]} = \int_0^1 u(x)[B_y(x) - B_y''(x) + B_y^{(4)}(x) - B_y^{(6)}(x)]dx.$$

If we have the following equations:

$$\begin{aligned}
&1) - B_y'(0) + B_y'''(0) - B_y^{(5)}(0) = 0,\\
&2) - B_y''(0) + B_y^{(4)}(0) = 0,\\
&3) B_y'''(0) = 0,\\
&4) B_y'(1) - B_y'''(1) + B_y^{(5)}(1) = 0,\\
&5) B_y''(1) - B_y^{(4)}(1) = 0,\\
&6) B_y''(1) = 0.
\end{aligned}$$

Then by reproducing property, we will get

$$\langle u, B_y \rangle_{S_2^3[0,1]} = \int_0^1 u(x)[B_y(x) - B_y''(x) + B_y^{(4)}(x) - B_y^{(6)}(x)]dx = u(y).$$

Therefore, by Dirac-Delta function we get

$$B_y(x) - B_y''(x) + B_y^{(4)}(x) - B_y^{(6)}(x) = \delta(x - y).$$

When $x \neq y$, we have $\delta(x - y) = 0$. Thus, we reach

$$B_y(x) - B_y''(x) + B_y^{(4)}(x) - B_y^{(6)}(x) = 0.$$

Then, we get

$$1 - \lambda^2 + \lambda^4 - \lambda^6 = 0.$$

If we solve the above equation, we will get

$$\begin{aligned}
\lambda_1 &= 1,\\
\lambda_2 &= -1,\\
\lambda_3 &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},\\
\lambda_4 &= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2},\\
\lambda_5 &= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2},
\end{aligned}$$

$$\lambda_6 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$$

Therefore, we obtain the reproducing kernel function $B_y(x)$ as:

$$B_y(x) = \begin{cases} c_1e^x + c_2e^{-x} + c_3e^{\frac{\sqrt{2}}{2}x}\cos(\frac{\sqrt{2}}{2}x) + c_4e^{\frac{\sqrt{2}}{2}x}\sin(\frac{\sqrt{2}}{2}x) \\ +c_5e^{-\frac{\sqrt{2}}{2}x}\cos(\frac{\sqrt{2}}{2}x) + c_6e^{-\frac{\sqrt{2}}{2}x}\sin(\frac{\sqrt{2}}{2}x), \quad x \le y \\ \\ d_1e^x + d_2e^{-x} + d_3e^{\frac{\sqrt{2}}{2}x}\cos(\frac{\sqrt{2}}{2}x) + d_4e^{\frac{\sqrt{2}}{2}x}\sin(\frac{\sqrt{2}}{2}x) \\ +d_5e^{-\frac{\sqrt{2}}{2}x}\cos(\frac{\sqrt{2}}{2}x) + d_6e^{-\frac{\sqrt{2}}{2}x}\sin(\frac{\sqrt{2}}{2}x), \quad x > y. \end{cases}$$

If we solve the above equations, we will get the coefficients as:

$$c_1 = \frac{1}{4}\frac{e^{-y}e + e^{-1}e^y}{e^ye^{-y}(e - e^{-1})},$$

$$c_2 = \frac{1}{4}\frac{e^{-y}e + e^{-1}e^y}{e^ye^{-y}(e - e^{-1})},$$

$$c_3 = \frac{1}{4}\frac{\left(\sin(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}\sqrt{2}} + \cos(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}\right)\sqrt{2}}{\left(\sin(\frac{1}{2}y\sqrt{2})^2 + \cos(\frac{1}{2}y\sqrt{2})^2\right)\left(e^{\frac{1}{2}\sqrt{2}} - e^{-\frac{1}{2}\sqrt{2}}\right)e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}}$$

$$c_4 = -\frac{1}{4}\frac{\left(\cos(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}\sqrt{2}} - \sin(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}\right)\sqrt{2}}{\left(\sin(\frac{1}{2}y\sqrt{2})^2 + \cos(\frac{1}{2}y\sqrt{2})^2\right)\left(e^{\frac{1}{2}\sqrt{2}} - e^{-\frac{1}{2}\sqrt{2}}\right)e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}}$$

$$c_5 = \frac{1}{4}\frac{\left(\cos(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}\sqrt{2}} - \sin(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}\right)\sqrt{2}}{\left(\sin(\frac{1}{2}y\sqrt{2})^2 + \cos(\frac{1}{2}y\sqrt{2})^2\right)\left(e^{\frac{1}{2}\sqrt{2}} - e^{-\frac{1}{2}\sqrt{2}}\right)e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}}$$

$$c_6 = \frac{1}{4}\frac{\left(\sin(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}\sqrt{2}} + \cos(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}\right)\sqrt{2}}{\left(\sin(\frac{1}{2}y\sqrt{2})^2 + \cos(\frac{1}{2}y\sqrt{2})^2\right)\left(e^{\frac{1}{2}\sqrt{2}} - e^{-\frac{1}{2}\sqrt{2}}\right)e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}}$$

$$d_1 = \frac{1}{4}\frac{e^{-1}(e^{-y} + e^y)}{e^ye^{-y}(e - e^{-1})},$$

$$d_2 = \frac{1}{4}\frac{e(e^{-y} + e^y)}{e^ye^{-y}(e - e^{-1})},$$

$$d_3 = \frac{1}{4}\frac{\left(\cos(\frac{1}{2}y\sqrt{2})e^{\frac{1}{2}y\sqrt{2}}e^{-\frac{1}{2}\sqrt{2}} + \sin(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}\sqrt{2}}e^{-\frac{1}{2}y\sqrt{2}}\right)\sqrt{2}}{\left(\sin(\frac{1}{2}y\sqrt{2})^2 + \cos(\frac{1}{2}y\sqrt{2})^2\right)\left(e^{\frac{1}{2}\sqrt{2}} - e^{-\frac{1}{2}\sqrt{2}}\right)e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}}$$

$$d_4 = \frac{1}{4}\frac{\left(\sin(\frac{1}{2}y\sqrt{2})e^{\frac{1}{2}y\sqrt{2}}e^{-\frac{1}{2}\sqrt{2}} - \cos(\frac{1}{2}y\sqrt{2})e^{-\frac{1}{2}\sqrt{2}}e^{-\frac{1}{2}y\sqrt{2}}\right)\sqrt{2}}{\left(\sin(\frac{1}{2}y\sqrt{2})^2 + \cos(\frac{1}{2}y\sqrt{2})^2\right)\left(e^{\frac{1}{2}\sqrt{2}} - e^{-\frac{1}{2}\sqrt{2}}\right)e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}}$$

$$d_5 = -\frac{1}{4}\frac{\left(\sin(\frac{1}{2}y\sqrt{2})e^{\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}\sqrt{2}} - \cos(\frac{1}{2}y\sqrt{2})e^{\frac{1}{2}\sqrt{2}}e^{-\frac{1}{2}y\sqrt{2}}\right)\sqrt{2}}{\left(\sin(\frac{1}{2}y\sqrt{2})^2 + \cos(\frac{1}{2}y\sqrt{2})^2\right)\left(e^{\frac{1}{2}\sqrt{2}} - e^{-\frac{1}{2}\sqrt{2}}\right)e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}}$$

$$d_6 = \frac{1}{4}\frac{\left(\cos(\frac{1}{2}y\sqrt{2})e^{\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}\sqrt{2}} + \sin(\frac{1}{2}y\sqrt{2})e^{\frac{1}{2}\sqrt{2}}e^{-\frac{1}{2}y\sqrt{2}}\right)\sqrt{2}}{\left(\sin(\frac{1}{2}y\sqrt{2})^2 + \cos(\frac{1}{2}y\sqrt{2})^2\right)\left(e^{\frac{1}{2}\sqrt{2}} - e^{-\frac{1}{2}\sqrt{2}}\right)e^{-\frac{1}{2}y\sqrt{2}}e^{\frac{1}{2}y\sqrt{2}}}$$

Then, we will get the reproducing kernel function $B_y(x)$ for $x \leq y$ as:

$$\begin{aligned}
B_y(x) = \frac{1}{4}\frac{1}{(e^{\sqrt{2}} - 1)(e^2 - 1)}e^{\frac{1}{2}\sqrt{2}+1}\\
\Big(e^{1-\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
+e^{1-\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
+e^{1+\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
-e^{1+\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
-e^{1-\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
+e^{1-\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
+e^{1+\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
+e^{1+\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
-e^{-1-\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
-e^{-1-\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}\\
-e^{-1+\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}
\end{aligned}$$

$$+e^{-1+\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$+e^{-1-\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{-1-\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{-1+\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{-1+\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{1-y+x-\frac{1}{2}\sqrt{2}}+e^{1-y+x+\frac{1}{2}\sqrt{2}}-e^{1-y-x-\frac{1}{2}\sqrt{2}}+e^{1-y-x+\frac{1}{2}\sqrt{2}}$$

$$-e^{-1+y+x-\frac{1}{2}\sqrt{2}}+e^{-1+y+x+\frac{1}{2}\sqrt{2}}-e^{-1+y-x-\frac{1}{2}\sqrt{2}}+e^{-1+y-x+\frac{1}{2}\sqrt{2}}\Big).$$

We obtain the reproducing kernel function $B_y(x)$ for $x > y$ as:

$$B_y(x) = -\frac{1}{4}\frac{1}{(e^{\sqrt{2}}-1)(e^2-1)}e^{\frac{1}{2}\sqrt{2}+1}$$

$$\Big(e^{1+\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$+e^{-1-\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{-1+\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{1-\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{1-\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{1+\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$+e^{-1-\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$+e^{-1+\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\sin(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{1-\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{1+\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$+e^{-1-\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$+e^{-1+\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\cos(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{1+\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$-e^{-1-\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$+e^{-1+\frac{1}{2}y\sqrt{2}-\frac{1}{2}x\sqrt{2}+\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$+e^{1-\frac{1}{2}y\sqrt{2}+\frac{1}{2}x\sqrt{2}-\frac{1}{2}\sqrt{2}}\cos(\frac{1}{2}y\sqrt{2})\sin(\frac{1}{2}x\sqrt{2})\sqrt{2}$$

$$+e^{1+y-x-\frac{1}{2}\sqrt{2}}-e^{1+y-x+\frac{1}{2}\sqrt{2}}+e^{1-y-x-\frac{1}{2}\sqrt{2}}-e^{1-y-x+\frac{1}{2}\sqrt{2}}$$

$$+e^{-1+y+x-\frac{1}{2}\sqrt{2}}-e^{-1+y+x+\frac{1}{2}\sqrt{2}}+e^{-1-y+x-\frac{1}{2}\sqrt{2}}-e^{-1-y+x+\frac{1}{2}\sqrt{2}}\Big).$$

4. Conclusion

In this paper, we defined very useful reproducing kernel Sobolev spaces. We found very important reproducing kernel functions in these spaces. These kernel functions are very useful to solve many problems in the reproducing kernel Sobolev spaces.

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

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