AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(1): 185–203.
DOI:10.3934/math.2020011
Received: 01 August 2019
Accepted: 18 October 2019
Published: 28 October 2019

Research article

Non-local boundary value problem for a system of fractional partial differential equations of the type I

Murat O. Mamchuev*

Institute of Applied Mathematics and Automation of KBSC of RAS, Shortanova str. 89-A, Nalchik, 360000, Kabardino-Balkar Republic, Russia

* **Correspondence:** Email: mamchuev@rambler.ru.

Abstract: A non-local boundary value problem in a rectangular domain for a system of fractional partial differential equations is investigated, in the case when all the eigenvalues of the matrix coefficient in the main part are sign-definite. Conditions for unique solvability of the problem under studying are obtained.

Keywords: fractional order derivatives; fractional hyperbolic systems; non-local boundary value problem; conditions for unique solvability
Mathematics Subject Classification: 35A08, 35A09, 35C05, 35C15, 35F40, 35F45, 35F46, 35R11

1. Introduction

Consider the following system of equations

$$\frac{\partial}{\partial x}u(x,y) + AD_{0y}^{\beta}u(x,y) = Bu(x,y) + f(x,y), \quad 0 < \beta < 1, \tag{1.1}$$

in the domain $\Omega = \{(x,y) : l_1 < x < l_2, 0 < y < T\}$, where $u(x,y)$ and $f(x,y)$ are unknown and given n-vectors, respectively; A and B are the given constant $n \times n$ matrices, D_{0y}^{β} is the Riemann – Liouville fracional integro-differentiation operator of order β [1, p. 9].

Let us review the papers associated with the investigation of the systems with fractional partial derivatives of order that is not higher than one including the scalar case $n = 1$. In paper [2] for the equation

$$D_{0x}^{\alpha}(u - h_1(y)) + D_{0y}^{\beta}(u - h_2(x)) = f, \quad 0 < \alpha, \beta < 1, \quad x, y > 0, \tag{1.2}$$

the solvability of the boundary value problem with the initial conditions $u(0,y) = h_1(y)$, $u(x,0) = h_2(x)$ is studied in a class of Hölder's continuous functions. The authors obtained a fundamental solution of

Eq. (1.2) in the form

$$\psi_{\alpha,\beta}(x,y) = \int\limits_0^\infty \tau^{-\frac{1}{\alpha}-\frac{1}{\beta}} \varphi_\alpha\left(x\tau^{-\frac{1}{\alpha}}\right)\varphi_\beta\left(y\tau^{-\frac{1}{\beta}}\right)d\tau, \tag{1.3}$$

where

$$\varphi_\mu(t) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}\sin(\pi\mu k)\Gamma(\mu k+1)t^{-\mu k-1}.$$

Note that the function $\psi_{\alpha,\beta}(x,y)$ can be represented (it is shown in [3]) as

$$\psi_{\alpha,\beta}(x,y) = \frac{1}{xy}\int\limits_0^\infty \phi(-\alpha,0;-\tau x^{-\alpha})\phi(-\beta,0;-\tau y^{-\beta})d\tau.$$

Paper [4] was devoted to study of Hölder's smoothness of solution for the following equation

$$D_{0x}^{\alpha}(u(x,y)-u_0(y)) + c(x,y)u_y(x,y) = f(x,y), \quad x,y>0,$$

satisfying the boundary conditions $u(0,y) = u_0(y)$ and $u(x,0) = u_1(x)$.

The uniqueness and existence theorems for a boundary value problem regular solution for the equation

$$D_{0x}^{\alpha}u(x,y) + \lambda D_{0y}^{\beta}u(x,y) + \mu u(x,y) = f(x,y), \quad 0<\alpha,\beta<1,\ \lambda>0,\ x,y>0 \tag{1.4}$$

are proved in papers [5, 6]. The fundamental solution has the form

$$w(x,y) = \frac{1}{xy}\int\limits_0^\infty e^{-\mu\tau}\phi(-\alpha,0;-\tau x^{-\alpha})\phi(-\beta,0;-\tau y^{-\beta})d\tau,$$

in the case when $\lambda = 1$, and, when $\mu = 0$ it has the form

$$w(x,y) = \frac{x^{\alpha-1}}{y}e_{\alpha,\beta}^{\alpha,0}\left(-\lambda\frac{x^\alpha}{y^\beta}\right),$$

where

$$e_{\alpha,\beta}^{\mu,\nu}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+\alpha k)\Gamma(\nu-\beta k)}$$

is the Wright type function [6]. In addition a boundary value problem with negative coefficient $\lambda < 0$ was studied for Eq. (1.4) in the case $\alpha = 1, \mu = 0$.

Equation (1.4) with variable coefficients $\lambda \equiv \lambda(x)$ and $\mu \equiv \mu(x)$, where $\alpha = 1$ and the function $\lambda(x)$ may have a zero of order $m \geq 0$ at the point $x = 0$, was investigated in the papers [7–9]. In that case the fundamental solution

$$w(x,y;t,s) = \frac{\exp(\Lambda(x,t))}{y-s}\phi\left(-\beta,0;-M(x,t)(y-s)^{-\beta}\right),$$

was constructed. In the last expression $\Lambda(x,t) = \int_t^x \lambda(\xi)d\xi$, $M(x,t) = \int_t^x \mu(\xi)d\xi$. The existence and uniqueness theorems for a boundary value problem and the Cauchy problem were proved.

We also note the papers [10] and [11], where the equation

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \lambda \frac{\partial^\beta u(x,t)}{\partial t^\beta}$$

was studied. The fractional derivatives are understood in the sense of Caputo and Riesz in paper [10], and in the sense of Caputo, Riemann–Liouville and Riesz in paper [11].

For the system

$$D_{0x}^\alpha u(x,y) + A D_{0y}^\beta u(x,y) = Bu(x,y) + f(x,y), \tag{1.5}$$

the boundary value problem was solved explicitly in [12] when A was an identity matrix, and in [13] when A was positive defined matrix. The fundamental solution of system (1.5) was constructed in terms of the introduced Wright function of the matrix argument in paper [13]. Article [3] used a similar approach to solve the problem with the boundary conditions in the multidimensional case.

Among the works devoted to the study of systems of equations with fractional partial derivatives, we also distinguish papers [14–16]. In [16] A. N. Kochubei described a class of first order systems of equations with constant coefficients containing a fractional derivative with respect to one of the independent variables, for which the Cauchy problem is solvable, and the fundamental solutions of which grow exponentially outside the set $\{|x|y^{-\beta} \le 1\}$. Such systems were called fractional hyperbolic systems. System (1.1) also belongs to this class of systems.

Note that, the systems of type (1.1) are differ significantly at the formulations of initial and boundary-value problems, depending on the sign-determinacy or sign-indeterminacy of the eigenvalues of the matrix coefficient in the main part of the system.

In papers [12, 13, 17] boundary value problems in rectangular domains were studied for systems with sign-determined eigenvalues, including systems with partial derivatives, of order strictly lower than one. For such systems the formulation of boundary value problems is similar to the case with a single equation. We call this type of systems the type I systems.

In papers [18–23] the Cauchy problem, mixed and non-local problems were investigated for a system of the type II, i.e., for the systems, where the matrix coefficient in the main part has an eigenvalues of the different signs.

In this paper, we first solve in explicit form an auxiliary problem for system (1.1) with $B = 0$. To do this, we use the properties of the Wright function of the matrix argument, which are studied in [13]. Next we investigate a non-local boundary value problem for system (1.1) by reducing to the auxiliary problem by using a system of integral equations. We prove the existence and uniqueness theorem. At the end, we give an example of the non-local boundary value problem and construct the graphs of its solution.

2. Preliminaries

The Riemann-Liouville fractional integro-differentiation operator D_{ay}^{ν} of order ν is defined as [1, p. 9]:

$$D_{ay}^{\nu}g(y) = \frac{\operatorname{sgn}(y-a)}{\Gamma(-\nu)}\int_a^y \frac{g(s)ds}{|y-s|^{\nu+1}},$$

for $\nu < 0$, and for $\nu \geq 0$ the operator D_{ay}^{ν} can be determined by recursive relation

$$D_{ay}^{\nu}g(y) = \operatorname{sgn}(y-a)\frac{d}{dy}D_{ay}^{\nu-1}g(y), \quad \nu \geq 0,$$

where $\Gamma(z)$ is the Euler gamma-function.

The symbol ∂_{0y}^{ν} denotes the Caputo fractional differentiation operator of order ν, [1, p. 11]:

$$\partial_{ay}^{\nu}g(y) = \operatorname{sgn}^n(y-a)D_{ay}^{\nu-n}g^{(n)}(y), \quad n-1 < \nu \leq n, \quad n \in \mathbb{N}.$$

The Wright function [24, 25] is called an entire function, which is depended from two parameters ρ and μ, and represented by the series

$$\phi(\rho, \mu; z) = \sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\rho k+\mu)}, \quad \rho > -1, \quad \mu \in \mathbb{C}.$$

Here we present the determination and some properties of the Wright function of the matrix argument, which are studied in [13].

Let A be a square matrix of order n. In view of the function $\phi(\rho, \mu; z)$ is analytic everywhere in $\mathbb{C}$, following series

$$\phi(\rho, \mu; A) = \sum_{k=0}^{\infty}\frac{A^k}{k!\Gamma(\rho k+\mu)}, \quad \rho > -1, \quad \mu \in \mathbb{C}$$

is converges for any matrix A given over the field of complex numbers $\mathbb{C}$, and determine the Wright function of the matrix argument.

The following equality holds

$$\phi(\rho, \mu; Az)\big|_{z=0} = \frac{1}{\Gamma(\mu)}I, \tag{2.1}$$

here I is an identity matrix of order n.

Following differentiation formula holds

$$\frac{d}{dz}\phi(\rho, \mu; Az) = A\phi(\rho, \rho+\mu; Az). \tag{2.2}$$

Now and further we assume that all of the eigenvalues of the matrix A are positive.

The next fractional integro-differentiation formula holds:

$$D_{0y}^{\delta}y^{\mu-1}\phi(-\beta, \mu; -Axy^{-\beta}) = y^{\mu-\delta-1}\phi(-\beta, \mu-\delta; -Axy^{-\beta}). \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\left(\frac{\partial}{\partial x} + AD_{0y}^{\beta}\right) y^{\mu-1}\phi(-\beta,\mu;-Axy^{-\beta}) = 0. \tag{2.4}$$

The following equality holds

$$\int_{0}^{\infty} \phi(-\beta,\mu;-Az)dz = \frac{1}{\Gamma(\mu+\beta)}A^{-1}. \tag{2.5}$$

We denote by $|A(x,y)|_*$ the scalar function that takes at each point (x,y) the largest of the values of the moduli of the elements of the matrix $A(x,y) = \|a_{ij}(x,y)\|$, that is $|A(x,y)|_* = \max\limits_{i,j} |a_{ij}(x,y)|$. Similarly, for the vector $b(x,y)$ with components $b_i(x,y)$ we denote $|b(x,y)|_* = \max\limits_{i} |b_i(x,y)|$.

Following estimates are hold:

$$|y^{\mu-1}\phi(-\beta,\mu;-Axy^{-\beta})|_* \le Cx^{-\theta}y^{\mu+\beta\theta-1}, \quad x>0, \quad y>0, \tag{2.6}$$

where $\beta \in (0,1)$; and $\theta \ge 0$ for $\mu \ne 0,-1,-2,...$, and $\theta \ge -1$ for $\mu = 0,-1,-2,...$; and

$$|\phi(-\beta,\mu;-Az)|_* \le C\exp\left(-\sigma z^{\frac{1}{1-\beta}}\right), \quad z \ge 0, \tag{2.7}$$

where $\beta \in (0,1)$, $\mu \in \mathbb{R}$, $\sigma < (1-\beta)\left(\lambda\beta^{\beta}\right)^{\frac{1}{1-\beta}}$, $\lambda = \min\limits_{1\le i\le p}\{\lambda_i\}$, $\lambda_1,...,\lambda_p$ are eigenvalues of the matrix A.

3. Auxiliary problem

A regular solution of system (1.1) in the domain Ω is defined as the vector function $u = u(x,y)$ satisfying system (1.1) at all points $x \in \Omega$, such that $\frac{\partial u}{\partial x}$, $D_{0y}^{\beta}u \in C(\Omega)$, $y^{1-\beta}u(x,y) \in C(\bar{\Omega})$.

Before turning to the presentation of the main results, we solve the following auxiliary problem for the case of system (1.1) with $B = 0$.

Problem 1. In the domain Ω find a solution of the system

$$\frac{\partial}{\partial x}u(x,y) + AD_{0y}^{\beta}u(x,y) = f(x,y), \quad 0<\beta<1, \tag{3.1}$$

with the conditions

$$\lim_{y\to 0} D_{0y}^{\beta-1}u(x,y) = \varphi(x), \quad l_1 \le x \le l_2, \tag{3.2}$$

$$u(l_1,y) = \psi(y), \quad 0<y<T, \tag{3.3}$$

where $\varphi(x)$ and $\psi(y)$ are given n-vectors.

Theorem 1. *Let all the eigenvalues of the matrix A be positive,* $\varphi(x) \in C[l_1,l_2]$, $y^{1-\beta}\psi(y) \in C[0,T]$, $y^{1-\beta}f(x,y) \in C(\overline{\Omega})$, $f(x,y)$ *satisfies the Hölder condition with respect to* y, *and the matching condition*

$$\lim_{y\to 0} D_{0y}^{\beta-1}\psi(y) = \varphi(l_1) \tag{3.4}$$

holds. Then there exists a unique regular in the domain Ω, *solution of Problem 1. Solution can be represented as*

$$u(x,y)=\int_{l_1}^{x}G(x-t,y)A\varphi(t)dt+\int_{0}^{y}G(x-l_1,y-s)\psi(s)ds+$$

$$+\int_{0}^{y}\int_{l_1}^{x}G(x-t,y-s)f(t,s)dtds, \tag{3.5}$$

where

$$G(x,y)=y^{-1}\phi(-\beta,0;-Axy^{-\beta}).$$

Remark 1. Without loss of generality, we prove Theorem 1 for the domain Ω with $l_1=0$ and $l_2=l$. A more general case reduces to this case by replacing the independent variables $x=\xi+l_1$, $y=\eta$.

To prove Theorem 1, we need the following assertions.

Lemma 1. *Any regular in the domain* Ω *solution* $u(x,y)$ *of Problem 1 can be represented in form* (3.5).

Proof. Let $u(x,y)$ be a solution of Problem 1. The function $V(x,y)$ is the solution of the equation

$$\frac{\partial}{\partial x}V(x,y)+\partial_{0y}^{\beta}V(x,y)A=I, \tag{3.6}$$

with the conditions

$$V(0,y)=0,\quad V(x,0)=0, \tag{3.7}$$

where I is the identity matrix.

Using (2.2), (2.3), (2.1), (2.7) and the relation

$$D_{0y}^{\alpha}\frac{y^{\beta}}{\Gamma(1+\beta)}=\frac{y^{\beta-\alpha}}{\Gamma(1+\beta-\alpha)},$$

it is easy to see that

$$V(x,y)=-A^{-1}y^{\beta}\phi\left(-\beta,1+\beta;-Axy^{-\beta}\right)+\frac{A^{-1}}{\Gamma(1+\beta)}y^{\beta}$$

is the solution of problems (3.6), (3.7).

From (2.2) and (2.3) it follows that

$$V_{xy}(x,y)=G(x,y). \tag{3.8}$$

Let $\varepsilon>0$. Integration by parts taking into account Eqs. (2.1), (2.7) and (3.7) leads to

$$\int_{\varepsilon}^{x}\int_{\varepsilon}^{y}V(x-t,y-s)\frac{\partial}{\partial t}u(t,s)dsdt=$$

$$=\int_{\varepsilon}^{x}\int_{\varepsilon}^{y}\frac{\partial}{\partial t}V(x-t,y-s)u(t,s)dtds-\int_{\varepsilon}^{y}V(x-t,y-s)u(t,s)\Big|_{t=\varepsilon}ds,$$

$$\int_{\varepsilon}^{x}\int_{\varepsilon}^{y} V(x-t,y-s)AD_{0s}^{\beta}u(t,s)dsdt =$$

$$= \int_{\varepsilon}^{x}\int_{\varepsilon}^{y} \frac{\partial}{\partial s}V(x-t,y-s)AD_{0s}^{\beta-1}u(t,s)dtds - \int_{\varepsilon}^{x} V(x-t,y-s)AD_{0s}^{\beta-1}u(t,s)\Big|_{s=\varepsilon}dt.$$

From the last two relations we get

$$\int_{\varepsilon}^{x}\int_{\varepsilon}^{y} V(x-t,y-s)\left(\frac{\partial}{\partial t}+AD_{0s}^{\beta}\right)u(t,s)dsdt =$$

$$= \int_{\varepsilon}^{x}\int_{\varepsilon}^{y}\left(\frac{\partial}{\partial t}V(x-t,y-s)u(t,s)+\frac{\partial}{\partial s}V(x-t,y-s)AD_{0s}^{\beta-1}u(t,s)\right)dsdt-$$

$$-\int_{\varepsilon}^{y} V(x-t,y-s)u(t,s)\Big|_{t=\varepsilon}ds - \int_{\varepsilon}^{x} V(x-t,y-s)AD_{0s}^{\beta-1}u(t,s)\Big|_{s=\varepsilon}dt.$$

Passing to the limit as $\varepsilon \to 0$, by using (3.1), (3.2), (3.3), (3.6) and analogue of the integration by parts formula in fractional calculus [1, p. 34]

$$\int_{0}^{y} g(y-s)D_{0y}^{\nu}h(s)ds = \int_{0}^{y} h(s)D_{ys}^{\nu}g(y-s)ds, \quad \nu < 0,$$

we obtain

$$\int_{0}^{x}\int_{0}^{y} u(t,s)dsdt = \int_{0}^{x}\int_{0}^{y} V(x-t,y-s)f(t,s)dsdt+$$

$$+\int_{0}^{y} V(x,y-s)\varphi(s)ds + \int_{0}^{x} V(x-t,y)A\psi(t)dt. \tag{3.9}$$

Differentiating (3.9) by x and by y, with (3.7) and (3.8), we get (3.5). Lemma 1 is proved.

Lemma 2. *Following estimates*

$$|G(x,y)|_{*} \le Cx^{-\theta}y^{\beta\theta-1}, \quad \theta \ge -1, \tag{3.10}$$

$$\left|D_{0y}^{\beta-1}G(x,y)\right|_{*} \le Cx^{-\theta}y^{\beta\theta-\beta}, \quad \theta \ge 0, \tag{3.11}$$

$$\left|\frac{\partial}{\partial x}G(x,y)\right|_{*} \le Cx^{-\theta}y^{\beta\theta-\beta-1}, \quad \theta \ge 0, \tag{3.12}$$

$$\left|D_{0y}^{\beta}G(x,y)\right|_{*} \le Cx^{-\theta}y^{\beta\theta-\beta-1}, \quad \theta \ge 0, \tag{3.13}$$

are hold, here C is a positive constant.

The validity of Lemma 2 follows from the relations (2.2), (2.3) and (2.6).

Lemma 3. *Let all the eigenvalues of the matrix A be positive,* $\varphi(x) \in C[0, l]$, $y^{1-\beta}\psi(y) \in C[0, T]$, *then the relations*

$$\lim_{x\to 0} \int_0^x G(x-t, y)A\psi(t)dt = 0, \quad y > \varepsilon > 0, \tag{3.14}$$

$$\lim_{y\to 0} D_{0y}^{\beta-1} \int_0^y G(x, y-s)\varphi(s)ds = 0, \quad x > \varepsilon > 0, \tag{3.15}$$

$$\lim_{y\to 0} D_{0y}^{\beta-1} \int_0^x G(x-t, y)A\psi(t)dt = \psi(x), \quad x > \varepsilon > 0, \tag{3.16}$$

$$\lim_{x\to 0} \int_0^y G(x, y-s)\varphi(s)ds = \varphi(y), \quad y > \varepsilon > 0 \tag{3.17}$$

are valid, and limits (3.15) and (3.16) are uniform on any closed subset of $(0; l)$, *and limits (3.14) and (3.17) on any closed subset of* $(0; T)$.

Proof. The validity of relations (3.14) and (3.15) follows from estimates (3.10), (3.11), $|\psi(x)|_* \le C$ and $|\varphi(y)|_* \le Cy^{\beta-1}$.

Let us transform the following integral

$$D_{0y}^{\beta-1} \int_0^x G(x-t, y)A\psi(t)dt = \left(\int_0^\varepsilon + \int_\varepsilon^x\right) D_{0y}^{\beta-1} G(t, y)A\psi(x-t)dt. \tag{3.18}$$

The limit of the second integral in the right-hand side of (3.18) with $y \to 0$, due to estimate (3.11) and the boundedness of the function $\psi(x)$, is zero for $x > \varepsilon > 0$. Denote by $I_1(x, y)$ the first integral in the right-hand side of (3.18), then

$$I_1(x, y) = \int_0^\varepsilon D_{0y}^{\beta-1} G(t, y)A[\psi(x-t) - \psi(x)]dt + \left[\int_0^\varepsilon D_{0y}^{\beta-1} G(t, y)dt\right] A\psi(x). \tag{3.19}$$

Taking advantage of the fact that, by virtue of (2.2),

$$Ay^{-\beta}\phi(-\beta, 1-\beta; -Axy^{-\beta}) = -\frac{\partial}{\partial x}\phi(-\beta, 1; -Axy^{-\beta}),$$

we obtain that

$$A \int_0^\varepsilon D_{0y}^{\beta-1} G(t, y)dt = I - \phi(-\beta, 1; -A\varepsilon y^{-\beta}). \tag{3.20}$$

Passing to the limit at $y \to 0$, taking into account formula (2.7), we get

$$\lim_{y\to 0} A \int_0^\varepsilon D_{0y}^{\beta-1} G(t, y)dt = I. \tag{3.21}$$

The function $\psi(t)$ is continuous on $[x-\varepsilon, x]$, therefore $\omega(\varepsilon) = \sup|\psi(x-t) - \psi(x)| \to 0$ with $\varepsilon \to 0$. Since ε can be chosen arbitrary, then the first term in (3.19) is arbitrarily small for any fixed y, that is, tends to zero, with $y \to 0$.

The second term, by virtue of (3.21), tends to $\psi(x)$. Thus $\lim\limits_{y\to 0} I_1(x, y) = \psi(x)$. From the latter, together with (3.18) follows (3.16). The relation (3.17) can be proved similarly. Lemma 3 is proved.

Lemma 4. *Under the conditions of Theorem 1, function (3.5) is a solution of system (3.1), such that* $\frac{\partial}{\partial x}u, D^{\beta}_{0y}u \in C(\Omega)$.

Proof. It follows from (3.12), (3.13) that the estimates

$$\left|\frac{\partial}{\partial x}G(x,y)\right|_* < Cx^{-\theta-1}, \quad |D^{\beta}_{0y}G(x,y)|_* < Cx^{-\theta-1}, \quad \theta \geq -1,$$

are valid for any fixed $y > \varepsilon > 0$ and the estimates

$$\left|\frac{\partial}{\partial x}G(x,y)\right|_* < Cy^{\beta\theta-1}, \quad |D^{\beta}_{0y}G(x,y)|_* < Cy^{\beta\theta-1}, \quad \theta \geq 0,$$

for $x > \varepsilon > 0$. From these estimates, taking into account relations (2.4), we can see that the first two terms (we denote their sum $u_0(x,y)$) on the right-hand side of (3.5) there are solutions of the homogeneous system

$$\frac{\partial}{\partial x}u_0(x,y) + AD^{\beta}_{0y}u_0(x,y) = 0,$$

at that $\frac{\partial}{\partial x}u_0$, $D^{\beta}_{0y}u_0 \in C(\Omega)$.

Denote by $u_f(x,y)$ the third term on the right-hand side of (3.5). Under the condition of Theorem 1, the function $f(x,y)$ satisfies the Hölder condition in the variable y, that is,

$$|f(x,y) - f(x,s)|_* \leq K|y-s|^q, \quad 0 < q < 1, \tag{3.22}$$

here K is positive number. Then

$$\frac{\partial}{\partial x}u_f(x,y) = \frac{\partial}{\partial x}\int\limits_0^x dt\int\limits_0^y G(x-t,y-s)f(t,s)ds = \lim_{t\to x}\int\limits_0^y G(x-t,y-s)f(t,s)ds+$$

$$+\int\limits_0^x dt\int\limits_0^y \frac{\partial}{\partial x}G(x-t,y-s)[f(t,s)-f(t,y)]ds + \int\limits_0^x dt\int\limits_0^y \frac{\partial}{\partial x}G(x-t,y-s)f(t,y)ds. \tag{3.23}$$

Taking into account estimate (3.12) and condition (3.22), we obtain the estimate for the integrand in the second term of (3.23)

$$\left|\frac{\partial}{\partial x}G(x-t,y-s)[f(t,s)-f(t,y)]\right|_* \leq nMC(x-t)^{-\theta-1}(y-s)^{\beta\theta-1+q}, \tag{3.24}$$

choosing $\theta \in [-1;0)$ with $q > \beta$ and $\theta \in (-q/\beta;0)$ with $q \leq \beta$, it is easy to see that the integral converges uniformly over all x and y for any $q \in (0,1)$. Transforming the last term of (3.23) with (2.4), we get

$$\frac{\partial}{\partial x}u_f(x,y) = f(x,y) + \int\limits_0^x dt\int\limits_0^y \frac{\partial}{\partial x}G(x-t,y-s)[f(t,s)-f(t,y)]ds-$$

$$-A\int_0^x D_{0y}^{\beta-1}G(x-t,y)f(t,y)dt. \tag{3.25}$$

From (3.10), (3.11), (3.24) and (3.25) it follows that $\frac{\partial}{\partial x}u_f \in C(\Omega)$.

Consider the function $F_\varepsilon(x,y) = \int_0^x dt \int_0^{y-\varepsilon} D_{ys}^{\beta-1}G(x-t,y-s)f(t,s)ds$. From estimate (3.11) we see that $\lim_{\varepsilon\to 0} F_\varepsilon(x,y) = D_{0y}^{\beta-1}u_f(x,y) \in C(\Omega)$. In view of (3.11) and

$$\left|D_{ys}^{\beta}G(x-t,y-s)[f(t,s)-f(t,y)]\right|_* \leq nCK(x-t)^{-\theta-1}(y-s)^{\beta\theta-1+q}, \tag{3.26}$$

we get that the derivative

$$\frac{\partial}{\partial y}F_\varepsilon(x,y) = \int_0^x D_{0\varepsilon}^{\beta-1}G(x-t,\varepsilon)f(t,y-\varepsilon)dt - \int_0^x D_{0\varepsilon}^{\beta-1}G(x-t,\varepsilon)f(t,y)dt+$$

$$+\int_0^x dt\int_0^{y-\varepsilon} D_{ys}^{\beta}G(x-t,y-s)[f(t,s)-f(t,y)]ds + \int_0^x D_{0s}^{\beta-1}G(x-t,y)f(t,y)dt$$

is continuous in Ω for $\varepsilon \to 0$. Therefore

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial y}F_\varepsilon(x,y) = \frac{\partial}{\partial y}\lim_{\varepsilon\to 0}F_\varepsilon(x,y) = D_{0y}^{\beta}u_f(x,y),$$

that is

$$D_{0y}^{\beta}u_f(x,y) = \int_0^x dt\int_0^y D_{ys}^{\beta}G(x-t,y-s)[f(t,s)-f(t,y)]ds+$$

$$+\int_0^x D_{0y}^{\beta-1}G(x-t,y)f(t,y)dt. \tag{3.27}$$

From (2.4), (3.25) and (3.27) we get

$$\left(\frac{\partial}{\partial x} + AD_{0y}^{\beta}\right)u_f(x,y) = f(x,y).$$

Lemma 4 is proved.

3.1. Proof of Theorem 1

Using estimates (3.10) and $|f(x,y)| \leq Cy^{\beta-1}$, we get

$$|u_f(x,y)|_* \leq Cx^{1-\theta}y^{\beta\theta+\beta-1}, \quad \theta \in (0;1), \tag{3.28}$$

where $u_f(x,y)$ is the third term on the right-hand side of equality (3.5). From (3.28) follow relations

$$\lim_{x\to 0}u_f(x,y) = 0, \quad \lim_{y\to 0}D_{0y}^{\beta-1}u_f(x,y) = 0, \tag{3.29}$$

and the inclusion $y^{1-\beta}u_f \in C(\overline{\Omega})$. Relations (3.14) – (3.17) and (3.29) imply the fulfillment of boundary conditions (3.2) and (3.3).

Denote by $u_\psi(x,y)$ and $u_\varphi(x,y)$, respectively, the first and second term on the right-hand side of Eq. (3.5). Using estimate (3.10) and the conditions of Theorem 1 on the functions $\psi(x)$ and $\varphi(y)$, we get estimates

$$|u_\psi(x,y)|_* \le Cx^{1-\theta}y^{\beta\theta-1}, \quad \theta \in [-1,1),$$

$$|u_\varphi(x,y)|_* \le Cx^{-\theta}y^{\beta\theta+\beta-1}, \quad \theta \in (0,2).$$

From the last two inequalities we get that $y^{1-\beta}(u_\psi + u_\varphi) \in C(\Omega)$.

Let us show the validity of the inclusion $y^{1-\beta}(u_\psi + u_\varphi) \in C(\overline{\Omega})$. For this purpose we represent $u_\psi(x,y)$ in the form

$$u_\psi(x,y) = A\int_0^x G(x-t,y)\psi(t)dt = A\int_0^x G(t,y)\psi(x-t)dt =$$

$$= A\int_0^x G(t,y)[\psi(x-t)-\psi(x)]dt + A\left[\int_0^x G(t,y)dt\right]\psi(x). \tag{3.30}$$

In view of (2.2) and (2.1) we obtain

$$A\int_0^x G(t,y)dt = A\int_0^x y^{-1}\phi\left(-\beta,0;-Aty^{-\beta}\right)dt = -\int_0^x y^{\beta-1}\frac{\partial}{\partial t}\phi\left(-\beta,\beta;-Aty^{-\beta}\right)dt =$$

$$= \frac{y^{\beta-1}}{\Gamma(\beta)}I - y^{\beta-1}\phi\left(-\beta,\beta;-Axy^{-\beta}\right). \tag{3.31}$$

Similarly we get

$$u_\varphi(x,y) = \int_0^y G(x,s)\varphi(y-s)ds =$$

$$= \int_0^y G(x,s)(y-s)^{\beta-1}[\varphi^*(y-s)-\varphi^*(y)]ds + \left[\int_0^y G(x,s)(y-s)^{\beta-1}ds\right]\varphi^*(y), \tag{3.32}$$

where $\varphi^*(y) = y^{1-\beta}\varphi(y)$, and

$$\int_0^y G(x,s)(y-s)^{\beta-1}ds = \Gamma(\beta)D_{0y}^{-\beta}y^{-1}\phi\left(-\beta,0;-Axy^{-\beta}\right)dt =$$

$$= \Gamma(\beta)y^{\beta-1}\phi\left(-\beta,\beta;-Axy^{-\beta}\right). \tag{3.33}$$

Using (3.30)–(3.33), (2.1), (2.5), (2.7), we get

$$\lim_{x\to 0} y^{1-\beta}u_\psi(x,y) = 0, \quad \lim_{y\to 0} y^{1-\beta}u_\psi(x,y) = \frac{1}{\Gamma(\beta)}\psi(x), \tag{3.34}$$

$$\lim_{x\to 0} y^{1-\beta} u_\varphi(x,y) = \frac{1}{\Gamma(\beta)} \varphi^*(y), \quad \lim_{y\to 0} y^{1-\beta} u_\varphi(x,y) = 0. \tag{3.35}$$

Relations (3.34) and (3.35) imply that $y^{1-\beta}(u_\psi + u_\varphi) \in C(\overline{\Omega} \setminus \{(0,0)\})$.

Let $\lim\limits_{\substack{x\to 0\\ y\to 0}} xy^{-\beta} = c$, $0 \le c \le \infty$. Then from relation (3.31) we obtain

$$\lim_{\substack{x\to 0\\ y\to 0}} y^{1-\beta} u_\psi(x,y) = [I - \phi(-\beta,\beta;-Ac)]\,\psi(0), \tag{3.36}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} y^{1-\beta} u_\varphi(x,y) = \Gamma(\beta)\phi(-\beta,\beta;-Ac)\,\varphi^*(0). \tag{3.37}$$

In view of (3.36) and (3.37) we obtain

$$\lim_{\substack{x\to 0\\ y\to 0}} y^{1-\beta}[u_\psi(x,y) + u_\varphi(x,y)] = \psi(0) + [\Gamma(\beta)\varphi^*(0) - \psi(0)]\phi(-\beta,\beta;-Ac)\,.$$

This limit does not depend on c, if

$$\lim_{x\to 0} \psi(x) = \Gamma(\beta) \lim_{y\to 0} y^{1-\beta}\varphi(y),$$

that is, under condition (3.4).

The above together with Lemma 4 proves the existence of the solution to problems (3.1), (3.2), (3.3) from the class specified in Theorem 1. The uniqueness of the solution to Problem 1 follows from Lemma 1. Theorem 1 is proved.

4. Non-local boundary value problem

In this section, we investigate the following non-local boundary value problem in a rectangular domain for system (1.1) of the type I.

Problem 2. Find a solution of system (1.1) in the domain Ω with conditions (3.2) and

$$Mu(l_1,y) + Nu(l_2,y) = \rho(y), \quad 0 < y < T, \tag{4.1}$$

where $\varphi(x)$ and $\rho(y)$ are given n-vectors, M and N are the given constant $n \times n$ matrix,

Theorem 2. *Let all the eigenvalues of the matrix A be positive,* $\varphi(x) \in C[l_1,l_2]$, $y^{1-\beta}\rho(y) \in C[0,T]$, $y^{1-\beta} f(x,y) \in C(\overline{\Omega})$, $f(x,y)$ *satisfies the Hölder condition with respect to y, and the matching condition*

$$\lim_{y\to 0} D_{0y}^{\beta-1}\rho(y) = M\varphi(l_1) + N\varphi(l_2), \tag{4.2}$$

holds, matrix M is nonsingular. Then there exists a unique regular in the domain Ω, *solution of Problem 2.*

Proof. By virtue of Theorem 1, the solution to Problem 1 for system (1.1) is a solution to the system of the integral equations

$$u(x,y) - \int_0^y \int_{l_1}^x G(x-t,y-s)Bu(t,s)dtds = F(x,y), \tag{4.3}$$

where

$$F(x,y)=\int_0^y G(x-l_1,y-s)\psi(s)ds+\Phi(x,y),$$

$$\Phi(x,y)=\int_{l_1}^x G(x-t,y)A\varphi(t)dt+\int_0^y\int_{l_1}^x G(x-t,y-s)f(t,s)dtds.$$

Due to estimate (3.10) we get the inclusion $y^{1-\beta}F(x,y)\in C(\overline{\Omega})$.

The solution of the system of integral Eqs. (4.3) can be obtained by an iterative method. This solution has the form

$$u(x,y)=F(x,y)+\int_0^y\int_{l_1}^x R(x-t,y-s)F(t,s)dtds, \tag{4.4}$$

where

$$R(x,y)=\sum_{n=1}^{\infty}K_n(x,y), \tag{4.5}$$

$$K_1(x,y)=K(x,y)=G(x,y)B,$$

$$K_n(x,y)=\int_0^y\int_{l_1}^x K_{n-1}(x-t,y-s)K_1(t,s)dtds.$$

For iterated kernels, in view of (3.10), the estimate

$$|K_m(x,y)|_*\le C^m|B|_*^m\frac{\Gamma^m(\varepsilon)\Gamma^m(\delta)}{\Gamma(m\varepsilon)\Gamma(m\delta)}x^{m\varepsilon-1}y^{m\delta-1},\quad \varepsilon=1-\theta,\quad \delta=\beta\theta,\quad 0<\theta<1.$$

is valid. Using this estimate, we obtain the convergence of series (4.5) and the estimate for the resolvent

$$|R(x,y)|_*\le\sum_{m=1}^{\infty}\frac{[C|B|_*\Gamma(\varepsilon)\Gamma(\delta)]^m}{\Gamma(m\varepsilon)\Gamma(m\delta)}x^{m\varepsilon-1}y^{m\delta-1}=x^{\varepsilon-1}y^{\delta-1}\sum_{m=0}^{\infty}\frac{[C_1x^{\varepsilon}y^{\delta}]^m}{\Gamma(m\varepsilon+\varepsilon)\Gamma(m\delta+\delta)}\le$$

$$\le Cx^{\varepsilon-1}y^{\delta-1}\sum_{m=0}^{\infty}\frac{[C_1x^{\varepsilon}y^{\delta}]^m}{m!\Gamma(m\delta+\delta)}=Cx^{\varepsilon-1}y^{\delta-1}\phi(\delta,\delta;C_1x^{\varepsilon}y^{\delta}),$$

where $C_1=C|B|_*\Gamma(\varepsilon)\Gamma(\delta)$, and C is a large enough number. Due to the continuity of the function $\phi(\delta,\delta;z)$, the following estimate is valid

$$|R(x,y)|_*\le Cx^{-\theta}y^{\beta\theta-1},\quad 0<\theta<1. \tag{4.6}$$

Thus, solution (4.4) can be represented as

$$u(x,y)=\Psi(x,y)+\int_0^y R_1(x,y-s)u(l_1,s)ds, \tag{4.7}$$

where

$$\Psi(x,y)=\Phi(x,y)+\int\limits_0^y\int\limits_{l_1}^x R(x-t,y-s)\Phi(t,s)dtds,$$

$$R_1(x,y-s)=G(x-l_1,y-s)+\int\limits_s^y\int\limits_{l_1}^x R(x-\xi,y-\eta)G(\xi-l_1,\eta)d\xi d\eta.$$

It is easy to show that function (4.4) is the solution to Problem 2. Now let $u(x,y)$ be a regular solution of Problem 1 in the domain Ω, then equality (4.7) also holds. Using representation (4.7), we express the boundary value:

$$u(l_2,y)=\overline{\Psi}(y)+\int\limits_0^y \overline{K}(y-s)u(l_1,s)ds, \tag{4.8}$$

where $\overline{\Psi}(y)=\Psi(l_2,y)$, $\overline{K}(y-s)=R_1(l_2,y-s)$.

Since the matrix M is invertible, condition (4.2) can be rewritten as

$$u(l_1,y)+M^{-1}Nu(l_2,y)=M^{-1}\rho(y),\quad 0<y<T.$$

Using (4.8), from the last equality we get

$$u(l_1,y)+\int\limits_0^y \tilde{K}(y-s)u(l_1,s)ds=P(y), \tag{4.9}$$

where

$$\tilde{K}(y)=M^{-1}N\overline{K}(y),\quad P(y)=M^{-1}\rho(y)+M^{-1}N\overline{\Psi}(y).$$

From (3.10) and (4.5) follow the estimate

$$|R_1(x,y-s)|_*\leq C(x-l_1)^{-\theta}(y-s)^{\beta\theta-1},\quad 0<\theta<1,$$

and the following inclusions

$$y^{1-\beta}\overline{\Psi}(y),\ y^{1-\beta}\tilde{K}(y)\in C[0,T]. \tag{4.10}$$

It follows from (4.10) and the conditions of Theorem 2 on the function $\rho(y)$, that $y^{1-\beta}P(y)\in C[0,T]$.

From relations (4.9) and (4.10), it follows that system (4.8) is a system of the Volterra integral equations of the second kind with a weak singularity in the kernel, and has the unique solution $u(l_1,y)$ such that $y^{1-\beta}u(l_1,y)\in C[0,T]$. After the value of $u(l_1,y)$ is found, the solution to Problem 2 can be obtained from representation (4.7).

From Theorem 1 it follows that for the inclusion $y^{1-\beta}u(x,y)\in C(\overline{\Omega})$ the condition

$$\lim_{y\to 0}D_{0y}^{\beta-1}u(l_1,y)=\varphi(l_1) \tag{4.11}$$

should be met. Taking into account equality (4.9), we rewrite condition (4.11) as

$$\lim_{y\to 0}D_{0y}^{\beta-1}u(l_1,y)=\lim_{y\to 0}\int\limits_0^y \tilde{K}(y-s)D_{0s}^{\beta-1}u(l_1,s)ds+$$

$$+ M^{-1} \lim_{y\to 0} D_{0y}^{\beta-1}\rho(y) + M^{-1}N \lim_{y\to 0} D_{0y}^{\beta-1}\overline{\Psi}(y) = \varphi(l_1). \tag{4.12}$$

From inclusions (4.10), estimates (3.10), (4.5) and $|u(l_1, y)|_* \leq Cy^{\beta-1}$ we obtain the relations

$$\lim_{y\to 0} \int_0^y \tilde{K}(y-s) D_{0s}^{\beta-1} u(l_1, s) ds = 0, \tag{4.13}$$

$$|\Phi(x, y)|_* \leq Cy^{\beta-1},$$

$$\left| D_{0s}^{\beta-1} \int_0^y \int_{l_1}^x R(x-t, y-s)\Phi(t, s) dt ds \right|_* \leq Cy^{\beta\theta}, \quad 0 < \theta < 1. \tag{4.14}$$

By virtue of (4.14) and the relation

$$\lim_{y\to 0} D_{0y}^{\beta-1} \int_{l_1}^x G(x-t, y) A\varphi(t) dt = \varphi(x),$$

which follows from (3.16) and Remark 1, we obtain

$$\lim_{y\to 0} D_{0y}^{\beta-1}\overline{\Psi}(y) = \lim_{y\to 0} D_{0y}^{\beta-1}\Phi(l_2, y) = \lim_{y\to 0} D_{0y}^{\beta-1} \int_{l_1}^{l_2} G(l_2 - t, y) A\varphi(t) dt = \varphi(l_2). \tag{4.15}$$

In view of (4.13) and (4.15), equality (4.12) takes the form

$$M^{-1} \lim_{y\to 0} D_{0y}^{\beta-1}\rho(y) - M^{-1}N\varphi(l_2) = \varphi(l_1).$$

Therefore, condition (4.2) is sufficient for $y^{1-\beta}u(x, y) \in C(\overline{\Omega})$. Theorem 2 is proved.

Remark 2. The case when all the eigenvalues of the matrix are negative, is reduced to the case with positive eigenvalues by changing the variables $\xi = x - l_1$, $\eta = y$, and the function $u(x, y) = u(\xi + l_1, \eta) = w(\xi, \eta)$. Moreover, for the solvability of Problem 2, the matrix N must be nonsingular.

5. Illustration

As example consider Problem 2 with $n = 2$, $AB = BA$, $l_1 = 0$, $l_2 = 1$, $T = 1$, $M = N = I$, $f(x, y) \equiv 0$, $\varphi(x) \equiv 0$, $\rho(y) = \frac{y^{\beta-1}}{\Gamma(\beta)} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, i.e., the system

$$\frac{\partial}{\partial x} u(x, y) + \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} D_{0y}^{\beta} u(x, y) = \begin{pmatrix} 1 & 2 \\ 6 & 5 \end{pmatrix} u(x, y), \tag{5.1}$$

with the conditions

$$\lim_{y\to 0} D_{0y}^{\beta-1} u(x, y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 0 \leq x \leq 1, \tag{5.2}$$

$$u(0,y)+u(1,y)=\rho(y), \quad 0<y<1. \tag{5.3}$$

Solution of problem (5.1)–(5.3) satisfies the following relation

$$u(x,y)=\int\limits_0^y G(x,y-s)u(0,s)ds, \tag{5.4}$$

where

$$G(x,y)=\frac{1}{y}H\begin{pmatrix} e^{-x}\phi\left(-\beta,0;-xy^{-\beta}\right) & 0 \\ 0 & e^{7x}\phi\left(-\beta,0;-5xy^{-\beta}\right)\end{pmatrix}H^{-1},$$

$$H=\begin{pmatrix} 1 & 1 \\ -1 & 3\end{pmatrix}, \quad H^{-1}=\frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1\end{pmatrix}.$$

From (5.4) we get

$$u(1,y)=\int\limits_0^y G(1,y-s)u(0,s)ds. \tag{5.5}$$

Substituting (5.5) into (5.4) we obtain following system of integral equations with respect to $u(0,y)$

$$u(0,y)+\int\limits_0^y K_1(y-s)u(0,s)ds=\rho(y), \tag{5.6}$$

where

$$K_1(y)=G(1,y).$$

Using the Wright functions convolution formula, we calculate the iterative kernels

$$K_n(y)=\int_0^y K_{n-1}(y-s)K_1(s)ds,$$

$$K_n(y)=\frac{1}{y}H\begin{pmatrix} e^{-n}\phi\left(-\beta,0;-ny^{-\beta}\right) & 0 \\ 0 & e^{7n}\phi\left(-\beta,0;-5ny^{-\beta}\right)\end{pmatrix}H^{-1},$$

and find the following solution of integral Eq. (5.5)

$$u(0,y)=\rho(y)+\int\limits_0^y R(y-s)\rho(s)ds, \tag{5.7}$$

where

$$R(y)=\sum_{n=1}^{\infty}(-1)^n K_n(y)=$$

$$=\frac{1}{y}H\sum_{n=1}^{\infty}(-1)^n\begin{pmatrix} e^{-n}\phi\left(-\beta,0;-ny^{-\beta}\right) & 0 \\ 0 & e^{7n}\phi\left(-\beta,0;-5ny^{-\beta}\right)\end{pmatrix}H^{-1}.$$

Put (5.7) into (5.5) we obtain the solution to problems (5.1)–(5.3) in the form

$$u(x,y)=\int_0^y G(x,y-s)\rho(s)ds+\int_0^y\left[\int_s^y G(x,y-\xi)R(\xi-s)d\xi\right]\rho(s)ds=$$

$$=\int_0^y G_0(x,y-s)\rho(s)ds, \tag{5.8}$$

where

$$G_0(x,y)=\frac{1}{y}H\sum_{n=0}^{\infty}(-1)^n\begin{pmatrix} e^{-(x+n)}\phi\left(-\beta,0;-(x+n)y^{-\beta}\right) & 0\\ 0 & e^{7(x+n)}\phi\left(-\beta,0;-5(x+n)y^{-\beta}\right)\end{pmatrix}H^{-1}.$$

After calculating the integrals, we write equality (5.8) in the form

$$u_1(x,y)=y^{\beta-1}\sum_{n=1}^{\infty}(-1)^n\left[e^{-(x+n)}\phi\left(-\beta,\beta;-(x+n)y^{-\beta}\right)+2e^{7(x+n)}\phi\left(-\beta,\beta;-5(x+n)y^{-\beta}\right)\right],$$

$$u_2(x,y)=y^{\beta-1}\sum_{n=1}^{\infty}(-1)^n\left[-e^{-(x+n)}\phi\left(-\beta,\beta;-(x+n)y^{-\beta}\right)+9e^{7(x+n)}\phi\left(-\beta,\beta;-5(x+n)y^{-\beta}\right)\right].$$

Figures 1 and 2 illustrate the solutions of problems (5.1)–(5.3) in cases $\beta=0.4$ and $\beta=0.6$.

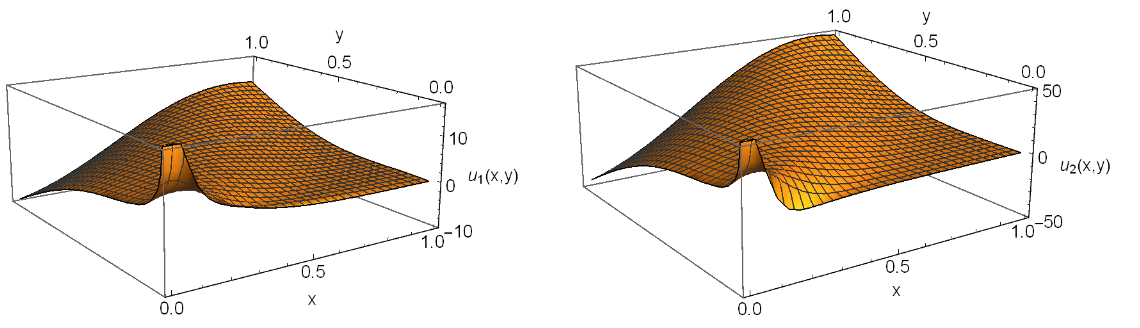

Figure 1. Surface solution of problems (5.1)–(5.3), with $\beta=0.4$.

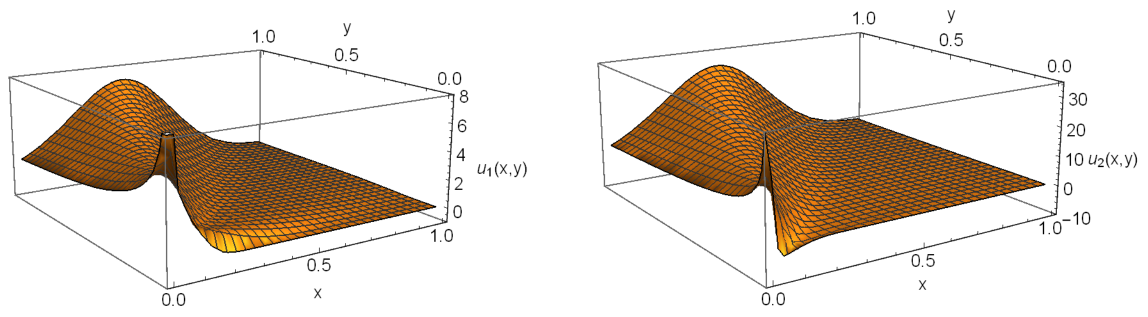

Figure 2. Surface solution of problems (5.1)–(5.3), with $\beta=0.6$.

6. Conclusion

We investigated the non-local boundary value Problem 2 for system (1.1). For this, we have written out an explicit solution of auxiliary Problem 1 for system (1.1) with the matrix B = 0 in terms of the matrix Wright function. Then, using the integral equations method, we reduced Problem 2 to Problem 1. Our approach is schematically illustrated by a particular example described in section 5. The system under study is of the type I. We previously studied some problems for a system of the type II, including Problem 2, which generalizes them. Comparing the results of [23] and the present work, we see that the conditions on the matrices M and N, for which Problem 2 is correct, depend on the distribution of the eigenvalues of the matrix A, that is, they are differ for systems of the type I and II.

Further research will be aimed at expanding the classes of systems and generalizing the described results.

Conflict of interest

The author declares no conflict of interest in this paper.

References

1. A. M. Nakhushev, *Fractional Calculus and Its Applications*, Moscow: Fizmatlit, 2003.
2. P. Clement, G. Gripenberg, S. O. Londen, *Schauder estimates for equations with fractional derivatives*, T. Am. Math. Soc., **352** (2000), 2239–2260.
3. M. O. Mamchuev, *Boundary value problem for a multidimensional system of equations with Riemann–Liouville fractional derivatives*, Sib. Electron. Math. Rep., **16** (2019), 732–747.
4. P. Clement, G. Gripenberg, S. O. Londen, *Hölder regularity for a linear fractional evolution equation*, In: Topics in Nonlinear Analysis, Basel: Birkhäuser, 1999, 62–82.
5. A. V. Pskhu, *Solution of a boundary value problem for a fractional partial differential equation*, Differ. Eq., **39** (2003), 1150–1158.
6. A. V. Pskhu, *Fractional Partial Differential Equations*, Moscow: Nauka, 2005.
7. M. O. Mamchuev, *A boundary value problem for a first-order equation with a partial derivative of a fractional order with variable coefficients*, Reports of Adyghe (Circassian) International Academy of Sciences, **11** (2009), 32–35.
8. M. O. Mamchuev, *Cauchy problem in non-local statement for first order equation with partial derivatives of fractional order with variable coefficients*, Reports of Adyghe (Circassian) International Academy of Sciences, **11** (2009), 21–24.
9. M. O. Mamchuev, *Boundary Value Problems for Equations and Systems with the Partial Derivatives of Fractional Order*, Nalchik: Publishing house KBSC of RAS, 2013.
10. R. Gorenflo, A. Iskenderov, Y. Luchko, *Mapping between solutions of fractional diffusion-wave equations*, Fract. Calc. Appl. Anal., **3** (2000), 75–86.
11. V. F. Morales-Delgado, M. A. Taneco-Hern'andez, J. F. G'omez-Aguilar, *On the solutions of fractional order of evolution equations*, EPJ Plus, **132** (2017), 47.

12. M. O. Mamchuev, *Boundary value problem for a system of fractional partial differential equations*, Differ. Eq., **44** (2008), 1737–1749.

13. M. O. Mamchuev, *Boundary value problem for a linear system of equations with the partial derivatives of fractional order*, Chelyabinsk Phys. Math. J., **2** (2017), 295–311.

14. A. Heibig, *Existence of solutions for a fractional derivative system of equations*, Integr. Equat. Oper. Th., **72** (2012), 483–508.

15. A. N. Kochubei, *Fractional-parabolic systems*, Potential Anal., **37** (2012), 1–30.

16. A. N. Kochubei, *Fractional-hyperbolic systems*, Fract. Calc. Appl. Anal., **16** (2013), 860–873.

17. M. O. Mamchuev, *Cauchy problem for the system of fractional partial equations of fractional order*, Bulletin KRASEC. Phys. Math. Sci., **23** (2018), 76–82.

18. M. O. Mamchuev, *Boundary value problems for a system of differential equations with partial derivatives of fractional order for unlimited domains*, Reports of Adyghe (Circassian) International Academy of Sciences, **6** (2003), 64–67.

19. M. O. Mamchuev, *Fundamental solution of a system of fractional partial differential equations*, Differ. Eq., **46** (2010), 1123–1134.

20. M. O. Mamchuev, *Cauchy problem in non-local statement for a system of fractional partial differential equations*, Differ. Eq., **48** (2012), 354–361.

21. M. O. Mamchuev, *Mixed problem for loaded system of equations with Riemann–Liouville derivatives*, Math. Notes, **97** (2015), 412–222.

22. M. O. Mamchuev, *Mixed problem for a system of fractional partial differential equations*, Differ. Eq., **52** (2016), 133–138.

23. M. O. Mamchuev, *Non-local boundary value problem for a system of equations with the partial derivatives of fractional order*, Math. Notes NEFU, **26** (2019), 23–31.

24. E. M. Wright, *On the coefficients of power series having exponential singularities*, J. London Math. Soc., **8** (1933), 71–79.

25. E. M. Wright, *The asymptotic expansion of the generalized Bessel function*, Proc. London Math. Soc. Ser. II, **38** (1934), 257–270.

26. R. Gorenflo, Y. Luchko, F. Mainardi, *Analytical properties and applications of the Wright function*, Fract. Calc. Appl. Anal., **2** (1999), 383–414.

AIMS Press