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Research article

New solitary wave solutions and stability analysis of the Benney-Luke and the Phi-4 equations in mathematical physics

Behzad Ghanbari[1], Mustafa Inc[2,*], Abdullahi Yusuf[3] and Dumitru Baleanu[4,5]

[1] Department of Engineering Science, Kermanshah University of Technology, Kermanshah, Iran
[2] Firat University, Science Faculty, Department of Mathematics, 23119 Elazig, Turkey
[3] Federal University Dutse, Science Faculty, Department of Mathematics, 7156 Dutse, Nigeria
[4] Cankaya University, Science Faculty, Department of Mathematics, 06530 Ankara, Turkey
[5] Institute of Space Sciences, Magurele, Romania

* **Correspondence:** Email: minc@firat.edu.tr; Tel: +905468036564.

Abstract: In this paper, we present new solitary wave solutions for the Benney-Luke equation (BLE) and Phi-4 equation (PE). The new generalized rational function method (GERFM) is used to reach such solutions. Moreover, the stability for the governing equations is investigated via the aspect of linear stability analysis. It is proved that, both the governing equations are stable. We can also solve other nonlinear system of PDEs which are involve in mathematical physics and many other branches of physical sciences with the help of this new method.

Keywords: Benney-Luke equation; Phi-4 equation; GERFM; stability analysis
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1. Introduction

Due to their very broad spectrum of applicability in nonlinear science, nonlinear evolution equations (NLEEs) were very significant elements. Nonlinear physical phenomena are among the most important areas of research in science and engineering, such as plasma physics, fluid mechanics, gas dynamics, elasticity, relativity, chemical responses, ecology, optical fiber, solid state physics, biomechanics to mention few. All these equations are fundamentally controlled by NLEEs [1–10]. NLEEs are frequently used to demonstrate separate wave motion.

It has been gaining more concentration ever since the arrival of the solitary wave in science aspects. Extracting precisely travelling wave alternatives to NLEEs is therefore essential. That's because getting accurate alternatives to NLEEs offers us the freedom to present data about the characteristics of complicated physical phenomenon. Thus, in the assessment of nonlinear physical phenomenon, the development of precise traveling wave solutions to NLEEs has become a concern. Several analytical methods were used to develop wave travel alternatives for NLEEs [11–31]. Full soliton stability is not yet mathematically or physically well understood. Although these solitons, owing to resonance with the continuous spectrum, have a natural tendency to leak energy, they can still withstand this inclination and stay strong. In this work, we will provide the exact travelling wave solutions and some dispersion relations for the governing equations [33].

As follows, the paper is organized. Section II presents the method descriptions. Section III discusses the method's applications to the governing equations. Analysis of stability is being studied in Section IV. The paper is concluded by Section V.

2. Description of the method

In this section, we will state the main steps of GERFM as follows [34]:

1. Let us take into account the NPDE in the form:

$$\mathcal{L}(\psi, \psi_x, \psi_t, \psi_{xx}, \ldots) = 0. \tag{2.1}$$

Using the transformations $\psi = \psi(\xi)$ and $\xi = \sigma x - lt$, Eq.(2.2) is reduced to following ODE as:

$$\mathcal{L}(\psi, \psi', \psi'', \ldots) = 0, \tag{2.2}$$

where the values of σ and l will be found later.

2. Suppose that solution of Eq. (2.2) is expressed by a finite series as:

$$\psi(\xi) = A_0 + \sum_{k=1}^{M} A_k \Theta(\xi)^k + \sum_{k=1}^{M} B_k \Theta(\xi)^{-k}. \tag{2.3}$$

where

$$\Theta(\xi) = \frac{p_1 e^{q_1 \xi} + p_2 e^{q_2 \xi}}{p_3 e^{q_3 \xi} + p_4 e^{q_4 \xi}}. \tag{2.4}$$

The values of constants $p_i, q_i (1 \leq i \leq 4)$, A_0, A_k and $B_k (1 \leq k \leq M)$ are determined, in such a way that solution (2.3) always persuade Eq. (2.2). By considering the homogenous balance principle the value of M is determined.

3. Putting Eq. (2.3) into Eq. (2.2) and rearranging the terms in Eq. (2.2) lead to an algebraic equations $P(Z_1, Z_2, Z_3, Z_4) = 0$ in terms of $Z_i = e^{q_i \xi}$ with $i = 1, \ldots, 4$. Equating the coefficients of P to zero, a system of nonlinear equations in terms of $p_i, q_i (1 \leq i \leq 4)$, and σ, l, A_0, A_k and $B_k (1 \leq k \leq M)$ is reached.
4. By solving the above system of equations using any symbolic computation software, the values of $p_i, q_i (1 \leq i \leq 4)$, A_0, A_k, and $B_k (1 \leq k \leq M)$ are determined, replacing these values in Eq. (2.3) provides us the soliton solutions of Eq. (2.1).

3. Application of the method

3.1. The Benny-Luke equation

Consider the BLE of the form [33]

$$u_{tt} - u_{xx} + \gamma u_{xxxx} - \delta u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0, \tag{3.1}$$

In order to find the solutions of Eq. (3.1), we utilize

$$u(x) = \mathbf{u}(\xi), \quad \xi = Kx + Lt, \tag{3.2}$$

where K and L are arbitrary constants to be determined.
If we use transformation (3.2) in Eq. (3.1), after an integration along with neglecting constant of integration, the following nonlinear ODE is obtained

$$\left(L^2 - K^2\right)\mathbf{u}' + K^2\left(\gamma K^2 - \delta L^2\right)\mathbf{u}''' + \frac{3}{2}LK^2(\mathbf{u}')^2 = 0, \tag{3.3}$$

Balancing the terms of $\mathbf{u}'''$ and $(\mathbf{u}')^2$ in Eq. (3.3) gives $M + 3 = 2(M + 1)$, and $M = 1$. Using $M = 1$ along with Eqs. (2.3) and (2.4), one gets:

$$\mathbf{u}(\xi) = A_0 + A_1\Phi(\xi) + \frac{B_1}{\Phi(\xi)}. \tag{3.4}$$

Using a methodology similar to the one adopted in Subsection 2, we get some solutions of (3.1), as bellows:

Family 1: We attain $p = [-1, -1, 1, -1]$ and $q = [1, -1, 1, -1]$, so we will obtain

$$\Phi(\xi) = -\frac{\cosh(\xi)}{\sinh(\xi)}. \tag{3.5}$$

Case 1:

$$K = K, L = -\frac{K\sqrt{4K^2\gamma - 1}}{\sqrt{4\delta K^2 - 1}}, A_0 = A_0, A_1 = -4\frac{K(\gamma - \delta)}{\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1}}, B_1 = 0.$$

So, the solitary wave solutions of Eq. (3.1) takes the form of

$$\mathbf{u}(\xi) = \frac{A_0\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1} - 4K\coth(\xi)(\delta - \gamma)}{\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1}}.$$

Thus the solution of (3.1) is obtained as

$$u_1(x, t) = \frac{A_0\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1} - 4K\coth(\xi)(\delta - \gamma)}{\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1}}, \tag{3.6}$$

where $\xi = K\left(x - \frac{\sqrt{4K^2\gamma-1}}{\sqrt{4\delta K^2-1}}t\right)$.

Case 2:

$$K = K, L = \frac{K\sqrt{16K^2\gamma - 1}}{\sqrt{16\delta K^2 - 1}},$$

$$A_0 = A_0, A_1 = \frac{4K(\gamma - \delta)}{\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}}, B_1 = \frac{4K(\gamma - \delta)}{\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}}.$$

So, the solitary wave solutions of Eq. (3.1) takes the form of

$$\mathbf{u}(\xi) = \frac{A_0\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}\coth(\xi) + 4K\left((\coth(\xi))^2 + 1\right)(\delta - \gamma)}{\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}\coth(\xi)}.$$

Therefore the solution of (3.1) is attained as

$$u_2(x,t) = \frac{A_0\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}\coth(\xi) + 4K\left((\coth(\xi))^2 + 1\right)(\delta - \gamma)}{\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}\coth(\xi)}, \tag{3.7}$$

where $\xi = K\left(x + \frac{\sqrt{16K^2\gamma-1}}{\sqrt{16\delta K^2-1}}t\right)$.

Case 3:

$$K = K, L = \frac{K\sqrt{4K^2\gamma - 1}}{\sqrt{4\delta K^2 - 1}}, A_0 = A_0, A_1 = 4\frac{K(\gamma - \delta)}{\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1}}, B_1 = 0.$$

So, the solitary wave solutions of Eq. (3.1) takes the form of

$$\mathbf{u}(\xi) = \frac{A_0\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1} + 4K\coth(\xi)(\delta - \gamma)}{\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1}}.$$

Thus we attain the solution of (3.1) as follows

$$u_3(x,t) = \frac{A_0\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1} + 4K\coth(\xi)(\delta - \gamma)}{\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1}}, \tag{3.8}$$

where $\xi = K\left(x + \frac{\sqrt{4K^2\gamma-1}}{\sqrt{4\delta K^2-1}}t\right)$.

Case 4:

$$K = K, L = -\frac{K\sqrt{16K^2\gamma - 1}}{\sqrt{16\delta K^2 - 1}},$$

$$A_0 = A_0, A_1 = -\frac{4K(\gamma - \delta)}{\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}}, B_1 = -\frac{4K(\gamma - \delta)}{\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}}.$$

So, the solitary wave solutions of Eq. (3.1) takes the form of

$$\mathbf{u}(\xi) = \frac{A_0\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}\coth(\xi) - 4K\left((\coth(\xi))^2 + 1\right)(\delta - \gamma)}{\sqrt{16\delta K^2 - 1}\sqrt{16K^2\gamma - 1}\coth(\xi)}.$$

And hence we attain the solution of (3.1) in the form

$$u_4(x,t) = \frac{A_0\sqrt{16\delta K^2-1}\sqrt{16K^2\gamma-1}\coth(\xi) - 4K\left((\coth(\xi))^2+1\right)(\delta-\gamma)}{\sqrt{16\delta K^2-1}\sqrt{16K^2\gamma-1}\coth(\xi)}, \tag{3.9}$$

where $\xi = K\left(x - \frac{\sqrt{16K^2\gamma-1}}{\sqrt{16\delta K^2-1}}t\right)$.

Family 2: We attain $p = [-i, -i, 1, 1]$ and $q = [i, -i, i, -i]$, so we will obtain

$$\Phi(\xi) = -\frac{\sin(\xi)}{\cos(\xi)}. \tag{3.10}$$

Case 1:

$$K = K, L = -\frac{K\sqrt{4K^2\gamma+1}}{\sqrt{4\delta K^2+1}}, A_0 = A_0, A_1 = -\frac{4K(\gamma-\delta)}{\sqrt{4\delta K^2+1}\sqrt{4K^2\gamma+1}}, B_1 = 0.$$

Inserting these values in Eqs.(3.4) and (3.10), one gets

$$\mathbf{u}(\xi) = \frac{A_0\sqrt{4\delta K^2+1}\sqrt{4K^2\gamma+1}\cos(\xi) - 4K\sin(\xi)(\delta-\gamma)}{\cos(\xi)\sqrt{4K^2\gamma+1}\sqrt{4\delta K^2+1}}.$$

Therefore we attain the solution of (3.1) as follows

$$u_5(x,t) = \frac{A_0\sqrt{4\delta K^2+1}\sqrt{4K^2\gamma+1}\cos(\xi) - 4K\sin(\xi)(\delta-\gamma)}{\cos(\xi)\sqrt{4K^2\gamma+1}\sqrt{4\delta K^2+1}}, \tag{3.11}$$

where $\xi = K\left(x - \frac{\sqrt{4K^2\gamma+1}}{\sqrt{4\delta K^2+1}}t\right)$.

Case 2:

$$K = K, L = \frac{K\sqrt{16K^2\gamma+1}}{\sqrt{16\delta K^2+1}},$$

$$A_0 = A_0, A_1 = \frac{4K(\gamma-\delta)}{\sqrt{16\delta K^2+1}\sqrt{16K^2\gamma+1}}, B_1 = -\frac{4K(\gamma-\delta)}{\sqrt{16\delta K^2+1}\sqrt{16K^2\gamma+1}}.$$

Plugging these values in Eqs.(3.4) and (3.10), one gets

$$\mathbf{u}(\xi) = \frac{A_0\sqrt{16\delta K^2+1}\sqrt{16K^2\gamma+1}\cos(\xi)\sin(\xi) - 8\left(\cos^2(\xi)-1/2\right)(\delta-\gamma)K}{\cos(\xi)\sin(\xi)\sqrt{16\delta K^2+1}\sqrt{16K^2\gamma+1}}.$$

Thus the solution of (3.1) is given by

$$u_6(x,t) = \frac{A_0\sqrt{16\delta K^2+1}\sqrt{16K^2\gamma+1}\cos(\xi)\sin(\xi) - 8\left(\cos^2(\xi)-1/2\right)(\delta-\gamma)K}{\cos(\xi)\sin(\xi)\sqrt{16\delta K^2+1}\sqrt{16K^2\gamma+1}}, \tag{3.12}$$

where $\xi = K\left(x + \frac{\sqrt{16K^2\gamma+1}}{\sqrt{16\delta K^2+1}}t\right)$.

Family 3: We attain $p = [-2 - i, - - 2 + i, 1, 1]$ and $q = [i, -i, i, -i]$, so we will obtain

$$\Phi(\xi) = \frac{\sin(\xi) - 2\cos(\xi)}{\cos(\xi)}. \tag{3.13}$$

Case 1:

$$K = K, L = -\frac{K\sqrt{4K^2\gamma+1}}{\sqrt{4\delta K^2+1}}, A_0 = A_0, A_1 = 0, B_1 = -\frac{20K(\gamma-\delta)}{\sqrt{4\delta K^2+1}\sqrt{4K^2\gamma+1}}.$$

Imposing these values in Eqs. (3.4) and (3.13), one gets

$$\mathbf{u}(\xi) = \frac{-20K\cos(\xi)\sqrt{4K^2\gamma+1}(\delta-\gamma)\sqrt{4\delta K^2+1} + E_1A_0\left(4K^2\gamma+1\right)\left(4\delta K^2+1\right)}{(4\delta K^2+1)(4K^2\gamma+1)(-\sin(\xi)+2\cos(\xi))},$$

where $E_1 = (2\cos(\xi) - \sin(\xi))$, we then reach the solution of (3.1 stated as

$$u_7(x,t) = \frac{-20K\cos(\xi)\sqrt{4K^2\gamma+1}(\delta-\gamma)\sqrt{4\delta K^2+1} + E_1A_0\left(4K^2\gamma+1\right)\left(4\delta K^2+1\right)}{(4\delta K^2+1)(4K^2\gamma+1)(-\sin(\xi)+2\cos(\xi))}, \tag{3.14}$$

where $\xi = K\left(x - \frac{\sqrt{4K^2\gamma+1}}{\sqrt{4\delta K^2+1}}t\right)$.

Case 2:

$$K = K, L = \frac{K\sqrt{4K^2\gamma+1}}{\sqrt{4\delta K^2+1}}, A_0 = A_0, A_1 = -\frac{4K(\gamma-\delta)}{\sqrt{4\delta K^2+1}\sqrt{4K^2\gamma+1}}, B_1 = 0.$$

Inserting these values in Eqs. (3.4) and (3.13), one reaches

$$\mathbf{u}(\xi) = \frac{-4(2\cos(\xi)-\sin(\xi))\sqrt{4K^2\gamma+1}(\delta-\gamma)K\sqrt{4\delta K^2+1} + \cos(\xi)E_2}{(4\delta K^2+1)(4K^2\gamma+1)\cos(\xi)},$$

where $E_2 = A_0\left(4K^2\gamma+1\right)\left(4\delta K^2+1\right)$. Therefore we attain the solution of (3.1) as

$$u_8(x,t) = \frac{-4(2\cos(\xi)-\sin(\xi))\sqrt{4K^2\gamma+1}(\delta-\gamma)K\sqrt{4\delta K^2+1} + \cos(\xi)E_2}{(4\delta K^2+1)(4K^2\gamma+1)\cos(\xi)}, \tag{3.15}$$

where $\xi = K\left(x + \frac{\sqrt{4K^2\gamma+1}}{\sqrt{4\delta K^2+1}}t\right)$.

Family 4: We attain $p = [1 - i, 1 + i, -1, 1]$ and $q = [i, -i, i, -i]$, so we will obtain

$$\Phi(\xi) = \frac{\cos(\xi) + \sin(\xi)}{\cos(\xi)}. \tag{3.16}$$

Case 1:

$$K = K, L = -\frac{K\sqrt{4K^2\gamma + 1}}{\sqrt{4\delta K^2 + 1}}, A_0 = A_0, A_1 = \frac{4K(\gamma - \delta)}{\sqrt{4\delta K^2 + 1}\sqrt{4K^2\gamma + 1}}, B_1 = 0.$$

Putting these values in Eqs.(3.4) and (3.16), one gets

$$\mathbf{u}(\xi) = \frac{-4K\sqrt{4K^2\gamma + 1}(\cos(\xi) + \sin(\xi))(\delta - \gamma)\sqrt{4\delta K^2 + 1} + \cos(\xi)E_3}{(4\delta K^2 + 1)(4K^2\gamma + 1)\cos(\xi)},$$

where $E_3 = A_0\left(4K^2\gamma + 1\right)\left(4\delta K^2 + 1\right)$. Therefore we attain the solution of (3.1) in the following form

$$u_9(x,t) = \frac{-4K\sqrt{4K^2\gamma + 1}(\cos(\xi) + \sin(\xi))(\delta - \gamma)\sqrt{4\delta K^2 + 1} + \cos(\xi)E_3}{(4\delta K^2 + 1)(4K^2\gamma + 1)\cos(\xi)}, \tag{3.17}$$

where $\xi = K\left(x - \frac{\sqrt{4K^2\gamma+1}}{\sqrt{4\delta K^2+1}}t\right)$.

Case 2:

$$K = K, L = -\frac{K\sqrt{4K^2\gamma + 1}}{\sqrt{4\delta K^2 + 1}}, A_0 = A_0, A_1 = 0, B_1 = -\frac{8K(\gamma - \delta)}{\sqrt{4\delta K^2 + 1}\sqrt{4K^2\gamma + 1}}.$$

So, the solitary wave solutions of Eq. (3.1) takes the form of

$$\mathbf{u}(\xi) = \frac{8K\cos(\xi)\sqrt{4K^2\gamma + 1}(\delta - \gamma)\sqrt{4\delta K^2 + 1} + (\cos(\xi) + \sin(\xi))E_4}{(4\delta K^2 + 1)(4K^2\gamma + 1)(\cos(\xi) + \sin(\xi))},$$

$E_4 = A_0\left(4K^2\gamma + 1\right)\left(4\delta K^2 + 1\right)$. Therefore one can reach the solution of (3.1) as comes next

$$u_{10}(x,t) = \frac{8K\cos(\xi)\sqrt{4K^2\gamma + 1}(\delta - \gamma)\sqrt{4\delta K^2 + 1} + (\cos(\xi) + \sin(\xi))E_4}{(4\delta K^2 + 1)(4K^2\gamma + 1)(\cos(\xi) + \sin(\xi))}, \tag{3.18}$$

where $\xi = K\left(x - \frac{\sqrt{4K^2\gamma+1}}{\sqrt{4\delta K^2+1}}t\right)$.

Family 5: We attain $p = [-1, 3, 1, -1]$ and $q = [1, -1, 1, -1]$, so we will obtain

$$\Phi(\xi) = \frac{\cosh(\xi) - 2\sinh(\xi)}{\sinh(\xi)}. \tag{3.19}$$

Case 1:

$$K = K, L = -\frac{K\sqrt{4K^2\gamma - 1}}{\sqrt{4\delta K^2 - 1}}, A_0 = A_0, A_1 = \frac{4K(\gamma - \delta)}{\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1}}, B_1 = 0.$$

Imposing these values in Eqs. (3.4) and (3.19), one gets

$$\mathbf{u}(\xi) = \frac{-4K\sqrt{4K^2\gamma - 1}(\cosh(\xi) - 2\sinh(\xi))(\delta - \gamma)\sqrt{4\delta K^2 - 1} + A_0E_5}{(4\delta K^2 - 1)(4K^2\gamma - 1)\sinh(\xi)},$$

where $E_5 = \sinh(\xi)\left(4K^2\gamma - 1\right)\left(4\delta K^2 - 1\right)$. Therefore we attain the solution of (3.1) as

$$u_{11}(x,t) = \frac{-4K\sqrt{4K^2\gamma - 1}\left(\cosh(\xi) - 2\sinh(\xi)\right)(\delta - \gamma)\sqrt{4\delta K^2 - 1} + A_0 E_5}{\left(4\delta K^2 - 1\right)\left(4K^2\gamma - 1\right)\sinh(\xi)}, \tag{3.20}$$

where $\xi = K\left(x - \frac{\sqrt{4K^2\gamma-1}}{\sqrt{4\delta K^2-1}}t\right)$.

Family 6: We attain $p = [-3, -1, 1, 1]$ and $q = [1, -1, 1, -1]$, so we will obtain

$$\Phi(\xi) = \frac{-2\cosh(\xi) - \sinh(\xi)}{\cosh(\xi)}. \tag{3.21}$$

Case 1:

$$K = K, L = \frac{K\sqrt{4K^2\gamma - 1}}{\sqrt{4\delta K^2 - 1}}, A_0 = A_0, A_1 = \frac{4K(\gamma - \delta)}{\sqrt{4\delta K^2 - 1}\sqrt{4K^2\gamma - 1}}, B_1 = 0.$$

Putting these values in Eqs.(3.4) and (3.21), one gets

$$\mathbf{u}(\xi) = \frac{4\left(2\cosh(\xi) + \sinh(\xi)\right)\sqrt{4K^2\gamma - 1}(\delta - \gamma)K\sqrt{4\delta K^2 - 1} + \cosh(\xi)}{\left(4\delta K^2 - 1\right)\left(4K^2\gamma - 1\right)\cosh(\xi)},$$

$E_6 = \left(4\delta K^2 - 1\right)A_0\left(4K^2\gamma - 1\right)$. Therefore we reach the solution of (3.1) as follows

$$u_{12}(x,t) = \frac{4\left(2\cosh(\xi) + \sinh(\xi)\right)\sqrt{4K^2\gamma - 1}(\delta - \gamma)K\sqrt{4\delta K^2 - 1} + \cosh(\xi)E_6}{\left(4\delta K^2 - 1\right)\left(4K^2\gamma - 1\right)\cosh(\xi)}, \tag{3.22}$$

where $\xi = K\left(x + \frac{\sqrt{4K^2\gamma-1}}{\sqrt{4\delta K^2-1}}t\right)$.

Family 7: We attain $p = [-3, -2, 1, 1]$ and $q = [0, 1, 0, 1]$, so we will obtain

$$\Phi(\xi) = \frac{-3 - 2e^{\xi}}{1 + e^{\xi}}. \tag{3.23}$$

Case 1:

$$K = K, L = -\frac{K\sqrt{K^2\gamma - 1}}{\sqrt{\delta K^2 - 1}}, A_0 = A_0, A_1 = 0, B_1 = -\frac{24K(\gamma - \delta)}{\sqrt{\delta K^2 - 1}\sqrt{K^2\gamma - 1}}.$$

Plugging these values in Eqs.(3.4) and (3.23), one attains

$$\mathbf{u}(\xi) = \frac{-24K\sqrt{K^2\gamma - 1}\left(1 + e^{\xi}\right)(\delta - \gamma)\sqrt{\delta K^2 - 1} + 2\left(\delta K^2 - 1\right)A_0 E_7}{\left(\delta K^2 - 1\right)\left(K^2\gamma - 1\right)\left(3 + 2e^{\xi}\right)},$$

where $E_7 = \left(K^2\gamma - 1\right)\left(3/2 + e^{\xi}\right)$. Therefore we attain the solution of (3.1) as

$$u_{13}(x,t) = \frac{-24K\sqrt{K^2\gamma - 1}\left(1 + e^{\xi}\right)(\delta - \gamma)\sqrt{\delta K^2 - 1} + 2\left(\delta K^2 - 1\right)A_0 E_7}{\left(\delta K^2 - 1\right)\left(K^2\gamma - 1\right)\left(3 + 2e^{\xi}\right)}, \tag{3.24}$$

where $\xi = K\left(x - \frac{\sqrt{K^2\gamma-1}}{\sqrt{\delta K^2-1}}t\right)$.

Family 8: We attain $p = [1, 0, 1, 1]$ and $q = [1, 0, 1, 0]$, so we will obtain

$$\Phi(\xi) = \frac{e^{\xi}}{e^{\xi} + 1}. \tag{3.25}$$

Case 1:

$$K = K, L = \frac{K\sqrt{K^2\gamma - 1}}{\sqrt{\delta K^2 - 1}}, A_0 = A_0, A_1 = -\frac{4K(\gamma - \delta)}{\sqrt{\delta K^2 - 1}\sqrt{K^2\gamma - 1}}, B_1 = 0.$$

Putting these values in Eqs.(3.4) and (3.25), one reaches

$$\mathbf{u}(\xi) = \frac{4Ke^{\xi}\sqrt{K^2\gamma - 1}(\delta - \gamma)\sqrt{\delta K^2 - 1} + A_0\left(\delta K^2 - 1\right)\left(K^2\gamma - 1\right)\left(1 + e^{\xi}\right)}{(\delta K^2 - 1)(K^2\gamma - 1)(1 + e^{\xi})}.$$

Therefore we attain the solution of (3.1) as comes next

$$u_{14}(x, t) = \frac{4Ke^{\xi}\sqrt{K^2\gamma - 1}(\delta - \gamma)\sqrt{\delta K^2 - 1} + A_0\left(\delta K^2 - 1\right)\left(K^2\gamma - 1\right)\left(1 + e^{\xi}\right)}{(\delta K^2 - 1)(K^2\gamma - 1)(1 + e^{\xi})}, \tag{3.26}$$

where $\xi = K\left(x + \frac{\sqrt{K^2\gamma-1}}{\sqrt{\delta K^2-1}}t\right)$.

Family 9: We attain $p = [2, 1, 1, 1]$ and $q = [1, 0, 1, 0]$, so we will obtain

$$\Phi(\xi) = \frac{2e^{\xi} + 1}{e^{\xi} + 1}. \tag{3.27}$$

Case 1:

$$K = K, L = -\frac{K\sqrt{K^2\gamma - 1}}{\sqrt{\delta K^2 - 1}}, A_0 = A_0, A_1 = \frac{4K(\gamma - \delta)}{\sqrt{\delta K^2 - 1}\sqrt{K^2\gamma - 1}}, B_1 = 0.$$

Putting these values in Eqs.(3.4) and (3.27), we have

$$\mathbf{u}(\xi) = \frac{-4\sqrt{K^2\gamma - 1}\left(2e^{\xi} + 1\right)(\delta - \gamma)K\sqrt{\delta K^2 - 1} + A_0\left(\delta K^2 - 1\right)\left(K^2\gamma - 1\right)\left(1 + e^{\xi}\right)}{(\delta K^2 - 1)(K^2\gamma - 1)(1 + e^{\xi})}.$$

Therefore we reach the solution of (3.1) as comes next

$$u_{15}(x, t) = \frac{-4\sqrt{K^2\gamma - 1}\left(2e^{\xi} + 1\right)(\delta - \gamma)K\sqrt{\delta K^2 - 1} + A_0\left(\delta K^2 - 1\right)\left(K^2\gamma - 1\right)\left(1 + e^{\xi}\right)}{(\delta K^2 - 1)(K^2\gamma - 1)(1 + e^{\xi})}, \tag{3.28}$$

where $\xi = K\left(x - \frac{\sqrt{K^2\gamma-1}}{\sqrt{\delta K^2-1}}t\right)$.

Family 10: We attain $p = [-1, 0, 1, 1]$ and $q = [0, 1, 0, 1]$, so we will obtain

$$\Phi(\xi) = -\frac{1}{e^{\xi} + 1}. \tag{3.29}$$

Case 1:

$$K = K, L = \frac{K\sqrt{K^2\gamma - 1}}{\sqrt{\delta K^2 - 1}}, A_0 = A_0, A_1 = -\frac{4K(\gamma - \delta)}{\sqrt{\delta K^2 - 1}\sqrt{K^2\gamma - 1}}, B_1 = 0.$$

Employing these values to Eqs. (3.4) and (3.29), one reaches

$$\mathbf{u}(\xi) = \frac{-4K\sqrt{K^2\gamma - 1}(\delta - \gamma)\sqrt{\delta K^2 - 1} + A_0\left(\delta K^2 - 1\right)\left(K^2\gamma - 1\right)\left(1 + e^{\xi}\right)}{(\delta K^2 - 1)(K^2\gamma - 1)(1 + e^{\xi})}.$$

Therefore we attain the solution of (3.1) is as comes next

$$u_{16}(x, t) = \frac{-4K\sqrt{K^2\gamma - 1}(\delta - \gamma)\sqrt{\delta K^2 - 1} + A_0\left(\delta K^2 - 1\right)\left(K^2\gamma - 1\right)\left(1 + e^{\xi}\right)}{(\delta K^2 - 1)(K^2\gamma - 1)(1 + e^{\xi})}, \tag{3.30}$$

where $\xi = K\left(x + \frac{\sqrt{K^2\gamma - 1}}{\sqrt{\delta K^2 - 1}}t\right)$.

3.2. *The Phi-4 equation*

Now, let us Consider the PE given by [33]

$$u_{tt} - u_{xx} + m^2 u + \sigma u^3 = 0, \tag{3.31}$$

To solve Eq. (3.31), we again apply the travelling wave transformation (3.2). Then Eq. (3.31) turns to the following nonlinear ODE as

$$\left(L^2 - K^2\right)\mathbf{u}'' + m^2\mathbf{u} + \sigma\mathbf{u}^3 = 0, \tag{3.32}$$

Balancing the terms of $\mathbf{u}''$ and $\mathbf{u}^3$ in Eq. (3.32) gives $3N = N + 2$, so $N = 1$. So, the solution of Eq. (3.31) will be as:

$$\mathbf{u}(\xi) = A_0 + A_1\Phi(\xi) + \frac{B_1}{\Phi(\xi)}. \tag{3.33}$$

Family 1: We attain $p = [1 + i, 1 - i, 1, -1]$ and $q = [i, -i, i, -i]$, so we will obtain

$$\Phi(\xi) = \frac{-\sin(\xi) + \cos(\xi)}{\cos(\xi)}. \tag{3.34}$$

Case 1:

$$K = K, L = \frac{\sqrt{4K^2 - 2m^2}}{2}, A_0 = -\frac{m}{\sqrt{\sigma}}, A_1 = 0, B_1 = \frac{2m}{\sqrt{\sigma}}.$$

Putting these values into Eqs. (3.33) and (3.34), one gets

$$\mathbf{u}(\xi) = \frac{m(\cos(\xi) + \sin(\xi))}{\sqrt{\sigma}(-\sin(\xi) + \cos(\xi))}.$$

Therefore we attain the solution of (3.31) given by

$$u_1(x,t) = \frac{m\left(\cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right) + \sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)\right)}{\sqrt{\sigma}\left(-\sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right) + \cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)\right)}. \tag{3.35}$$

Case 2:

$$K = K, L = \frac{\sqrt{4K^2 - 2m^2}}{2}, A_0 = \frac{m}{\sqrt{\sigma}}, A_1 = -\frac{m}{\sqrt{\sigma}}, B_1 = 0.$$

Inserting these values into Eqs. (3.33) and (3.34), one gets

$$\mathbf{u}(\xi) = \frac{m\sin(\xi)}{\sqrt{\sigma}\cos(\xi)}.$$

Therefore we attain the solution of (3.31) stated as

$$u_2(x,t) = \frac{m\sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)}{\sqrt{\sigma}\cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)}. \tag{3.36}$$

Family 2: We attain $p = [1-i, 1-i, 1, 1]$ and $q = [i, -i, i, -i]$, so we will obtain

$$\Phi(\xi) = \frac{\cos(\xi) + \sin(\xi)}{\cos(\xi)}. \tag{3.37}$$

Case 1:

$$K = K, L = \frac{\sqrt{4K^2 - 2m^2}}{2}, A_0 = \frac{m}{\sqrt{\sigma}}, A_1 = 0, B_1 = -2\frac{m}{\sqrt{\sigma}}.$$

Putting these values into Eqs. (3.33) and (3.37), one gets

$$\mathbf{u}(\xi) = -\frac{m(-\sin(\xi) + \cos(\xi))}{\sqrt{\sigma}(\cos(\xi) + \sin(\xi))}.$$

Therefore we attain the solution of (3.31) as comes next

$$u_3(x,t) = -\frac{m\left(-\sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right) + \cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)\right)}{\sqrt{\sigma}\left(\cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right) + \sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)\right)}. \tag{3.38}$$

Family 3: We attain $p = [-2 - i, -2 + i, 1, 1]$ and $q = [i, -i, i, -i]$, so we will obtain

$$\Phi(\xi) = \frac{-2\cos(\xi) + \sin(\xi)}{\cos(\xi)}. \tag{3.39}$$

Case 1:

$$K = K, L = \frac{\sqrt{4K^2 - 2m^2}}{2}, A_0 = -\frac{2m}{\sqrt{\sigma}}, A_1 = 0, B_1 = -\frac{5m}{\sqrt{\sigma}}$$

Imposing these values into Eqs. (3.33) and (3.39), one gets

$$\mathbf{u}(\xi) = \frac{m(2\sin(\xi) + \cos(\xi))}{\sqrt{\sigma}(2\cos(\xi) - \sin(\xi))}.$$

Hence we reach the solution of (3.31) as follows

$$u_4(x,t) = \frac{m\left(2\sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right) + \cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)\right)}{\sqrt{\sigma}\left(2\cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right) - \sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)\right)}. \tag{3.40}$$

Family 4: We attain $p = [1 - i, -1 - i, -1, 1]$ and $q = [i, -i, i, -i]$, so we will obtain

$$\Phi(\xi) = \frac{-\sin(\xi) + \cos(\xi)}{\sin(\xi)}. \tag{3.41}$$

Case 1:

$$K = K, L = \frac{\sqrt{4K^2 - 2m^2}}{2}, A_0 = \frac{m}{\sqrt{\sigma}}, A_1 = \frac{m}{\sqrt{\sigma}}, B_1 = 0.$$

Putting these values into Eqs. (3.33) and (3.41), one gets

$$\mathbf{u}(\xi) = u(\xi) = \frac{m\cos(\xi)}{\sqrt{\sigma}\sin(\xi)}.$$

Therefore we attain the solution of (3.31) as comes next

$$u_5(x,t) = u(\xi) = \frac{m\cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)}{\sqrt{\sigma}\sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)}. \tag{3.42}$$

Family 5: We attain $p = [2 - i, -2 - i, -1, 1]$ and $q = [i, -i, i, -i]$, so we will obtain

$$\Phi(\xi) = \frac{\cos(\xi) - 2\sin(\xi)}{\sin(\xi)}. \tag{3.43}$$

Case 1:

$$K = K, L = \frac{\sqrt{4K^2 - 2m^2}}{2}, A_0 = \frac{2m}{\sqrt{\sigma}}, A_1 = 0, B_1 = \frac{5m}{\sqrt{\sigma}}.$$

Inserting these values into Eqs. (3.33) and (3.43), one gets

$$\mathbf{u}(\xi) = \frac{m(\sin(\xi) + 2\cos(\xi))}{\sqrt{\sigma}(\cos(\xi) - 2\sin(\xi))}.$$

Therefore we attain the solution of (3.31) given as

$$u_6(x,t) = \frac{m\left(\sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right) + 2\cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)\right)}{\sqrt{\sigma}\left(\cos\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right) - 2\sin\left(Kx + \frac{\sqrt{4K^2-2m^2}}{2}\right)\right)}. \tag{3.44}$$

Family 6: We attain $p = [i, -i, 1, 1]$ and $q = [i, -i, i, -i]$, so we will obtain

$$\Phi(\xi) = -\frac{\sin(\xi)}{\cos(\xi)}. \tag{3.45}$$

Case 1:

$$K = K, L = \frac{\sqrt{16K^2 - 2m^2}}{4}, A_0 = 0, A_1 = -\frac{m}{2\sqrt{\sigma}}, B_1 = \frac{m}{2\sqrt{\sigma}}.$$

Employing these values into Eqs. (3.33) and (3.45), one gets

$$\mathbf{u}(\xi) = -\frac{\left(2(\cos(\xi))^2 - 1\right)m}{2\sqrt{\sigma}\cos(\xi)\sin(\xi)}.$$

Therefore we attain the solution of (3.31) is as comes next

$$u_7(x,t) = -\frac{\left(2\left(\cos\left(Kx + \frac{\sqrt{16K^2-2m^2}}{4}t\right)\right)^2 - 1\right)m}{2\sqrt{\sigma}\cos\left(Kx + \frac{\sqrt{16K^2-2m^2}}{4}t\right)\sin\left(Kx + \frac{\sqrt{16K^2-2m^2}}{4}t\right)}. \tag{3.46}$$

Family 7: We attain $p = [-1, -1, 1, -1]$ and $q = [1, -1, 1, -1]$, so we will obtain

$$\Phi(\xi) = -\frac{\cosh(\xi)}{\sinh(\xi)}. \tag{3.47}$$

Case 1:

$$K = K, L = \frac{\sqrt{4K^2 - m^2}}{2}, A_0 = 0, A_1 = 1/2\,\frac{\sqrt{2}m}{\sqrt{\sigma}}, B_1 = -1/2\,\frac{\sqrt{2}m}{\sqrt{\sigma}}.$$

Putting these values into Eqs. (3.33) and (3.47), one gets

$$\mathbf{u}(\xi) = -\frac{\sqrt{2}m}{2\sqrt{\sigma}\sinh(\xi)\cosh(\xi)}.$$

Therefore we attain the solution of (3.31) as follows

$$u_8(x,t) = -\frac{\sqrt{2}m}{2\sqrt{\sigma}\sinh\left(Kx + \frac{\sqrt{4K^2-m^2}}{2}t\right)\cosh\left(Kx + \frac{\sqrt{4K^2-m^2}}{2}t\right)}. \tag{3.48}$$

4. Stability analysis

In this section, the stability analysis [35–37] for the governing equations that is (3.1) and (3.31) will be analyzed.

4.1. Stability for Eq. (3.1)

Consider the perturbed solution of the form [37]

$$u(x,t) = \tau w(x,t) + P_0, \tag{4.1}$$

it can be easily seen that, any constant P_0 is a steady state solution for (3.1). Plugging (4.1) into (3.1), one arrives at

$$\tau w_{tt} + 2\tau^2 w_x w_{xt} - \tau w_{xx} + \tau^2 w_t w_{xx} - \delta\tau w_{xxtt} + \gamma\tau w_{xxxx} = 0, \tag{4.2}$$

linearizing the above equation in τ, we reach

$$\tau w_{tt} - \tau w_{xx} - \delta\tau w_{xxtt} + \gamma\tau w_{xxxx} = 0. \tag{4.3}$$

Assume that (4.3) has a solution of the form

$$w(x,t) = e^{i(kx+t\omega)}, \tag{4.4}$$

where k is the normalized wave number, plugging (4.4) into (4.3)

$$\gamma k^4 + k^2\left(1 - \delta\omega^2\right) - \omega^2 = 0, \tag{4.5}$$

solving for ω, we obtain

$$\omega = -\sqrt{\frac{\gamma k^4 + k^2}{\delta k^2 + 1}}. \tag{4.6}$$

From (4.6), one can see that the real part is negative for all k values, then any superposition of the solutions will appear to decay. Thus, the dispersion is stable.

4.2. Stability for Eq. (3.31)

In a similar manner, consider the perturbed solution of the form

$$u(x,t) = \tau w(x,t) + P_1, \tag{4.7}$$

it is plainly to see that in (4.7) any constant P_1 is a steady state solution for (3.31). Plugging (4.7) into (3.31), one gets

$$m^2(\tau w(x,t) + P_1) + \sigma(\tau w(x,t) + P_1)^3 + \tau w_{tt} - \tau w_{xx} = 0, \tag{4.8}$$

linearizing the above equation in τ, one gets

$$m^2\tau w(x,t) + 3\sigma\tau P_1^2 w(x,t) + \tau w_{tt} - \tau w_{xx} = 0. \tag{4.9}$$

Suppose again that (4.9) has a formal solution of the form (4.4), inserting (4.4) into (4.9), one gets

$$k^2 + m^2 + 3P_1^2\sigma - \omega^2 = 0, \tag{4.10}$$

solving for ω from the immediate equation, we acquire

$$\omega = -\sqrt{k^2 + m^2 + 3P_1^2\sigma}. \tag{4.11}$$

From (4.11), it can be seen that the real part is negative for all k, m, P_1^2 values. Thus, any superposition of the solutions will appear to decay. Hence, the dispersion is stable.

5. Conclusion

In this work, we present new solitary wave solutions for the BLE and PE. We applied the new GERFM to reach such solutions. Moreover, the stability for the governing equations is investigated via the aspect of linear stability analysis. It has been proved that, both the governing equations are stable. These new families of solutions are shown the power, effectiveness and fruitfulness of this method. These fresh solutions have many applications in physics and other physical sciences branches. Other nonlinear PDEs involving mathematical physics and other different branches of physical sciences can also be solved through this method.

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Conflict of interest

All authors declared that there is no conflict of interest in this paper.

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