**Mathematics**

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(3): 997–1018.
DOI:10.3934/math.2019.3.997
Received: 30 May 2019
Accepted: 02 July 2019
Published: 31 July 2019

*Research article*

# Application of the fixed point theorems on the existence of solutions for $q-$fractional boundary value problems

**Sina Etemad**[1] **and Sotiris K. Ntouyas**[2,3,*]

[1] Department of Mathematics, Azarbaijan Shahid Madani University, Azarshahr, Tabriz, Iran
[2] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
[3] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: sntouyas@uoi.gr.

**Abstract:** In this paper, we study the existence of solutions for nonlinear fractional $q-$difference equations and inclusions. We apply some known fixed point theorems to prove the existence results. Finally, some illustrative examples are presented to state the validity of our main results.

**Keywords:** fractional $q-$difference equation and inclusion; boundary value problem; existence; fixed point theorem
**Mathematics Subject Classification:** 34A08, 34A60, 39A12, 39A13

## 1. Introduction

The classical fractional calculus is one of the branches of applied mathematics which has merged in pure mathematics. In fact, the topic of fractional differential equations and inclusions, as one of the subjects of the fractional calculus, is an important and effective field of research which the basic techniques of the functional analysis and fixed point theory were used to proving the existence and uniqueness of solutions for this kind of differential equations and inclusions. On the other hand, the extensiveness and importance of this topic has been caused to publish the many works and papers by other researchers (for example, see [2, 3, 5, 7, 9, 12–15, 19, 32–34].

Later, $q-$difference calculus or quantum calculus, as a generalization of the classical calculus, has gained considerable attention of researchers and mathematicians. The first work on the subject of $q-$difference calculus dates back to Jackson's works [25].

In recent years, many researchers have studied and published various and distinct papers on the existence theory of fractional $q-$difference equations and inclusions (for examples, see [1, 4, 6, 17, 18, 20–23, 26, 29, 30, 35, 37, 39, 41]).

In [8], Bashir Ahmad *et al.* studied the existence of solutions for the nonlocal boundary value problem of fractional $q-$difference equation

$$
\begin{gathered}
{}^cD_q^\alpha x(t) = f(t, x(t)), \qquad 0 \le t \le 1, \qquad 1 < \alpha \le 2, \\
\alpha_1 x(0) - \beta_1 D_q x(0) = \gamma_1 x(\eta_1), \quad \alpha_2 x(1) + \beta_2 D_q x(1) = \gamma_2 x(\eta_2),
\end{gathered}
$$

where, ${}^cD_q^\alpha$ is the fractional $q-$derivative of the Caputo type and $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$. Authors in that paper, by applying the Banach contraction principle, Krasnoselskii's fixed point theorem and the Leray-Schauder nonlinear alternative studied the existence results.

In [40], Zhao *et al.* dealt with the following nonlinear fractional $q-$difference equation with the nonlocal $q-$integral boundary value conditions:

$$
\begin{gathered}
D_q^\alpha x(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \ 1 < \alpha \le 2, \\
x(0) = 0, \quad x(1) = \mu I_q^\beta x(\eta), \qquad 0 < \beta \le 2,
\end{gathered}
$$

where, $D_q^\alpha$ is the fractional $q-$derivative of Riemann-Liouville type and $\mu > 0$. They studied the existence of solutions for the above problem by using the generalized Banach contraction principle, the monotone iterative method and Krasnoselskii's fixed point theorem.

In 2015, Alsaedi, Ntouyas and Ahmad investigated the fractional $q-$difference inclusion with nonlocal and substrip type boundary conditions

$$
\begin{gathered}
{}^cD_q^\nu x(t) \in F(t, x(t)), \quad 0 \le t \le 1, \quad 1 < \nu \le 2, \\
x(0) = g(x), \qquad x(w) = b \int_\delta^1 x(s) d_q s \qquad 0 < w < \delta < 1,
\end{gathered}
$$

where ${}^cD_q^\nu$ denotes the Caputo fractional $q-$derivative of order $\nu$ [10].

Motivated by the above papers, in this paper, we discuss the existence of solutions for fractional $q-$difference equation

$$
\begin{cases}
{}^cD_q^\alpha u(t) = f(t, u(t), D_q u(t)), & 0 \le t \le 1, \ 0 < q < 1, \\
u(0) = 0, \quad D_q u(1) = 0, \quad D_q^2 u(1) = 0,
\end{cases} \tag{1.1}
$$

where, ${}^cD_q^\alpha$ denotes the fractional $q-$derivative of the Caputo type of order $\alpha$ and $\alpha \in (2, 3]$ and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous mapping.

Also, we study the existence of solutions for the following fractional $q-$difference inclusion

$$
\begin{cases}
{}^cD_q^\alpha u(t) \in F(t, u(t), D_q u(t)), & 0 \le t \le 1, \ 0 < q < 1, \\
u(0) = 0, \quad D_q u(1) = 0, \quad D_q^2 u(1) = 0,
\end{cases} \tag{1.2}
$$

where $F : [0, 1] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ is a compact multivalued map.

The rest of the paper is organized as follows: In section 2, we state some important definitions and lemmas on the fundamental concepts of $q-$fractional calculus and fixed point theory. In section 3, we state main results on the existence of solutions for $q-$fractional boundary value problem (1.1). Section 4 contains some main theorems on the existence of solutions for $q-$fractional boundary value problem (1.2). Finally, in section 5, we give some illustrative examples to show the validity and applicability of our results.

## 2. Preliminaries

In this section, we first recall some known definitions and lemmas about $q-$fractional calculus. For more details in this regard, see [22, 25–27].

Let $0 < q < 1$. For each $a \in \mathbb{R}$ we define $[a]_q = \dfrac{1-q^a}{1-q}$. The $q-$analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is given by

$$(a-b)^{(0)} = 1, \quad (a-b)^{(n)} = \prod_{k=0}^{n-1}(a - bq^k), \;\; n \in \mathbb{N}, a, b \in \mathbb{R}.$$

In general, if $\alpha$ is real number then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \frac{a - bq^k}{a - bq^{\alpha+k}}, \;\; a \neq 0.$$

It is clear that if $b = 0$, then $a^{(\alpha)} = a^\alpha$. The $q-$Gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}},$$

where $x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. Also, we have $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$. The $q-$derivative of a real-valued function $f$ is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x \to 0}(D_q f)(x).$$

The $q-$derivative of higher order of a function $f$ is given by

$$(D_q^0 f)(x) = f(x), \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \;\; (n \in \mathbb{N}).$$

The $q-$integral of a function $f$ defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(s) d_q s = x(1-q) \sum_{k=0}^{\infty} f(xq^k) q^k, \;\; x \in [0, b]$$

such that the sum is absolutely convergent. Now, if $a \in [0, b]$, then $q-$integral of $f$ from $a$ to $b$ is

$$\begin{aligned} \int_a^b f(s) d_q s &= I_q f(b) - I_q f(a) = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s \\ &= (1-q) \sum_{k=0}^{\infty} [b f(bq^k) - a f(aq^k)] q^k, \end{aligned}$$

provided that the series converges. Similar to $q-$derivatives, an operator $I_q^n$ is given by

$$(I_q^0 f)(x) = f(x), \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \;\; (n \in \mathbb{N}).$$

Note that $(D_qI_qf)(x) = f(x)$ and if $f$ is continuous at $x = 0$, then $(I_qD_qf)(x) = f(x) - f(0)$.

Let $\alpha \geq 0$ and $f$ be a function defined on $[0, 1]$. The fractional $q-$integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and for $x \in [0, 1]$

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha-1)} f(s) d_q s, \quad \alpha > 0.$$

The fractional $q-$derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$({}^c D_q^\alpha f)(x) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f)(x) = \frac{1}{\Gamma_q([\alpha] - \alpha)} \int_0^x (x - qs)^{([\alpha]-\alpha-1)} (D_q^{[\alpha]} f)(s) d_q s, \quad \alpha > 0,$$

where $[\alpha]$ is the smallest integer greater than or equal to $\alpha$ [16]. In the following lemmas, we bring some important properties of these $q-$operators.

**Lemma 2.1.** *[17] Let $\alpha, \beta \geq 0$ and $f$ be a function defined on* $[0, 1]$. *Then*

*(i)* $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,

*(ii)* $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

**Lemma 2.2.** *[17] Let $\alpha > 0$ and $n$ be a positive integer. Then the following equality holds:*

$$(I_q^\alpha D_q^n f)(x) = (D_q^n I_q^\alpha f)(x) - \sum_{k=0}^{n-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha + k - n + 1)} (D_q^k f)(0).$$

Now, we recall some definitions and lemmas on the multifunctions and fixed point theory which are needed in the sequal.

Consider the set $X$ with the metric $d$. Denote by $\mathcal{P}(X)$, $2^X$, $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cp}(X)$, $\mathcal{P}_b(X)$ and $\mathcal{P}_{cv}(X)$, the class of all subsets, the class of all nonempty subsets of $X$, the class of all closed subsets of $X$, the class of all compact subsets of $X$, the class of all bounded subsets of $X$ and the class of all convex subsets of $X$, respectively. Let $F : X \to 2^X$ be a multivalued map. If $u \in Fu$ then we say that $u \in X$ is a fixed point of the multifunction $F$ [16, 24, 38]. The fixed point set of the multivalued operator $F$ will be denoted by $Fix(F)$.

A multifunction $F : [0, 1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable whenever the function $(t) \mapsto d(y, F(t)) = \inf\{\|y - v\| : v \in F(t)\}$ is measurable for all $y \in \mathbb{R}$ [16, 24]. The Pompeiu-Hausdorff metric $H_d : 2^X \times 2^X \to [0, \infty)$ is defined by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(a, B) = \inf_{b \in B} d(a, b)$, $d(A, b) = \inf_{a \in A} d(a, b)$ [16, 24]. Then the space $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl,b}(X), H_d)$ is a generalized metric space, where $\mathcal{P}_{cl,b}(X)$ is the set of closed and bounded subsets of $X$. [16, 24].

A multi-valued mapping $F : X \to \mathcal{P}_{cl}(X)$ is called a contraction if there exists $\gamma \in (0, 1)$ such that $H_d(F(x), F(y)) \leq \gamma d(x, y)$ for all $x, y \in X$ [16].

$F$ is called upper semi-continuous (u.s.c.) on $X$ if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of $X$, and if for each open set $N$ of $X$ containing $F(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of $x_0$ such that $F(\mathcal{N}_0) \subseteq N$ [16]. The operator $F$ is said to be completely continuous if $F(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$.

An element $x \in X$ is called an endpoint of a multifunction $F : X \to \mathcal{P}(X)$ whenever $Fx = \{x\}$ [11]. Also, we say that $F$ has an approximate endpoint property whenever $\inf_{x\in X} \sup_{y\in Fx} d(x, y) = 0$ [11]. A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n\to\infty} f(\lambda_n) \leq f(\lambda)$ for all sequence $\{\lambda_n\}_{n\geq 1}$ with $\lambda_n \to \lambda$.

We denote by $\Psi$, the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ [36]. It is known that $\psi(t) < t$ for all $t > 0$ [36]. In 2012, Samet, Vetro and Vetro introduced the notion of $\alpha$-$\psi$-contractive type mappings [36]. We say that the selfmap $T : X \to X$ is an $\alpha$-$\psi$-contraction whenever $\alpha(u, v)d(Tu, Tv) \leq \psi(d(u, v))$ for all $u, v \in X$ [36]. Also, the selfmap $T$ is called $\alpha$-admissible whenever $\alpha(u, v) \geq 1$ implies $\alpha(Tu, Tv) \geq 1$ [36]. We say that $X$ has the property $(B)$ whenever for each sequence $\{u_n\}$ in $X$ with $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \geq 1$ and $u_n \to u$, we have $\alpha(u_n, u) \geq 1$ for all $n$ [36].

In 2013, Mohammadi, Rezapour and Shahzad generalized this notion to multifunctions [31]. A multifunction $F : X \to \mathcal{P}_{cl,b}(X)$ is called $\alpha - \psi-$contraction whenever

$$\alpha(u, v)H_d(Fu, Fv) \leq \psi(d(u, v))$$

for each $u, v \in X$ [31]. Similarly, the space $X$ has the property $(C_\alpha)$ whenever for each sequence $\{u_n\}$ in X with $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha(u_{n_k}, u) \geq 1$ for all $k \in \mathbb{N}$. The multi-valued map $F$ is $\alpha-$admissible whenever for each $u \in X$ and $v \in Fu$ with $\alpha(u, v) \geq 1$, we have $\alpha(v, w) \geq 1$ for all $w \in Fv$ [31].

Our main results based on the following fixed point theorems.

**Theorem 2.1.** *( [36]) Let $(X, d)$ be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to \mathbb{R}$ a map and $T$ an $\alpha-$admissible and $\alpha - \psi-$contractive selfmap on $X$ such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. If $X$ has the property $(B)$, then $T$ has a fixed point.*

**Theorem 2.2.** *( [28, 38], Krasnoselskii) Let $M$ be a closed, bounded, convex and nonempty subset of a Banach space $X$. Let $A$ and $B$ be two operators mapping $M$ to $X$ such that:*

*(i) $Ax + By \in M$ whenever $x, y \in M$;*
*(ii) $A$ is compact and continuous;*
*(iii) $B$ is a contraction mapping.*

*Then there exists $z \in M$ such that $z = Az + Bz$.*

**Theorem 2.3.** *( [31]) Let $(X, d)$ be a complete metric space, $\alpha : X\times X \to [0, \infty)$ a map, $\psi \in \Psi$ a strictly increasing map, $F : X \to \mathcal{P}_{cl,b}(X)$ an $\alpha$-admissible $\alpha$-$\psi$-contractive multifunction and $\alpha(u_0, u_1) \geq 1$ for some $u_0 \in X$ and $u_1 \in Fu_0$. If the space $X$ has the property $(C_\alpha)$, then $F$ has a fixed point.*

**Theorem 2.4.** *( [11]) Let $(X, d)$ be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be an upper semi-continuous function such that $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ for all $t > 0$. Suppose that $T : X \to \mathcal{P}_{cl,b}(X)$ is a multifunction such that $H_d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$. Then $T$ has a unique endpoint if and only if $T$ has approximate endpoint property.*

## 3. Existence results for fractional $q-$difference equation

Let $X = \{u : u, D_q u \in C([0,1],\mathbb{R})\}$. Then $X$ is a Banach space via the norm

$$\|u\| = \sup_{t\in[0,1]} |u(t)| + \sup_{t\in[0,1]} |D_q u(t)|.$$

Put

$$\begin{aligned} \Lambda_1 &= \frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(\alpha)} + \frac{2+q}{(1+q)\Gamma_q(\alpha-1)}, \quad & \Lambda_2 = \frac{2}{\Gamma_q(\alpha)} + \frac{2}{\Gamma_q(\alpha-1)}, \\ \Delta_1 &= \frac{1}{\Gamma_q(\alpha)} + \frac{2+q}{(1+q)\Gamma_q(\alpha-1)}, & \Delta_2 = \frac{1}{\Gamma_q(\alpha)} + \frac{2}{\Gamma_q(\alpha-1)}, \\ \Phi_1 &= \|m\|\Lambda_1, & \Phi_2 = \|m\|\Lambda_2. \end{aligned} \tag{3.1}$$

**Lemma 3.1.** *Let $y \in C([0,1],\mathbb{R})$. Then the integral solution of the $q-$fractional boundary value problem*

$$\begin{cases} {}^cD_q^\alpha u(t) = y(t), \\ u(0) = 0, \quad D_q u(1) = 0, \quad D_q^2 u(1) = 0 \end{cases} \tag{3.2}$$

*is given by*

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)d_qs - t\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} y(s)d_qs \\ &\quad - \frac{t^2 - t(1+q)}{1+q}\int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} y(s)d_qs. \end{aligned} \tag{3.3}$$

*Proof.* Choose the constants $c_0$, $c_1$ and $c_2 \in \mathbb{R}$ such that

$$u(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)d_qs + c_0 + c_1 t + c_2 t^2. \tag{3.4}$$

Thus, we have

$$D_q u(t) = \int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} y(s)d_qs + c_1 + c_2(1+q)t,$$

$$D_q^2 u(t) = \int_0^t \frac{(t-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} y(s)d_qs + c_2(1+q).$$

By using the boundary value conditions, we find that $c_0 = 0$ and

$$c_1 = -\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} y(s)d_qs + \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} y(s)d_qs,$$

and

$$c_2 = -\frac{1}{1+q}\int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} y(s)d_qs.$$

By substituting the values of $c_i$'s in Eq (3.3), we obtain the $q-$integral equation (3.2). The converse follows by direct computation. The proof is completed. □

In view of the above lemma, we define an operator $S : X \to X$ as follows:

$$
\begin{aligned}
(Su)(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),D_q u(s))d_q s - t\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s,u(s),D_q u(s))d_q s \\
&\quad -\frac{t^2-t(1+q)}{1+q}\int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} f(s,u(s),D_q u(s))d_q s. \qquad (3.5)
\end{aligned}
$$

It is evident that the solution of the problem (1.1) is a fixed point of an operator $S$; that is $Su = u$.

Now, we are ready to prove our main results.

**Theorem 3.1.** *Let $\psi \in \Psi$, $\xi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a map and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ a continuous function. Suppose that:*

$(H1)$ *For all $u_1, u_2, v_1, v_2 \in \mathbb{R}$,*

$$
\Big| f(t,u_1,v_1) - f(t,u_2,v_2) \Big| \le \psi\Big(|u_1-u_2| + |v_1-v_2|\Big)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big),
$$

*with $\xi\Big((u_1(t),v_1(t)),(u_2(t),v_2(t))\Big) \ge 0$ for $t \in [0,1]$.*

$(H2)$ *There exists $u_0 \in \mathbb{R}$ such that $\xi\Big((u_0(t),D_q u_0(t)),(Fu_0(t),D_q(Fu_0(t)))\Big) \ge 0$ for all $t \in [0,1]$ and $\xi\Big((u(t),D_q u(t)),(v(t),D_q v(t))\Big) \ge 0$ implies*

$$
\xi\Big((Fu(t),D_q(Fu(t))),(Fv(t),D_q(Fv(t)))\Big) \ge 0
$$

*for all $t \in [0,1]$ and $u, v \in \mathbb{R}$.*

$(H3)$ *For each convergent sequence $\{u_n\}_{n\ge1}$ in $\mathbb{R}$ with $u_n \to u$ and*

$$
\xi\Big((u_n(t),D_q u_n(t)),(u_{n+1}(t),D_q u_{n+1}(t))\Big) \ge 0
$$

*for all $n$ and $t \in [0,1]$, we have $\xi\Big((u_n(t),D_q u_n(t)),(u(t),D_q u(t))\Big) \ge 0$.*

*Then the fractional $q-$difference Eq (1.1) has at least one solution.*

*Proof.* Let $u, v \in \mathbb{R}$ with $\xi((u(t),D_q u(t)),((v(t),D_q v(t))) \ge 0$ for all $t \in [0,1]$. Then, we get

$$
\begin{aligned}
|Su(t)-Sv(t)| &\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),D_q u(s)) - f(s,v(s),D_q v(s))| d_q s \\
&\quad + t\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |f(s,u(s),D_q u(s)) - f(s,v(s),D_q v(s))| d_q s \\
&\quad + \frac{|t^2-t(1+q)|}{(1+q)}\int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} |f(s,u(s),D_q u(s)) - f(s,v(s),D_q v(s))| d_q s \\
&\le \frac{1}{\Gamma_q(\alpha+1)}\psi(|u(s)-v(s)| + |D_q u(s) - D_q v(s)|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big) \\
&\quad + \frac{1}{\Gamma_q(\alpha)}\psi(|u(s)-v(s)| + |D_q u(s) - D_q v(s)|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2+q}{(1+q)\Gamma_q(\alpha-1)}\psi(|u(s)-v(s)|+|D_qu(s)-D_qv(s)|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)\\
\leq\ & \frac{1}{\Gamma_q(\alpha+1)}\psi(\|u-v\|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)+\frac{1}{\Gamma_q(\alpha)}\psi(\|u-v\|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)\\
&+\frac{2+q}{(1+q)\Gamma_q(\alpha-1)}\psi(\|u-v\|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)\\
=\ & \Lambda_1\psi(\|u-v\|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
|D_qSu(t)-D_qSv(t)| \leq\ & \int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s),D_qu(s))-f(s,v(s),D_qv(s))|d_qs\\
&+\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s),D_qu(s))-f(s,v(s),D_qv(s))|d_qs\\
&+|t-1|\int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)}|f(s,u(s),D_qu(s))-f(s,v(s),D_qv(s))|d_qs\\
\leq\ & \frac{1}{\Gamma_q(\alpha)}\psi(|u(s)-v(s)|+|D_qu(s)-D_qv(s)|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)\\
&+\frac{1}{\Gamma_q(\alpha)}\psi(|u(s)-v(s)|+|D_qu(s)-D_qv(s)|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)\\
\leq\ & \frac{1}{\Gamma_q(\alpha)}\psi(\|u-v\|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)+\frac{1}{\Gamma_q(\alpha)}\psi(\|u-v\|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)\\
&+\frac{1}{\Gamma_q(\alpha-1)}\psi(\|u-v\|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)\\
=\ & \Lambda_2\psi(\|u-v\|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big).
\end{aligned}
$$

Hence

$$
\|Su-Sv\|\leq(\Lambda_1+\Lambda_2)\psi(\|u-v\|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)=\psi(\|u-v\|).
$$

Now, define the function $\alpha:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ as follows

$$
\alpha(u,v)=\begin{cases}1, & \text{if}\quad \xi\Big((u(t),D_qu(t)),(v(t),D_qv(t))\Big)\geq 0\\ 0, & \text{otherwise.}\end{cases}
$$

By definition of the above function, it is clear that

$$
\alpha(u,v)d(Su,Sv)\leq\psi(d(u,v))
$$

for each $u,v\in\mathbb{R}$. This means that $S$ is an $\alpha-\psi-$contractive operator. Also, it is easy to see that $S$ is an $\alpha-$admissible and $\alpha(u_0,Su_0)\geq 1$. Suppose that $\{u_n\}_{n\geq 1}$ is a sequence in $\mathbb{R}$ with $u_n\to u$ and $\alpha(u_n,u_{n+1})\geq 1$ for all $n$. By definition of the function $\alpha$, we have

$$
\xi\Big((u_n(t),D_qu_n(t)),(u_{n+1}(t),D_qu_{n+1}(t))\Big)\geq 0.
$$

Thus by the hypothesis, $\xi\Big((u_n(t), D_q u_n(t)), (u(t), D_q u(t))\Big) \geq 0$. This shows that for all $n$, $\alpha(u_n, u) \geq 1$. So $\mathbb{R}$ has the property ($B$). Finally, Theorem 2.1 implies that the operator $S$ has fixed point $u^* \in \mathbb{R}$ which is the solution for the $q-$fractional problem (1.1). This completes the proof. □

**Theorem 3.2.** *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Suppose that:*

($H4$) *There exists a continuous function $L : [0,1] \to \mathbb{R}$ such that for each $t \in [0,1]$ and for all $u_i, v_i \in \mathbb{R}$, $i = 1,2$, we have*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L(t)(|u_1 - v_1| + |u_2 - v_2|).$$

($H5$) *There exist a continuous function $\mu : [0,1] \to \mathbb{R}^+$ and a non-decreasing continuous function $\psi : [0,1] \to \mathbb{R}^+$ such that*

$$|f(t,u_1,u_2)| \leq \mu(t)\psi(|u_1| + |u_2|), \quad t \in [0,1],\ u_i \in \mathbb{R},\ i = 1,2.$$

*Then, the fractional $q-$difference equation (1.1) has at least one solution on $[0,1]$ if*

$$k := \|L\|(\Lambda_1 + \Lambda_2) < 1,$$

*where $\|L\| = \sup_{t\in[0,1]} |L(t)|$ and $\Lambda_1, \Lambda_2$ are given by Eq (3.1).*

*Proof.* Define $\|\mu\| = \sup_{t\in[0,1]} |\mu(t)|$ and choose a suitable constant $r$ such that

$$r \geq \psi(\|u\|)\|\mu\|\{\Delta_1 + \Delta_2\}, \tag{3.6}$$

where $\Delta_i$'s are given by Eq (3.1). We consider the set $B_r = \{u \in X : \|u\| \leq r\}$, where $r$ is given in Eq (3.6). It is clear that $B_r$ is a closed, bounded, convex and nonempty subset of the Banach space $X$. Now, define two operators $S_1$ and $S_2$ on $B_r$ as follows:

$$(S_1 u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),D_q u(s))d_q s, \tag{3.7}$$

and

$$\begin{aligned} (S_2 u)(t) &= -t\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s,u(s),D_q u(s))d_q s \\ &\quad -\frac{t^2 - t(1+q)}{1+q}\int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} f(s,u(s),D_q u(s))d_q s, \end{aligned} \tag{3.8}$$

for each $t \in [0,1]$. Put $a = \sup_{u\in X} \psi(\|u\|)$. For $u, v \in B_r$, we have

$$\begin{aligned} |(S_1 u + S_2 v)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),D_q u(s))| d_q s \\ &\quad + t\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |f(s,v(s),D_q v(s))| d_q s \\ &\quad + \frac{|t^2 - t(1+q)|}{(1+q)}\int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} |f(s,v(s),D_q v(s))| d_q s \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\mu(s)\psi(|u(s)|+|D_qu(s)|)d_qs \\
&\quad+\frac{t}{\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}\mu(s)\psi(|v(s)|+|D_qv(s)|)d_qs \\
&\quad+\frac{|t^2-t(1+q)|}{(1+q)\Gamma_q(\alpha-2)}\int_0^1 (1-qs)^{(\alpha-3)}\mu(s)\psi(|v(s)|+|D_qv(s)|)d_qs \\
&\le a\|\mu\|\Big[\frac{1}{\Gamma_q(\alpha+1)}+\frac{1}{\Gamma_q(\alpha)}+\frac{2+q}{(1+q)\Gamma_q(\alpha-1)}\Big] = a\|\mu\|\Lambda_1.
\end{aligned}
$$

Also,

$$
\begin{aligned}
|(D_qS_1u+D_qS_2v)(t)| &\le \int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s),D_qu(s))|d_qs \\
&\quad+\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,v(s),D_qv(s))|d_qs \\
&\quad+|t-1|\int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)}|f(s,v(s),D_qv(s))|d_qs \\
&\le \frac{1}{\Gamma_q(\alpha-1)}\int_0^t (t-qs)^{(\alpha-2)}\mu(s)\psi(|u(s)|+|D_qu(s)|)d_qs \\
&\quad+\frac{1}{\Gamma_q(\alpha-1)}\int_0^1 (1-qs)^{(\alpha-2)}\mu(s)\psi(|v(s)|+|D_qv(s)|)d_qs \\
&\quad+\frac{|t-1|}{\Gamma_q(\alpha-2)}\int_0^1 (1-qs)^{(\alpha-3)}\Gamma_q(\alpha-2)\mu(s)\psi(|v(s)|+|D_qv(s)|)d_qs \\
&\le a\|\mu\|\Big[\frac{2}{\Gamma_q(\alpha)}+\frac{2}{\Gamma_q(\alpha-1)}\Big] = a\|\mu\|\Lambda_2.
\end{aligned}
$$

Hence $\|S_1u+S_2v\|\le r$ and so, $S_1u+S_2v\in B_r$.

Clearly, the continuity of $S_1$ is follows from the function $f$. Also, for each $u\in B_r$, we have

$$
|(S_1u)(t)| \le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s),D_qu(s))|ds \le \frac{1}{\Gamma_q(\alpha+1)}\|\mu\|\psi(\|u\|),
$$

and

$$
|(D_qS_1u)(t)| \le \int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s),D_qu(s))|ds \le \frac{1}{\Gamma_q(\alpha)}\|\mu\|\psi(\|u\|).
$$

Thus $\|S_1u\|\le\Big\{\frac{1}{\Gamma_q(\alpha+1)}+\frac{1}{\Gamma_q(\alpha)}\Big\}\|\mu\|\psi(\|u\|)$. This proves that the operator $S_1$ is uniformly bounded on $B_r$.

Now, we show that the operator $S_1$ is compact on $B_r$. For each $t_1,t_2\in[0,1]$ with $t_1<t_2$, one can write

$$
|(S_1u)(t_2)-(S_1u)(t_1)|
$$

$$
\begin{aligned}
&= \Big|\int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),D_qu(s))ds - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),D_qu(s))ds\Big| \\
&\le \Big|\int_0^{t_1} \frac{(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),D_qu(s))ds\Big| \\
&\quad +\Big|\int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s),D_qu(s))ds\Big| \\
&\le \int_0^{t_1} \frac{(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),D_qu(s))|ds \\
&\quad +\int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),D_qu(s))|ds \\
&\le \left\{\frac{t_2^\alpha-t_1^\alpha-(t_2-t_1)^\alpha}{\Gamma_q(\alpha+1)}+\frac{(t_2-t_1)^\alpha}{\Gamma_q(\alpha+1)}\right\}\|\mu\|\psi(\|u\|).
\end{aligned}
$$

It is seen that $|(S_1u)(t_2)-(S_1u)(t_1)|\to 0$ as $t_2\to t_1$. Also, we have

$$
\begin{aligned}
&|(D_qS_1u)(t_2)-(D_qS_1u)(t_1)| \\
&= \Big|\int_0^{t_2} \frac{(t_2-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s,u(s),D_qu(s))ds - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s,u(s),D_qu(s))ds\Big| \\
&\le \Big|\int_0^{t_1} \frac{(t_2-qs)^{(\alpha-2)}-(t_1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s,u(s),D_qu(s))ds\Big| \\
&\quad +\Big|\int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s,u(s),D_qu(s))ds\Big| \\
&\le \int_0^{t_1} \frac{(t_2-qs)^{(\alpha-2)}-(t_1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |f(s,u(s),D_qu(s))|ds \\
&\quad +\int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |f(s,u(s),D_qu(s))|ds \\
&\le \left\{\frac{t_2^{\alpha-1}-t_1^{\alpha-1}-(t_2-t_1)^{\alpha-1}}{\Gamma_q(\alpha)}+\frac{(t_2-t_1)^{\alpha-1}}{\Gamma_q(\alpha)}\right\}\|\mu\|\psi(\|u\|).
\end{aligned}
$$

Again, we see that $|(D_qS_1u)(t_2)-(D_qS_1u)(t_1)|\to 0$ as $t_2\to t_1$. Hence $\|(S_1u)(t_2)-(S_1u)(t_1)\|$ tends to zero as $t_2\to t_1$. Thus, $S_1$ is equicontinuous and so $S_1$ is relatively compact on $B_r$. Consequently, the Arzelá-Ascoli theorem implies that $S_1$ is a compact operator on $B_r$.

Finally, we prove that $S_2$ is a contraction mapping. For each $u,v\in B_r$, we have

$$
\begin{aligned}
&|(S_2u)(t)-(S_2v)(t)| \\
&\le t\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |f(s,u(s),D_qu(s))-f(s,v(s),D_qv(s))|d_qs \\
&\quad +\frac{|t^2-t(1+q)|}{(1+q)}\int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} |f(s,u(s),D_qu(s))-f(s,v(s),D_qv(s))|d_qs \\
&\le t\int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} L(s)\Big(|u(s)-v(s)|+|D_qu(s)-D_qv(s)|\Big)d_qs
\end{aligned}
$$

$$+\frac{|t^2-t(1+q)|}{(1+q)}\int_0^1\frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)}L(s)\Big(|u(s)-v(s)|+|D_qu(s)-D_qv(s)|\Big)d_qs.$$

Also,

$$\begin{aligned}
&|(D_qS_2u)(t)-(D_qS_2v)(t)|\\
\leq\ & \int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s),D_qu(s))-f(s,v(s),D_qv(s))|d_qs\\
&+|t-1|\int_0^1\frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)}|f(s,u(s),D_qu(s))-f(s,v(s),D_qv(s))|d_qs\\
\leq\ & \int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}L(s)\Big(|u(s)-v(s)|+|D_qu(s)-D_qv(s)|\Big)d_qs\\
&+|t-1|\int_0^1\frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)}L(s)\Big(|u(s)-v(s)|+|D_qu(s)-D_qv(s)|\Big)d_qs.
\end{aligned}$$

Hence, we obtain

$$\sup_{t\in[0,1]}|(S_2u)(t)-(S_2v)(t)|\leq\|L\|\Delta_1\|u-v\|,$$

$$\sup_{t\in[0,1]}|(D_qS_2u)(t)-(D_qS_2v)(t)|\leq\|L\|\Delta_2\|u-v\|.$$

Thus, $\|S_2u-S_2v\|\leq\|L\|(\Delta_1+\Delta_2)\|u-v\|$ or $\|S_2u-S_2v\|\leq k\|u-v\|$. Thus $S_2$ is contraction on $B_r$ as $k<1$. Therefore, all the assumptions of Theorem 2.2 are satisfied. Hence, Theorem 2.2 implies that the $q-$fractional boundary value problem (1.1) has at least one solution on $[0,1]$. □

## 4. Existence results for fractional $q-$difference inclusion

In this section, we prove our main results about the existence of solutions for fractional $q-$difference inclusion (1.2).

**Definition 4.1.** *A function $u\in C([0,1],\mathbb{R})$ is called a solution for the fractional $q-$difference inclusion (1.2) whenever it satisfies the boundary value conditions and there exists a function $v\in L^1([0,1])$ such that $v(t)\in F(t,u(t),D_qu(t))$ for almost all $t\in[0,1]$ and*

$$\begin{aligned}
u(t)\ =\ & \int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}v(s)d_qs-t\int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}v(s)d_qs\\
&-\frac{t^2-t(1+q)}{1+q}\int_0^1\frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)}v(s)d_qs.
\end{aligned}$$

*for all $t\in[0,1]$.*

Let $X$ be a Banach space with the norm defined in the last section. For each $u\in X$, the set of selections of the operator $F$ is defined by

$$S_{F,u}=\Big\{v\in L^1([0,1]):\ v(t)\in F(t,u(t),D_qu(t))\text{ for almost all }t\in[0,1]\Big\}.$$

Also, we define the operator $\mathcal{N} : X \to \mathcal{P}(X)$ by

$$\mathcal{N}(u) = \Big\{h \in X : \text{ there exists } v \in S_{F,u} \text{ such that } h(t) = w(t) \text{ for all } t \in [0,1]\Big\}, \tag{4.1}$$

where

$$\begin{aligned} w(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s - t \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} v(s) d_q s \\ &\quad - \frac{t^2 - t(1+q)}{1+q} \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} v(s) d_q s. \end{aligned}$$

**Theorem 4.1.** *Let $F : [0,1] \times \mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ be a multifunction. Suppose that:*

($H6$) *The operator $F$ is integrably bounded and $F(\cdot, u, v) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for all $u, v \in \mathbb{R}$.*

($H7$) *Assume that there exists $m \in C([0,1],[0,\infty))$ such that*

$$H_d\Big(F(t,u_1,u_1'), F(t,u_2,u_2')\Big) \le m(t)\psi\Big(|u_1 - u_2| + |u_1' - u_2'|\Big)\Big(\frac{1}{\|m\|(\Lambda_1 + \Lambda_2)}\Big) \tag{4.2}$$

*for all $t \in [0,1]$ and $u_i, u_i' \in \mathbb{R}$ $(i = 1,2)$ and $\psi \in \Psi$. The constants $\Lambda_1$ and $\Lambda_2$ are given by Eq (3.1).*

($H8$) *There exists a function $\xi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with $\xi\Big((u_1,u_1'),(u_2,u_2')\Big) \ge 0$ for $u_i, u_i' \in \mathbb{R}$ $(i = 1,2)$.*

($H9$) *If $\{u_n\}_{n\ge1}$ is a sequence in $X$ such that $u_n \to u$ and $\xi\Big((u_n(t), D_q u_n(t)),(u(t), D_q u(t))\Big) \ge 0$ for all $t \in [0,1]$, then there exists a subsequence $\{u_{n_j}\}_{j\ge1}$ of $\{u_n\}$ such that*

$$\xi\Big((u_{n_j}(t), D_q u_{n_j}(t)),(u(t), D_q u(t))\Big) \ge 0$$

*for all $t \in [0,1]$.*

($H10$) *There exist $u_0 \in X$ and $h \in \mathcal{N}(u_0)$ such that $\xi\Big((u_0(t), D_q u_0(t)),(h(t), D_q h(t))\Big) \ge 0$, for all $t \in [0,1]$, where the operator $\mathcal{N} : X \to \mathcal{P}(X)$ is given by Eq (4.1).*

($H11$) *For each $u \in X$ and $h \in \mathcal{N}(u)$ with $\xi\Big((u(t), D_q u(t)),(h(t), D_q h(t))\Big) \ge 0$, there exists $w \in \mathcal{N}(u)$ such that $\xi\Big((h(t), D_q h(t)),(w(t), D_q w(t))\Big) \ge 0$.*

*Then the fractional $q-$difference inclusion (1.2) has a solution.*

*Proof.* It is evident that the fixed point of the operator $\mathcal{N} : X \to \mathcal{P}(X)$ is a solution of the inclusion problem (1.2). Since the multivalued map $t \mapsto F(t, u(t), D_q u(t))$ is measurable and it closed-valued for all $u \in X$, so $F$ has measurable selection and the set $S_{F,u}$ is not empty.

We prove that $\mathcal{N}(u)$ is a closed subset of $X$ for all $u \in X$, i.e., $\mathcal{N}(u) \in \mathcal{P}_{cl}(X)$. For this, let $\{u_n\}_{n\ge1}$ be a sequence in $\mathcal{N}(u)$ which converges to $u$. We should prove that $u \in \mathcal{N}(u)$. For each natural number $n$, there exists $v_n \in S_{F,u}$ such that

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_n(s) d_q s - t \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} v_n(s) d_q s \\ &\quad - \frac{t^2 - t(1+q)}{1+q} \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} v_n(s) d_q s, \end{aligned}$$

for almost all $t \in [0,1]$.

That $F$ has compact values, we pass into a subsequence (if necessary) to obtain that a subsequence $\{v_n\}_{n\geq 1}$ converges to some $v \in L^1([0,1])$. Thus $v \in S_{F,u}$ and we get

$$\begin{aligned} u_n(t) &\rightarrow u(t) \\ &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s - t \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} v(s) d_q s \\ &\quad - \frac{t^2 - t(1+q)}{1+q} \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} v(s) d_q s, \end{aligned}$$

for each $t \in [0,1]$. This shows that $u \in \mathcal{N}(u)$; that is, the operator $\mathcal{N}$ is closed-valued. Now, since $F$ is a multifunction with compact values, it is easy to check that $\mathcal{N}(u)$ is bounded set in $X$ for all $u \in X$. In the next step, we show that the operator $\mathcal{N}$ is an $\alpha - \psi-$contractive multivalued map. For this purpose, we define a function $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(u,u') = \begin{cases} 1, & \text{if} \quad \xi\Big((u(t), D_q u(t)), (u'(t), D_q u'(t))\Big) \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $u, u' \in X$. Let $u, u' \in X$ and $h_1 \in \mathcal{N}(u')$. We choose $v_1 \in S_{F,u'}$ such that

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_1(s) d_q s - t \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} v_1(s) d_q s \\ &\quad - \frac{t^2 - t(1+q)}{1+q} \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} v_1(s) d_q s, \end{aligned}$$

for all $t \in [0,1]$. Since by (4.2), we have

$$H_d\Big(F(t,u,D_q u), F(t,u',D_q u')\Big) \leq m(t)\psi\Big(|u-u'| + |D_q u - D_q u'|\Big)\Big(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}\Big)$$

for all $u, u' \in X$ with $\xi\Big((u(t), D_q u(t)), (u'(t), D_q u'(t))\Big) \geq 0$ for almost all $t \in [0,1]$, so there exists $w \in F(t, u(t), D_q u(t))$ such that

$$|v_1(t) - w| \leq m(t)\psi\Big(|u(t) - u'(t)| + |D_q u(t) - D_q u'(t)|\Big)\Big(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}\Big).$$

Now, consider the multivalued map $A : [0,1] \to \mathcal{P}(\mathbb{R})$ which is given by

$$\begin{aligned} A(t) = \Big\{ w \in \mathbb{R} \ : \ & |v_1(t) - w| \leq m(t)\psi\Big(|u(t) - u'(t)| \\ & + |D_q u(t) - D_q u'(t)|\Big)\Big(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}\Big)\Big\} \end{aligned}$$

for all $t \in [0,1]$. Clearly, the multifunction $A(\cdot) \cap F(\cdot, u(\cdot), D_q u(\cdot))$ is measurable, because $v_1$ and $\varphi = m\psi(|u-u'| + |D_q u - D_q u'|)\Big(\frac{1}{\Lambda_1+\Lambda_2}\Big)$ are measurable. In this step, we can choose $v_2 \in F(t, u(t), D_q u(t))$ such that

$$|v_1(t) - v_2(t)| \leq m(t)\psi\Big(|u(t) - u'(t)| + |D_q u(t) - D_q u'(t)|\Big)\Big(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}\Big),$$

for all $t \in [0,1]$. Define the element $h_2 \in \mathcal{N}(u)$ by follows:

$$\begin{aligned} h_2(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_2(s) d_q s - t \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} v_2(s) d_q s \\ &\quad - \frac{t^2 - t(1+q)}{1+q} \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} v_2(s) d_q s \end{aligned}$$

for all $t \in [0,1]$. Letting $\sup_{t\in[0,1]} |m(t)| = \|m\|$, we have

$$\begin{aligned} &|h_1 - h_2| \\ \leq\ & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |v_1(s) - v_2(s)| d_q s + t \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |v_1(s) - v_2(s)| d_q s \\ & + \frac{|t^2 - t(1+q)|}{(1+q)} \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} |v_1(s) - v_2(s)| d_q s \\ \leq\ & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} m(s) \psi\Big(|u(s) - u'(s)| + |D_q u(s) - D_q u'(s)|\Big)\Big(\frac{1}{\|m\|(\Lambda_1 + \Lambda_2)}\Big) d_q s \\ & + t \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} m(s) \psi\Big(|u(s) - u'(s)| + |D_q u(s) - D_q u'(s)|\Big)\Big(\frac{1}{\|m\|(\Lambda_1 + \Lambda_2)}\Big) d_q s \\ & + \frac{|t^2 - t(1+q)|}{(1+q)} \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} m(s) \psi\Big(|u(s) - u'(s)| \\ & + |D_q u(s) - D_q u'(s)|\Big)\Big(\frac{1}{\|m\|(\Lambda_1 + \Lambda_2)}\Big) d_q s \\ \leq\ & \frac{1}{\Gamma_q(\alpha+1)} \|m\| \psi(\|u - u'\|)\Big(\frac{1}{\|m\|(\Lambda_1 + \Lambda_2)}\Big) \\ & + \frac{1}{\Gamma_q(\alpha)} \|m\| \psi(\|u - u'\|)\Big(\frac{1}{\|m\|(\Lambda_1 + \Lambda_2)}\Big) \\ & + \frac{(2+q)}{(1+q)\Gamma_q(\alpha-1)} \|m\| \psi(\|u - u'\|)\Big(\frac{1}{\|m\|(\Lambda_1 + \Lambda_2)}\Big) \\ =\ & \Big[\frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{\Gamma_q(\alpha)} + \frac{(2+q)}{(1+q)\Gamma_q(\alpha-1)}\Big] \|m\| \psi(\|u - u'\|)\Big(\frac{1}{\|m\|(\Lambda_1 + \Lambda_2)}\Big) \\ =\ & \Lambda_1 \psi(\|u - u'\|)\Big(\frac{1}{(\Lambda_1 + \Lambda_2)}\Big), \end{aligned}$$

and

$$\begin{aligned} &|D_q h_1 - D_q h_2| \\ \leq\ & \int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |v_1(s) - v_2(s)| d_q s + \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |v_1(s) - v_2(s)| d_q s \\ & + |t-1| \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} |v_1(s) - v_2(s)| d_q s \\ \leq\ & \int_0^t \frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} m(s) \psi\Big(|u(s) - u'(s)| + |D_q u(s) - D_q u'(s)|\Big)\Big(\frac{1}{\|m\|(\Lambda_1 + \Lambda_2)}\Big) d_q s \end{aligned}$$

$$
\begin{aligned}
& + \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} m(s)\psi\Big(|u(s)-u'(s)| + |D_q u(s) - D_q u'(s)|\Big)\Big(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}\Big) d_q s \\
& + |t-1| \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} m(s)\psi\Big(|u(s)-u'(s)| + |D_q u(s) - D_q u'(s)|\Big)\Big(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}\Big) d_q s \\
\leq \; & \frac{1}{\Gamma_q(\alpha)}\|m\|\psi(\|u-u'\|)(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}) + \frac{1}{\Gamma_q(\alpha)}\|m\|\psi(\|u-u'\|)\Big(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}\Big) \\
& + \frac{2}{\Gamma_q(\alpha-1)}\|m\|\psi(\|u-u'\|)\Big(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}\Big) \\
= \; & \Big[\frac{2}{\Gamma_q(\alpha)} + \frac{2}{\Gamma_q(\alpha-1)}\Big]\|m\|\psi(\|u-u'\|)\Big(\frac{1}{\|m\|(\Lambda_1+\Lambda_2)}\Big) \\
= \; & \Lambda_2\psi(\|u-u'\|)\Big(\frac{1}{(\Lambda_1+\Lambda_2)}\Big),
\end{aligned}
$$

for all $t \in [0,1]$. Hence, we obtain

$$
\begin{aligned}
\|h_1 - h_2\| &= \sup_{t\in[0,1]} |h_1(t) - h_2(t)| + \sup_{t\in[0,1]} |D_q h_1(t) - D_q h_2(t)| \\
&\leq (\Lambda_1+\Lambda_2)\psi(\|u-u'\|)\Big(\frac{1}{(\Lambda_1+\Lambda_2)}\Big) = \psi(\|u-u'\|).
\end{aligned}
$$

Therefore $\alpha(u,u')H_d(\mathcal{N}(u),\mathcal{N}(u')) \leq \psi(\|u-u'\|)$ for all $u,u' \in X$. This means that $\mathcal{N}$ is an $\alpha-\psi-$contractive multivalued mapping. Now, let $u \in X$ and $u' \in \mathcal{N}(u)$ be such that $\alpha(u,u') \geq 1$. Thus, by definition of $\alpha$, we have $\xi\Big((u(t), D_q u(t)), (u'(t), D_q u'(t))\Big) \geq 0$ and by the hypothesis there exists $w \in \mathcal{N}(u')$ such that $\xi\Big((u'(t), D_q u'(t)), (w(t), D_q w(t))\Big) \geq 0$. This implies that $\alpha(u', w) \geq 1$ and so, this proves that the operator $\mathcal{N}$ is an $\alpha-$admissible.

Finally, we choose $u_0 \in X$ and $u' \in \mathcal{N}(u_0)$ such that

$$
\xi\Big((u_0(t), D_q u_0(t)), (u'(t), D_q u'(t))\Big) \geq 0.
$$

Hence $\alpha(u_0, u') \geq 1$. Consequently, Theorem 2.3 implies that $\mathcal{N}$ has a fixed point. In the other words, there exists $u^* \in X$ such that $u^* \in \mathcal{N}(u^*)$ which $u^*$ is the solution of the fractional $q-$difference inclusion (1.2) and the proof is completed. □

**Theorem 4.2.** *Let $F : [0,1] \times \mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ be a multifunction. Suppose that:*

($H12$) *The function $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing upper semi-continuous mapping such that $\liminf_{t\to\infty}(t - \psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$.*

($H13$) *The operator $F : [0,1] \times \mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ be an integrably bounded multifunction such that $F(\cdot, u_1, u_2) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for all $u_1, u_2 \in \mathbb{R}$.*

($H14$) *There exists a function $m \in C([0,1],[0,\infty))$ such that*

$$
H_d(F(t,u_1,u_2) - F(t,u_1',u_2')) \leq m(t)\psi(|u_1 - u_1'| + |u_2 - u_2'|)\Big(\frac{1}{\Phi_1+\Phi_2}\Big)
$$

*for all $t \in [0,1]$ and $u_i, u_i' \in \mathbb{R}$ $(i = 1,2)$, where $\Phi_i$'s are given in Eq (3.1).*

($H15$) *The operator $\mathcal{N}$ has the approximate endpoint property where $\mathcal{N}$ is defined in Eq (4.1).*

*Then the $q-$fractional inclusion problem (1.2) has a solution.*

*Proof.* We show that the multifunction $\mathcal{N} : X \to \mathcal{P}(X)$ has an endpoint. For this, we prove that $\mathcal{N}(u)$ is a closed subset of $\mathcal{P}(X)$ for all $u \in X$. First of all, since the multivalued map $t \mapsto F(t, u(t), D_q u(t))$ is measurable and has closed values for all $u \in X$, so it has measurable selection and thus $S_{F,u}$ is nonempty for all $u \in X$.

Similar to the first part of the proof of Theorem 4.1, one can see that the operator $\mathcal{N}(u)$ is closed-valued. Also, $\mathcal{N}(u)$ is a bounded set for all $u \in X$, because $F$ is a compact multivalued map.

Finally, we show that $H_d(\mathcal{N}(u), \mathcal{N}(w)) \leq \psi(\|u - w\|)$. Let $u, w \in X$ and $h_1 \in \mathcal{N}(w)$. Choose $v_1 \in S_{F,w}$ such that

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_1(s) d_q s - t \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} v_1(s) d_q s \\ &\quad - \frac{t^2 - t(1+q)}{1+q} \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} v_1(s) d_q s, \end{aligned}$$

for almost all $t \in [0, 1]$. Since

$$H_d(F(t, u(t), D_q u(t)) - F(t, w(t), D_q w(t))) \leq m(t)\psi(|u(t) - w(t)| + |D_q u(t) - D_q w(t)|)\Big(\frac{1}{\Phi_1 + \Phi_2}\Big)$$

for all $t \in [0, 1]$, there exist $z \in F(t, u(t), D_q u(t))$ such that

$$|v_1(t) - z| \leq m(t)\psi(|u(t) - w(t)| + |D_q u(t) - D_q w(t)|)\Big(\frac{1}{\Phi_1 + \Phi_2}\Big)$$

for all $t \in [0, 1]$. Now, consider the multivalued map $U : [0, 1] \to \mathcal{P}(\mathbb{R})$ which is defined by

$$U(t) = \{z \in \mathbb{R} : \ |v_1(t) - z| \leq m(t)\psi(|u(t) - w(t)| + |D_q u(t) - D_q w(t)|)\Big(\frac{1}{\Phi_1 + \Phi_2}\Big).$$

Since $v_1$ and $\varphi = m\psi(|u - w| + |D_q u - D_q w|)\Big(\frac{1}{\Phi_1 + \Phi_2}\Big)$ are measurable, the multifunction $U(\cdot) \cap F(\cdot, u(\cdot), D_q u(\cdot))$ is measurable. Choose $v_2(t) \in F(t, u(t), D_q u(t))$ such that

$$|v_1(t) - v_2(t)| \leq m(t)\psi(|u(t) - w(t)| + |D_q u(t) - D_q w(t)|)\Big(\frac{1}{\Phi_1 + \Phi_2}\Big),$$

for all $t \in [0, 1]$. We define the element $h_2 \in \mathcal{N}(u)$ as follows:

$$\begin{aligned} h_2(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v_2(s) d_q s - t \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} v_2(s) d_q s \\ &\quad - \frac{t^2 - t(1+q)}{1+q} \int_0^1 \frac{(1-qs)^{(\alpha-3)}}{\Gamma_q(\alpha-2)} v_2(s) d_q s, \end{aligned}$$

for all $t \in [0, 1]$. Therefore, similar to the proof of Theorem 4.1, we get

$$\begin{aligned} \|h_1 - h_2\| &= \sup_{t\in[0,1]} |h_1(t) - h_2(t)| + \sup_{t\in[0,1]} |D_q h_1(t) - D_q h_2(t)| \\ &\leq (\Phi_1 + \Phi_2)\psi(\|u - w\|)\Big(\frac{1}{\Phi_1 + \Phi_2}\Big) = \psi(\|u - w\|). \end{aligned}$$

Hence $H_d(\mathcal{N}(u), \mathcal{N}(w)) \leq \psi(\|u - w\|)$ for all $u, w \in X$. By hypothesis $(H15)$, since the multifunction $\mathcal{N}$ has approximate endpoint property, by Theorem 2.4, there exists $u^* \in X$ such that $\mathcal{N}(u^*) = \{u^*\}$. Consequently, the $q-$fractional inclusion (1.2) has the solution $u^*$ and the proof is completed. □

## 5. Examples

Now, in this section, we present some illustrative examples to show the validity of our main results.

**Example 5.1.** *Consider the fractional q−difference equation*

$$ {}^cD_{1/2}^{5/2}u(t) = \frac{t}{100}|\arcsin u| + \frac{t|\arctan(D_{1/2}u)|}{100 + 100|\arctan(D_{1/2}u)|}, \qquad t \in [0,1] \tag{5.1}$$

*via the boundary value conditions*

$$u(0) = 0, \qquad D_{1/2}u(1) = u(0), \qquad D_{1/2}^2 u(1) = u(0) \tag{5.2}$$

*where* ${}^cD_{1/2}^{5/2}$ *denotes the Caputo q−fractional derivative of order* $5/2$. *Clearly,* $\alpha = 5/2$ *and* $q = 1/2$. *We define* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *by*

$$f(t, x(t), y(t)) = \frac{t}{100}|\arcsin x(t)| + \frac{t|\arctan y(t)|}{100 + 100|\arctan y(t)|}.$$

*In this case, for each* $x_i, y_i \in \mathbb{R}$ $(i = 1, 2)$, *we have*

$$\begin{aligned} |f(t, x_1(t), y_1(t)) - f(t, x_2(t), y_2(t))| &\leq \frac{t}{100}|\arcsin x_1(t) - \arcsin x_2(t)| \\ &\quad + \frac{t}{100}|\arctan y_1(t) - \arctan y_2(t)| \\ &\leq \frac{t}{100}\Big(|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|\Big). \end{aligned}$$

*Hence* $L(t) = t/100$ *and so* $\|L\| = \sup_{t\in[0,1]} |L(t)| = 1/100$. *On the other hand, we define continuous and nondecreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}$ *by* $\psi(x) = x$ *for all* $x \in \mathbb{R}^+$. *We have*

$$|f(t, u(t), (D_{1/2}u)(t))| \leq \frac{t}{100}(|u| + |D_{1/2}u|) = \frac{t}{100}\psi(|u| + |D_{1/2}u|).$$

*Clearly,* $\mu : [0,1] \to \mathbb{R}$ *is given by* $\mu(t) = t/100$ *which is continuous function. Then, we have* $\Lambda_1 + \Lambda_2 \approx 6.0085$ *and so* $k \approx 0.06 < 1$. *Since, all assumptions of Theorem 3.2 hold, thus the fractional q−difference equation (5.1)–(5.2) has at least one solution on* $[0,1]$.

**Example 5.2.** *We consider the fractional q−difference inclusion*

$$ {}^cD_{1/2}^{5/2}u(t) \in \Big[0, \frac{0.025t|\cos u(t)|}{2(1 + |\cos u(t)|)} + \frac{25t|\sin(\pi/2)t||D_{1/2}u(t)|}{2000(1 + |D_{1/2}u(t)|)}\Big], \qquad t \in [0,1] \tag{5.3}$$

*via the boundary value conditions*

$$u(0) = 0, \qquad D_{1/2}u(1) = u(0), \qquad D_{1/2}^2 u(1) = u(0). \tag{5.4}$$

*Put* $\alpha = 5/2$ *and* $q = 1/2$. *By these values, we get* $\Lambda_1 \approx 2.5596$ *and* $\Lambda_2 \approx 3.4489$. *We define multifunction* $F : [0,1] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ *by follows:*

$$F(t, x(t), y(t)) = \Big[0, \frac{0.025t|\cos x(t)|}{2(1 + |\cos x(t)|)} + \frac{25t|\sin(\pi/2)t||y(t)|}{2000(1 + |y(t)|)}\Big]$$

*for each* $t \in [0,1]$. *By above definition, there exists a continuous function* $m : [0,1] \to [0,\infty)$ *by* $m(t) = 5t/200$ *for all* $t$. *Then* $\|m\| = 5/200$. *Also, we define upper semi-continuous and nondecreasing function* $\psi : (0,\infty) \to [0,\infty)$ *by* $\psi(t) = t/2$ *for all* $t > 0$. *It is clear that* $\liminf_{t\to\infty}(t - \psi(t)) > 0$ *and* $\psi(t) < t$ *for all* $t > 0$. *On the other hand, we have* $\Phi_1 \approx 0.06399$ *and* $\Phi_2 \approx 0.08622$ *and* $\frac{1}{\Phi_1 + \Phi_2} \approx 6.6577 > 0$. *For every* $x_i, y_i \in \mathbb{R}$ $(i = 1,2)$, *we have*

$$\begin{aligned} & H_d\Big(F(t, x_1(t), y_1(t)) - F(t, x_2(t), y_2(t))\Big) \\ \leq \; & \frac{5t}{200}.\frac{1}{2}(|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|) \\ = \; & \frac{5t}{200}\psi(|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|) \\ \leq \; & m(t)\psi(|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|)\Big(\frac{1}{\Phi_1 + \Phi_2}\Big). \end{aligned}$$

*Now, put* $X = \{u : u, D_{1/2}u \in C([0,1],\mathbb{R})\}$. *Define* $\mathcal{N} : X \to \mathcal{P}(\mathbb{R})$ *as follows:*

$$\mathcal{N}(u) = \Big\{h \in X : \text{ there exists } v \in S_{F,u} \text{ such that } h(t) = w(t) \text{ for all } t \in [0,1]\Big\},$$

*where*

$$\begin{aligned} w(t) \;=\; & \int_0^t \frac{(t - \frac{1}{2}s)^{(\frac{5}{2}-1)}}{\Gamma_{1/2}(\frac{5}{2})} v(s) d_{\frac{1}{2}} s - t \int_0^1 \frac{(1 - \frac{1}{2}s)^{(\frac{5}{2}-2)}}{\Gamma_{1/2}(\frac{5}{2} - 1)} v(s) d_{\frac{1}{2}} s \\ & - \frac{t^2 - \frac{3}{2}t}{\frac{3}{2}} \int_0^1 \frac{(1 - \frac{1}{2}s)^{(\frac{5}{2}-3)}}{\Gamma_{1/2}(\frac{5}{2} - 2)} v(s) d_{\frac{1}{2}} s. \end{aligned}$$

*Also, the operator* $\mathcal{N}$ *has the approximate endpoint property, because* $\sup_{u\in\mathcal{N}(0)} \|u\| = 0$ *and so* $\inf_{u\in X} \sup_{z\in\mathcal{N}(u)} \|u - z\| = 0$. *All assumptions of Theorem 4.2 hold. Therefore, by Theorem 4.2, the fractional* $q-$*difference inclusion (5.3)–(5.4) has a solution.*

## Conflict of interest

All authors declare no conflicts of interest in this paper.

## References


1. R. P. Agarwal, *Certain fractional q-integrals and q-derivatives*, Math. Proc. Cambridge Philos. Soc., **66** (1969), 365–370.

2. R. P. Agarwal, D. Baleanu, V. Hedayati, et al. *Two fractional derivative inclusion problems via integral boundary condition*, Appl. Math. Comput., **257** (2015), 205–212.

3. R. P. Agarwal, M. Belmekki and M. Benchohra, *A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative*, Adv. Differ. Equations, **2009** (2009), Article ID 981728.

4. B. Ahmad, *Boundary-value problems for nonlinear third-order q-difference equations*, Electron. J. Differ. Equations, **94** (2011), 1–7.

5. B. Ahmad, J. J. Nieto, A. Alsaedi, et al. *A study of nonlinear Langevin equation involving two fractional orders in different intervals*, Nonlinear Anal. Real World Appl., **13** (2012), 599–606.

6. B. Ahmad and S. K. Ntouyas, *Boundary value problems for q-difference inclusions*, Abstr. Appl. Anal., **2011** (2011), Article ID 292860.

7. B. Ahmad and S. K. Ntouyas, *Nonlinear fractional differential equations and inclusions of arbitrary order and multi-strip boundary conditions*, Electron. J. Differ. Equations, **98** (2012), 1–22.

8. B. Ahmad, S. K. Ntouyas and I. K. Purnaras, *Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations*, *Adv. Differ. Equations,* **2012** (2012), 1–15.

9. B. Ahmad and S. Sivasundaram, *On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order*, Appl, Math. Comput., **217** (2010), 480–487.

10. A. Alsaedi, S. K. Ntouyas and B. Ahmad, *An existence theorem for fractional q-difference inclusions with nonlocal substrip type boundary conditions*, Sci. World J., **2015** (2015), Article ID 424306.

11. A. Amini-Harandi, *Endpoints of set-valued contractions in metric spaces*, Nonlinear Anal., **72** (2010), 132–134.

12. D. Baleanu, R. P. Agarwal, H. Mohammadi, et al. *Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces*, Bound. Value Prob., **2013** (2013), 1–8.

13. D. Baleanu, H. Mohammadi and Sh. Rezapour, *Positive solutions of a boundary value problem for nonlinear fractional differential equations*, Abstr. Appl. Anal., **2012** (2012), Article ID 837437.

14. D. Baleanu, H. Mohammadi and Sh. Rezapour, *On a nonlinear fractional differential equation on partially ordered metric spaces*, Adv. Differ. Equations, **2013** (2013), 1–10.

15. D. Baleanu, Sh. Rezapour, S. Etemad,et al. *On a time-fractional integro-differential equation via three-point boundary value conditions*, Math. Probl. Eng., **2015** (2015), Article ID 785738.

16. K. Deimling, *Multi-valued Differential Equations*, Berlin: Walter de Gruyter, 1992.

17. M. El-Shahed and F. Al-Askar, *Positive solutions for boundary value problem of nonlinear fractional q-difference equation*, ISRN Math. Anal., **2011** (2011), Article ID 385459.

18. S. Etemad, M. Ettefagh and Sh. Rezapour, *On the existence of solutions for nonlinear fractional q-difference equations with q-integral boundary conditions*, J. Adv. Math. Stud., **8** (2015), 265–285.

19. S. Etemad and Sh. Rezapour, *On the existence of solutions for three variables fractional partial differential equation and inclusion*, J. Adv. Math. Stud., **8** (2015), 224–231.

20. R. A. C. Ferreira, *Nontrivial solutions for fractional q-difference boundary value problems*, Electron. J. Qual. Theory Differ. Equations, **70** (2010), 1–10.

21. R. A. C. Ferreira, *Positive solutions for a class of boundary value problems with fractional q-differences*, Comput. Math. Appl., **61** (2011), 367–373.

22. C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Springer, Cham, 2016.

23. J. R. Graef and L. Kong, *Positive solutions for a class of higher order boundary value problems with fractional q-derivatives*, Appl. Math. Comput., **218** (2012), 9682–9689.

24. A, Granas and J. Dugundji, *Fixed Point Theory*, New York: Springer, 2005.

25. F. H. Jackson, *On q-functions and a certain difference operator*, *Trans. R. Soc. Edinburgh*, **46** (1908), 253–281.

26. F. H. Jackson, *On q-definite integrals*, Quart. J. Pure Appl. Math., **41** (1910), 193–203.

27. V. Kac and P. Cheung, *Quantum Calculus*, New York: Springer-Verlag, 2002.

28. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Series of North-Holland Mathematics Studies, vol. 204. Amsterdam: Elsevier, 2006.

29. S. Liang and J. Zhang, *Existence and uniqueness of positive solutions for three-point boundary value problem with fractional q-differences*, J. Appl. Math. Comput., **40** (2012), 277–288.

30. J. Ma and J. Yang, *Existence of solutions for multi-point boundary value problem of fractional q-difference equation*, *Electron. J. Qual. Theory Differ. Equations,* **92** (2011), 1–10.

31. B. Mohammadi, Sh. Rezapour and N. Shahzad, *Some results on fixed points of $\alpha-\psi-$ciric generalized multifunctions*, Fixed Point Theory Appl., **2013** (2013), 1–10.

32. S. K. Ntouyas and S. Etemad, *On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions*, Appl. Math. Comput., **266** (2015), 235–243.

33. S. K. Ntouyas, S. Etemad and J. Tariboon, *Existence of solutions for fractional differential inclusions with integral boundary conditions*, Bound. Value Prob., **2015** (2015), 1–14.

34. S. K. Ntouyas, S. Etemad and J. Tariboon, *Existence results for multi-term fractional differential inclusions*, *Adv. Differ. Equations,* **2015** (2015), 1–15.

35. N. Patanarapeelert, U. Sriphanomwan and T. Sitthiwirattham, *On a class of sequential fractional q-integrodifference boundary value problems involving different numbers of q in derivatives and integrals*, Adv. Differ. Equations, **2016** (2016), 1–16.

36. B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for $\alpha-\psi-$ contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165.

37. T. Sitthiwirattham, *On nonlocal fractional q-integral boundary value problems of fractional q-difference and fractional q-integrodifference equations involving different numbers of order and q*, Bound. Value Probl., **2016** (2016), 1–19.

38. D. R. Smart, *Fixed Point Theorems*, Cambridge: Cambridge University Press, 1980.

39. C. B. Zhai and J. Ren, *Positive and negative solutions of a boundary value problem for a fractional q-difference equation*, Adv. Differ. Equations, **2017** (2017), 1–13.

40. Y. Zhao, H. Chen and Q. Zhang, *Existence results for fractional q-difference equations with nonlocal q-integral boundary conditions*, Adv. Differ. Equations, **2013** (2013), 1–15.

41. Y. Zhao, G. Ye and H. Chen, *Multiple positive solutions of a singular semipositone integral boundary value problem for fractional q-derivative equation*, *Abstr. Appl. Anal.,* **2013** (2013), Article ID 643571.

AIMS Press