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Research article

Time-varying delays in electrophysiological wave propagation along cardiac tissue and minimax control problems associated with uncertain bidomain type models

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Abstract: Motivated by topics and issues critical to human health, the problem studied in this work derives from the modeling and stabilizing control of electrical cardiac activity in order to maximize the efficiency and safety of treatment for cardiac disease.
In this paper we consider nonlinear minimax control problems constrained by an uncertain modified bidomain model of cardiac tissue electrophysiology system, in order to take into account the influence of noises in data and time-delays in signal transmission. The state system is a degenerate nonlinear coupled system of reaction-diffusion equations in the shape of a set of delay differential equations coupled with a set of delay partial differential equations with multiple time-varying delays. The concept of our minimax control approach consists in setting the problem in the worst-case disturbances which leads to the game theory in which the controls and disturbances play antagonistic roles. The proposed strategy consists in controlling these instabilities by acting on certain data to maintain the system in a desired state. First, the mathematical model is introduced and its well-posedness is studied. Second, the minimax control problem is formulated. Afterwards the Fréchet differentiability of nonlinear solution map from the couple control-disturbance input to the solution of state system is assessed as well as stability of the derived sensitive system. The existence of an optimal solution is proved and first-order necessary optimality conditions are established by using sensitivity and adjoint calculus.

Keywords: minimax control; multiple time-varying delays; electrotherapy; reaction-diffusion system; bidomain type model; parabolic-elliptic coupling; ionic model; cardiac electrophysiology; fluctuation; adjoint model; sensitive model; necessary conditions of optimality
Mathematics Subject Classification: 49J35, 49K35, 35Q92, 35K57, 35B30, 92C50

1. Introduction and mathematical setting of the problem

1.1. Motivation and study system

The heart is an electrically controlled mechanical pump which drives blood flow through the circulatory system vessels (through deformation of its walls), where electrical impulses trigger mechanical contraction (of various chambers of heart) and whose dysfunction is incompatible with life. The electrical system of a normal heart is highly organized in a steady rhythmic pattern. This normal heartbeat is called sinus rhythm. Irregular or abnormal heartbeats, called arrhythmias, are caused by a change in the propagation and/or formation of electrical impulses, that regulate a steady heartbeat, causing a heartbeat that is too fast or too slow, that can remain stable or become chaotic (irregular and disorganized). Many times, arrhythmias are harmless and can occur in healthy people without heart disease; however, some of these rhythms can be serious and require special and efficiency treatments. Fibrillation is one type of arrhythmia and is considered the most serious cardiac rhythm disturbance. It occurs when the heart beats with rapid, erratic electrical impulses (highly disorganized almost chaotic activation). This causes the heart's chambers to quiver (or fibrillate) uselessly instead of contracting normally. Then the heart loses its ability to pump enough blood through the circulatory system. The treatment therapy of these diseases, when it becomes troublesome or when it can present a danger, often uses electrical impulses to stabilize cardiac function and restore the sinus rhythm, by implanting the patients with active cardiac devices (electrotherapy). For example, in case of cardiac rhythms that are too slow, the devices transmit electronic impulses and ensure that periodic contractions of heart are maintained at a hemodynamically sufficient rate; and in the case of a fast heart rate or irregular, the devices monitor the heart rate and, if needed, treat episodes of tachyarrhythmia (including tachycardia and/or fibrillation) by transmitting automatically impulses to either give defibrillation shocks or cause overstimulation (via an ICDs[*]) or synchronize the contraction of left and right ventricles. Although ICD electrotherapy has been shown to be an effective treatment against lethal cardiac arrhythmias, it remains a highly non-optimal therapy since the administrated strong shocks required for defibrillation can cause significant extra-cardiac stimulation, resulting in (physical and psychological) pains and long-term tissue damage. It is then necessary to optimize the defibrillation shock impulse in order to achieve the lowest energy necessary to successfully cardiovert a patient and, consequently, a maximum result with minimal detrimental side effects.

Then, efficient tools for the assistance of patient specific treatment of cardiac disorders is of great scientific and socio-economical interest. The evaluation of the bioelectrical activity in the heart is a very complex process which uses different phenomenological mechanism and subject to various perturbations, and physiological and pathophysiological variations. Consequently, this has greatly emphasized the need for methodologies capable of predicting, understanding and optimizing different complex phenomena occurring in these fields, despite different sources of uncertainty like the absence of complete or reliable data (e.g., stimulus currents, measurement data), neglected dynamics, or intrinsic physical variability. The challenge here is e.g., to reduce the uncertainty and increase the reliability of model predictions in treatment of cardiac disease.

The goal of the present paper is to investigate minimax control problems for a bidomain type system, commonly used for modeling the propagation of electrophysiological waves in the myocardium, with

[*]The so-called implantable cardioverter defibrillators

disturbances (perturbation or noise) and controls in which multiple time-varying delays appear in the state system. The objective of a minimax control is to compensate the undesirable effects of system disturbances through control actions such that a cost function achieves its minimum for the worst disturbances: i.e. to find the best control which takes into account the worst-case disturbance. From the standpoint of our specific application, the main goal is to regulate and stabilize the optimal external applied current via transmembrane potential sensor.

Tissue-level cardiac electrophysiology, which can provide a bridge between electrophysiological cell models at smaller scales, and tissue mechanics, metabolism and blood flow at larger scales, is usually modeled using the coupled bidomain equations, originally derived in [67], which represent a homogenization of the intracellular and extracellular medium, where electrical currents are governed by Ohm's law (see also e.g. [44] for a review and an introduction to this field). The model was modified and extended to include heart tissue surrounded by a conductive bath or a conductive body (see e.g. [56] and [65]). From mathematical viewpoint, the classical bidomain system (Figure 1) is commonly formulated in terms of intracellular and extracellular electrical potentials of anisotropic cardiac tissue (macroscale), ϕ_i and ϕ_e, (or, equivalently, extracellular potential ϕ_e and the transmembrane voltage $\phi = \varphi_i - \varphi_e$) coupled with cellular state variables u describing cellular membrane dynamics. This is a system of non-linear partial differential equations (PDEs) coupled with ordinary differential equations (ODEs), in the physical region Ω (occupied by excitable cardiac tissue, which is an open, bounded, and connected subset of d-dimensional Euclidean space $\mathbf{R}^d$, $d \leq 3$). The PDEs describe the propagation of the electrical potentials and ODEs describe the electrochemical processes.

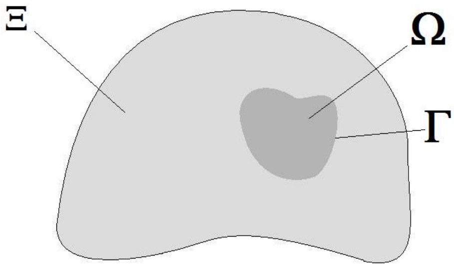

Figure 1. Bidomain system is defined on heart domain Ω, while Ξ is the rest of the body.

Time delays in signal transmission are inevitable and a small delay can affect considerably the resulting electrical activity in heart and thus the cardiac disorders therapeutic treatment. It is then necessary to introduce the impact of delays on dynamical behaviors of such a system. Delay terms can lead to change the stability of dynamics and give rise to highly complex behavior including oscillations and chaos. Motivated by above discussions, to take into account the effect of time-delays in propagation of electrophysiological waves in heart, together with other critical cardiac material parameters, we have developed a new bidomain model by incorporating multiple time delays in [6].

In this new model, in order to take into account the influence of time-delays in signal transmission and inward movement of u into the cell which prolongs the depolarization phase of action potential, classical bidomain model has been modified by using multiple time-delays functions in operators representing the ionic activity in myocardium. More precisely, the derived system, is a nonlinear coupled reaction-diffusion model in shape of a set of delay differential equations (DDE) coupled with a set of delay partial differential equations, in the heart's spatial domain Ω which is a bounded open subset with a sufficiently regular boundary $\Gamma = \partial\Omega$, and during the final fixed time horizon $T > 0$, as follows (for more detail to derive this model see [6])

$$
\begin{aligned}
&\mathfrak{c}_m \frac{\partial \phi}{\partial t} + \mathcal{I}(.;\phi,u) - div(\mathcal{K}_i \nabla \phi) = div(\mathcal{K}_i \nabla \varphi_e) + \mathcal{H}(.;\phi_\tau,u_\tau) + I_i, \text{ in } Q = \Omega \times (0,T)\\
&-div((\mathcal{K}_e + \mathcal{K}_i)\nabla \varphi_e) = div(\mathcal{K}_i \nabla \phi) + I, \text{ in } Q\\
&\frac{\partial u}{\partial t} + \mathcal{G}(.;\phi,u) = \mathcal{E}(.;\phi_\tau,u_\tau), \text{ in } Q\\
&\text{subject to initial and past conditions}\\
&\phi(.,t=0) = \phi_0, \;\; u(.,t=0) = u_0, \text{ in } \Omega\\
&\phi = \phi_{past}, \;\; u = u_{past}, \text{ in } Q_0 = \Omega \times [-\delta(0), 0[\\
&\text{and boundary conditions}\\
&(\mathcal{K}_i \nabla(\phi + \varphi_e)) \cdot \mathbf{n} = 0, \text{ on } \Sigma = \partial\Omega \times (0,T)\\
&(\mathcal{K}_e \nabla \varphi_e) \cdot \mathbf{n} = 0, \text{ on } \Sigma
\end{aligned}
\tag{1.1}
$$

where $\phi = \varphi_i - \varphi_e$, φ_e and φ_i are the transmembrane, extracellular and intracellular potentials, respectively; $\mathcal{K}_i$ and $\mathcal{K}_e$ are the conductivity tensors describing anisotropic intracellular and extracellular conductive media and $\mathfrak{c}_m(x) = \kappa C_m(x) > 0$, where C_m is the membrane capacitance per unit area and κ is the surface area-to-volume ratio. The tissue is assumed to be passive, so the capacitance C_m can be assumed to be not a function of the state variables. The function $\mathfrak{c}_m$ is assumed to be space variable and satisfies $0 < \underline{c}_m \le \mathfrak{c}_m = \mathfrak{b}_m^2 \le \overline{c}_m$ (where $\underline{c}_m$ and $\overline{c}_m$ are positive constants). The electrophysiological ionic state u describes a cumulative way of the effects of the ion transport through the cell membranes (which describe e.g., the dynamics of ion-channel and ion concentrations in different cellular compartments). The operator $\mathcal{I} = \kappa \mathcal{I}_{ion}$, where the nonlinear operator $\mathcal{I}_{ion}$ describes the sum of transmembrane ionic currents across cell membrane with u. The nonlinear operator $\mathcal{G}$ is representing the ionic activity in myocardium. Functional forms for $\mathcal{I}$ and $\mathcal{G}$ are determined by an electrophysiological cell model (which can found in the CellMl Repository[†]). The source terms are $I_i = \kappa f_i$, $I_e = \kappa f_e$ and $I = -I_i - I_e$, where f_i and f_e describe intracellular and extracellular stimulation currents, respectively. The operators $\mathcal{H}$ and $\mathcal{E}$ are time-delay operators and the functions ϕ_τ and u_τ are delayed states corresponding to ϕ and u respectively, and $\mathbf{n}$ is the outward normal to $\Gamma = \partial\Omega$. Here, the unknowns are the potentials ϕ, φ_e and a single ionic variable u (e.g. gating variable, concentration, etc.).

In absence of a grounded electrode, the bidomain equations are a naturally singular problem since φ_e only appears in the equations and boundary conditions through its gradient. Moreover, the state φ_e is only defined up to a constant. Such problems have compatibility conditions determining whether there are any solution to the PDEs. This is easily found by integrating the second equation of (1.1) over the domain and using the divergence theorem with boundary conditions. Then the following conservation of the total current is derived (*a.e. in* $(0,T)$)

$$\int_\Omega I d\mathbf{x} = 0. \tag{1.2}$$

Consequently, we must choose I such that the compatibility condition (1.2) is satisfied. Moreover, the function φ_e is defined within a class of equivalence, regardless of a time-dependent function. This

[†]http: //models.cellml.org

function can be fixed, for example by setting the Gauge condition (a.e. in (0, T))

$$\int_{\Omega} \varphi_e d\mathbf{x} = 0 \ a.e. \ in \ (0, T). \tag{1.3}$$

Remark 1.1. *1. Condition (1.3) is a common condition for pressure in fluid mechanics (in Navier-Stokes systems).*

2. The functions $\mathcal{K}_i$, $\mathcal{K}_e$, $\mathcal{H}$, $\mathcal{G}$ and $\mathfrak{c}_m$ depend on the fiber extension ratio.

3. If we assume that I is only dependent on time and is of the form

$$I(\mathbf{x}, t) = \theta(t)(\chi_{\Omega_1}(\mathbf{x}) - \chi_{\Omega_2}(\mathbf{x})), \tag{1.4}$$

where χ_{Ω_i} is the characteristic function of set Ω_i, $i = 1, 2$, then condition (1.2) is satisfied if $mes(\Omega_1) = mes(\Omega_2)$. The support regions Ω_1 and Ω_2 can be considered to represent an anode (positive electrode) and a cathode (negative electrode) respectively. □

In recent years, various problems concerning biological rhythmic phenomena and delayed processes have been studied (see e.g., [8, 13, 14, 16, 20–23, 38, 42, 43, 55, 60, 61, 64, 72] and the references therein). For problems associated with bidomain models with time-delay, the literature is limited, to our knowledge, to [6, 30]. Concerning problems associated with bidomain models without time-delays various methods and technique, as evolution variational inequalities approach, semi-group theory, Faedo-Galerkin method and others, the studies of the well-posedness of solutions have been derived in the literature (see e.g., [9, 15, 18, 27, 69] and the references therein); for development of multiscale mathematical and computational modeling of bioelectrical activity in myocardial tissue and their numerical simulations, which are based on methods as finite difference method, finite element method or lattice Boltzmann method, have been receiving a significant amount of attention (see e.g., [4, 17, 24–26, 28, 29, 31–34, 36, 40, 44, 46, 57, 63, 65, 70, 71] and the references therein), with a particular attention to the formation of cardiac disorders (as arrhythmias) and their therapeutic treatment (see e.g., [3, 41, 58, 68] and the references therein). For control problems associated with the electrocardiology, we can mention [2, 9, 19, 45, 53].

The new feature introduced in this work concerns the study of nonlinear minimax control problem for a bidomain model with time-delays of cardiac tissue electrophysiology system, in order to take into account the influence of noises in data. The minimax control problem and the necessary optimality conditions are new for these types of equations studied here. This study is motivated by the applications, for example, in determining the best optimal current to be applied (taking into account the influence of disturbances in data), so that the peaks in the transmembrane potential are damped. In this context, it is possible to consider the specific application of implantable Cardioverter defibrillators, which are used to treat patients with life-threatening ventricular arrhythmias, in order to maximize both cardiac performance and additionally the lifetime of the device. Our approach is based on the results of existence and characterization of saddle points in infinite-dimensional (for more details see [10] and for minimax control see [11, 12]).

The paper is organized as follows. In next section, first we give some preliminaries and well-posedness of the state equations results. Then some regularity results of the solution as well as the input-to-state stability estimate are derived, under extra assumptions. In Section 3, first we formulate the minimax control problem and we study rigorously the Fréchet differentiability of the solution operator of the problem. Second, we study the minimax control problem corresponding to obtain the saddle point of cost function $\mathcal{J}$. The functional $\mathcal{J}$ is depending on disturbance and control in the domain Ω over the time interval under consideration $[0, T]$. We prove the existence of an optimal solution and give necessary optimality conditions. The optimality system is corresponding to identify the gradient of the cost function that is necessary to develop a numerical computation in order to solve the minimax control problem.

2. Well-Posedness and regularity of the state system

2.1. Assumptions, notations and some fundamental inequalities

We use the standard notation for Sobolev spaces (see [1]), denoting the norm of $W^{m,p}(\Omega)$ ($m \in \mathbb{N}$, $p \in [1, \infty]$) by $\| \,.\, \|_{W^{m,p}}$. In the special case $p = 2$, we use $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. The duality pairing of a Banach space X with its dual space X' is given by $\langle ., . \rangle_{X',X}$. For a Hilbert space Y the inner product is denoted by $(., .)_Y$ and the inner product in $L^2(\Omega)$ is denoted by $(., .)$. For any pair of real numbers $r, s \geq 0$, we introduce the Sobolev space $H^{r,s}(Q)$ defined by $H^{r,s}(Q) = L^2(0, T; H^r(\Omega)) \cap H^s(0, T; L^2(\Omega))$, which is a Hilbert space normed by

$$\left(\| v \|^2_{L^2(0,T;H^r(\Omega))} + \| v \|^2_{H^s(0,T;L^2(\Omega))}\right)^{1/2},$$

where $H^s(0, T; L^2(\Omega))$ denotes the Sobolev space of order s of functions defined on $(0, T)$ and taking values in $L^2(\Omega)$, and defined by, for $\theta \in (0, 1), s = (1 - \theta)m$ with m an integer, (see e.g., [48]) $H^s(0, T; L^2(\Omega)) = [H^m(0, T, L^2(\Omega)), L^2(Q)]_\theta$, $H^m(0, T; L^2(\Omega)) = \left\{v \in L^2(Q) | \ \frac{\partial^j v}{\partial t^j} \in L^2(Q), \text{for } 1 \leq j \leq m\right\}$. For a given Banach space $\mathbb{X}$, with norm $\|.\|_{\mathbb{X}}$, of functions integrable on Ω, we define its subspace $\mathbb{X}|_{\mathbb{R}} = \left\{u \in \mathbb{X}, \displaystyle\int_\Omega u = 0\right\}$ that is a Banach space with norm $\|.\|_{\mathbb{X}}$, and we denote by $[u]$ the projection of $u \in \mathbb{X}$ on $\mathbb{X}|_{\mathbb{R}}$ such that $[u] = u - \dfrac{1}{mes(\Omega)} \displaystyle\int_\Omega u d\mathbf{x}$ (with $mes(\Omega)$ standing for Lebesgue measure of the domain Ω). Finally, we introduce the spaces:

- $\mathbb{H} = L^2(\Omega)$ and $\mathbb{V} = H^1(\Omega)$ endowed with their usual norms,
- $\mathbb{U} = \mathbb{V}|_{\mathbb{R}}$.

We will denote by $\mathbb{V}'$ (resp. $\mathbb{U}'$) the dual of $\mathbb{V}$ (resp. of $\mathbb{U}$). We have the following continuous embeddings (see e.g. [1,47]), where $p \geq 2$ if $d = 2$ and $2 \leq p \leq 6$ if $d = 3$, p' is such that $\dfrac{1}{p'} + \dfrac{1}{p} = 1$

$$\begin{aligned} &\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}', \ \mathbb{U} \subset \mathbb{H}|_{\mathbb{R}} \subset \mathbb{U}', \\ &\mathbb{V} \subset L^p(\Omega) \subset \mathbb{H} \equiv (\mathbb{H})' \subset L^{p'}(\Omega) \subset \mathbb{V}' \end{aligned} \tag{2.1}$$

and the injections $\mathbb{V} \subset \mathbb{H}$ and $\mathbb{U} \subset \mathbb{H}|_{\mathbb{R}}$ are compact. We can now introduce the following spaces: $\mathcal{H}(Q) = L^\infty(0, T; L^2(\Omega))$, $\mathcal{V}(Q) = L^2(0, T; \mathbb{V})$, $\tilde{\mathcal{V}}(Q) = L^2(0, T; \mathbb{U})$ and, for $q > 1$, the space $\mathcal{W}_q(Q) = \left\{w \in \mathcal{V}(Q) | \frac{\partial w}{\partial t} \in L^q(0, T, \mathbb{V}')\right\}$.

Remark 2.1. *If $u \in \mathcal{W}_q(Q) \cap \mathcal{H}(Q)$, then u is a weakly continuous function on $[0, T]$ with values in $L^2(\Omega)$ i.e. $u \in C_w([0, T]; L^2(\Omega))$ (see e.g. [47]).* □

Remark 2.2. *Let $\Omega \subset I\!R^m$, $m \geq 1$, be an open and bounded set with a smooth boundary and q be a nonnegative integer. We have the following results (see e.g. [1])*

(i) $H^q(\Omega) \subset L^p(\Omega)$, $\forall p \in [1, \frac{2m}{m-2q}]$, with continuous embedding (with the exception that if $2q = m$, then $p \in [1, +\infty[$ and if $2q > m$, then $p \in [1, +\infty]$).

(ii) (Gagliardo-Nirenberg inequalities) There exists $C > 0$ such that

$$\| v \|_{L^p} \leq C \| v \|_{H^q}^{\theta} \| v \|_{L^2}^{1-\theta}, \forall v \in H^q(\Omega),$$

where $0 \leq \theta < 1$ and $p = \frac{2m}{m-2\theta q}$ (with the exception that if $q - m/2$ is a nonnegative integer, then θ is restricted to 0). □

Remark 2.3. *The spaces $\mathcal{W}^{(i)} = H^1(0, T; H^{i-2}(\Omega)) \cap L^2(0, T; H^i(\Omega))$ satisfy the following embedding:*
(i) $\mathcal{W}^{(i)}$, for $i = 1, 3$, is compactly embedded into $L^2(0, T, H^{i-1}(\Omega))$ (see e.g. [66]).
(ii) $\mathcal{W}^{(i)} \subset C^0([0, T]; H^{i-1}(\Omega))$, for $i = 1, 3$ (see e.g. [48]). □

Definition 2.1. *A real valued function $\mathcal{H}$ defined on $D \times I\!R^q$, $q \geq 1$, is a Carathéodory function iff $\mathcal{H}(.; \mathbf{v})$ is measurable for all $\mathbf{v} \in I\!R^q$ and $\mathcal{H}(y; .)$ is continuous for almost all $y \in D$.*

Lemma 2.1. *(**PoincarWirtinger inequality**) Assume that $1 \leq p \leq \infty$ and that Ω is a bounded connected open subset of $I\!R^d$ with a sufficiently regular boundary $\partial\Omega$ (e.g., a Lipschitz boundary). Then there exists a Poincar constant C , depending only on Ω and p, such that for every function u in Sobolev space $W^{1,p}(\Omega)$*

$$\|[u]\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

Remark 2.4. *From the PoincarWirtinger inequality, we can deduce that the H^1 semi-norm and the H^1 norm are equivalent in the space $\mathbb{U}$.* □

Our study involves the following fundamental inequalities, which are repeated here for review:

(i) *Hölder's inequality*: $\displaystyle\int_D \Pi_{i=1,k} f_i dx \leq \Pi_{i=1,k} \| f_i \|_{L^{q_i}(D)}$,

where $\| f_i \|_{L^{q_i}(D)} = \left(\displaystyle\int_D | f_i |^{q_i} dx \right)^{1/q_i}$ and $\displaystyle\sum_{1 \leq i \leq k} \frac{1}{q_i} = 1$.

(ii) *Young's inequality* ($\forall a, b > 0$ and $\epsilon > 0$): $ab \leq \frac{\epsilon}{p} a^p + \frac{\epsilon^{-q/p}}{q} b^q,\ for\ p, q \in]1, +\infty[\ and\ \frac{1}{p} + \frac{1}{q} = 1.$

(iii) *Minkowski's integral inequality*:

$$\left[\int_\Omega \left(\int_0^t | f(x,s) | \, ds \right)^p dx \right]^{1/p} \leq \int_0^t \left(\int_\Omega | f(x,s) |^p \, dx \right)^{1/p} ds, \text{ for } p \in]1, +\infty[\text{ and } t > 0.$$

(iv) *Gronwall's Lemma*:

$$\text{If } \frac{d\psi}{dt} \leq g(t)\psi(t) + h(t),\ \forall t \geq 0\ \ then$$
$$\psi(t) \leq \psi(0) exp \left(\int_0^t g(s) ds \right) + \int_0^t h(s) exp \left(\int_s^t g(\tau) d\tau \right) ds,\ \forall t \geq 0.$$

Finally, we denote by $\mathfrak{L}(A; B)$ the set of linear and continuous operators from a vectorial space A into a vectorial space B, and by $\mathcal{R}^*$ the adjoint operator to a linear operator $\mathcal{R}$ between Banach spaces.

From now on, we assume that the following assumptions hold for the nonlinear operators and tensor functions appearing in our model.

(H1) We assume that the conductivity tensor functions $\mathcal{K}_\theta \in W^{1,\infty}(\overline{\Omega})$, $\theta \in \{i, e\}$ are symmetric, positive definite matrix functions and that they are uniformly elliptic, i.e., there exist constants $0 < K_1 < K_2$ such that

$$K_1\|\psi\|^2 \leq \psi^T \mathcal{K}_\theta \psi \leq K_2\|\psi\|^2 \ \text{ in } \overline{\Omega}, \quad \forall \psi \in \mathbb{R}^d. \tag{2.2}$$

Remark 2.5. *We can emphasize a specificity of the tensors $\mathcal{K}_e$ and $\mathcal{K}_i$ (see e.g., [29]).*

1. *The tensors $\mathcal{K}_e(\mathbf{x})$ and $\mathcal{K}_i(\mathbf{x})$ have the same basis of eigenvectors $Q(\mathbf{x}) = (q_k(\mathbf{x}))_{1\leq k\leq d}$ in $\mathbb{R}^d$, which reflect the organization of muscle in fibers, and consequently $\mathcal{K}_i(\mathbf{x}) = Q(\mathbf{x})\Lambda_i(\mathbf{x})Q(\mathbf{x})^T$ and $\mathcal{K}_e(\mathbf{x}) = Q(\mathbf{x})\Lambda_e(\mathbf{x})Q(\mathbf{x})^T$, where $\Lambda_i(\mathbf{x}) = diag((\lambda_{i,k})_{1\leq k\leq d})$ and $\Lambda_e(\mathbf{x}) = diag((\lambda_{e,k})_{1\leq k\leq d})$.*
2. *The muscle fibers are tangent to Γ so that (for $\theta \in \{i, e\}$) : $\mathcal{K}_\theta \mathbf{n} = \lambda_{\theta,d}\mathbf{n}$, a.e., in Γ, with $\lambda_{\theta,d}(\mathbf{x}) \geq \lambda > 0$, λ a constant.* □

The operators $\mathcal{I}$ and $\mathcal{G}$ which describe electrophysiological behavior of the system can be taken as follows (affine functions with respect to u)

$$\begin{aligned} \mathcal{I}(\mathbf{x}, t; \phi, u) &= \mathcal{I}_0(\mathbf{x}, t; \phi) + \mathcal{I}_1(\mathbf{x}, t; \phi)u, \\ \mathcal{G}(\mathbf{x}, t; \phi, u) &= \mathcal{I}_2(\mathbf{x}, t; \phi) + \hbar(\mathbf{x}, t)u, \end{aligned} \tag{2.3}$$

where $\hbar$ is a sufficiently regular function. Moreover, the operators $\mathcal{I}_0$, $\mathcal{I}_1$ and $\mathcal{I}_2$ appearing in $\mathcal{I}$ and $\mathcal{G}$, are supposed to satisfy the following assumptions.

(H2) The operators $\mathcal{I}_0, \mathcal{I}_1$ and $\mathcal{I}_2$ are Carathodory functions from $(\Omega\times\mathbb{R})\times\mathbb{R}$ into $\mathbb{R}$ and continuous on ϕ (as in Belmiloudi [9]). Furthermore, for some $p \geq 2$ if $d = 2$ and $p \in [2, 6]$ if $d = 3$ (for more details see [18]), the following requirements hold

(i) there exist constants $\beta_i \geq 0$ $(i = 1, \ldots, 6)$ such that for any $v \in \mathbb{R}$

$$|\mathcal{I}_0(.; v)| \leq \beta_1 + \beta_2|v|^{p-1}, \tag{2.4}$$

$$|\mathcal{I}_1(.; v)| \leq \beta_3 + \beta_4|v|^{p/2-1}, \tag{2.5}$$

$$|\mathcal{I}_2(.; v)| \leq \beta_5 + \beta_6|v|^{p/2}. \tag{2.6}$$

(ii) there exist constants $\mu_1 > 0, \mu_2 > 0, \mu_3 \geq 0, \mu_4 \geq 0$ such that for any $(v, w) \in \mathbb{R}^2$:

$$\mu_1 v\mathcal{I}(.; v, w) + w\mathcal{G}(.; v, w) \geq \mu_2|v|^p - \mu_3\left(\mu_1|v|^2 + |w|^2\right) - \mu_4. \tag{2.7}$$

In order to assure the uniqueness of solution we assume that

(H3) The Nemytskii operators $\mathcal{I}$ and $\mathcal{G}$ satisfy Carathodory conditions and there exists some $\mu > 0$ such the operator $F_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F_\mu(.; \mathbf{v}) = \begin{pmatrix} \mu(\mathcal{I}(.; \mathbf{v})) \\ \mathcal{G}(.; \mathbf{v}) \end{pmatrix}, \ \forall \mathbf{v} = (v, w) \in \mathbb{R}^2, \tag{2.8}$$

satisfies a one-sided Lipschitz condition (see e.g. Seidman et al. [62], Belmiloudi [14]): there exists a constant $C_L > 0$ such that $(\forall \mathbf{v}_i = (v_i, w_i) \in \mathbb{R}^2,\ i = 1, 2)$

$$\Big(F_\mu(.; \mathbf{v}_1) - F_\mu(.; \mathbf{v}_2)\Big) \cdot (\mathbf{v}_1 - \mathbf{v}_2) \geq -C_L\|\mathbf{v}_1 - \mathbf{v}_2\|^2. \tag{2.9}$$

Finally, we assume that the operators $\mathcal{H}$ and $\mathcal{E}$ which describe multiple time-delays related to ϕ and u are defined as in Belmiloudi [14] i.e.,

$$\begin{aligned}\mathcal{H}(\mathbf{x},t;\phi_\tau,u_\tau) &= \sum_{k=1}^{n_1} a_k(\mathbf{x},t)\phi(\mathbf{x},t-\xi_k(t)) + \sum_{l=1}^{n_2} b_l(\mathbf{x},t)u(\mathbf{x},t-\eta_l(t)),\\ \mathcal{E}(\mathbf{x},t;\phi_\tau,u_\tau) &= \sum_{k=1}^{n_1} c_k(\mathbf{x},t)\phi(\mathbf{x},t-\xi_k(t)) + \sum_{l=1}^{n_2} d_l(\mathbf{x},t)u(\mathbf{x},t-\eta_l(t)),\end{aligned} \tag{2.10}$$

where a_k, c_k, b_l and d_l (for $1 \le k \le n_1$ and $1 \le l \le n_2$) are C^∞ functions. For the functions ξ_k and η_l (for $1 \le k \le n_1$ and $1 \le l \le n_2$), we suppose that (as in [14]):

(RC) $t \in [0,T) \to (r_k(t) = t - \xi_k(t),\ 1 \le k \le n_1)$ and $t \in [0,T) \to (p_l(t) = t - \eta_l(t),\ 1 \le l \le n_2)$ are strictly increasing functions and $(\xi_k(t),\ 1 \le k \le n_1)$ and $(\eta_l(t),\ 1 \le l \le n_2)$ are C^1 non-negative functions on $[0,T)$. So we have the existence of inverse functions $(e_k)_k$ of $(r_k)_k$ and $(q_l)_l$ of $(p_l)_l$, respectively. We also define the following subdivision: $s_{-1} = -\delta(0) = \max_{1\le k\le n_1}\max_{1\le l\le n_2}(\xi_k(0),\eta_l(0))$, $s_0 = 0$ and $\forall j \in \mathbb{N}^*$, $s_j = \min_{1\le k\le n_1}\min_{1\le l\le n_2}\left(e_k(s_{j-1}), q_l(s_{j-1})\right)$, and we denote T_j as $T_j = s_j - s_{j-1}$, $\forall j \in \mathbb{N}^*$. We introduce the following notations: $I_j = (s_{-1}, s_j)$ and $Q_j = \Omega \times I_j$ for $j \in \mathbb{N}$.

Remark 2.6. *According to hypotheses (RC), we prove easily that:*
(i) the sequences $(s_j)_{j\in\mathbb{N}}$ *is strictly increasing and* $s_j \le T, \forall j \ge 0$,
(ii) for $j \ge 2$, *if* $t \in (s_{j-1}, s_j)$ *then* $\forall i = 1,n$, $r_i(t) \le s_{j-1}$, $p_i(t) \le s_{j-1}$,
(iii) if $t \in (s_0, s_1)$ *then* $\forall i = 1,n$, $r_i(t) \in (s_{-1}, s_0)$, $p_i(t) \in (s_{-1}, s_0)$. □

Remark 2.7. *The functions* a_k, c_k, b_k *and* d_k *are diffusion coefficients which represent the strength of each associated time-delay. A zero coefficient means that the associated previous state doesn't impact the system. Time-delays come from biological inhomogeneous properties of heart region. Electrical waves go through muscles, bones or fat which induce time-delays in their interaction in regards to ionical channels behavior.* □

Lemma 2.2. *([6]) Assume that* F_μ *is differentiable with respect to* (ϕ,u) *and denote by* $\lambda_1(\phi,u) \le \lambda_2(\phi,u)$ *the eigenvalues of the symmetrical part of Jacobian matrix* $\nabla F_\mu(\phi,u)$:

$$Q_\mu(\phi,u) = \frac{1}{2}\left(\nabla F_\mu(\phi,u)^T + \nabla F_\mu(\phi,u)\right).$$

If there exist a constant C_F *independent of* ϕ *and* u *such as:*

$$C_F \le \lambda_1(\phi,u) \le \lambda_2(\phi,u), \tag{2.11}$$

then F_μ *satisfies the hypothesis (H3).*

Lemma 2.3. *([6]) Let assumptions* (2.3)*, (H1) and (H2) be fulfilled. For* $(\phi,u) \in L^p(\Omega)\times\mathbb{H}$ *and a.e., t, there exist constants* $C_i > 0$ *(*$i = 1,6$*) such that*

$$\|\mathcal{I}(.,t;\phi,u)\|_{L^{p'}(\Omega)} \le C_1 + C_2\|\phi\|_{L^p(\Omega)}^{p/p'} + C_3\|u\|_{\mathbb{H}}^{2/p'}, \tag{2.12}$$

$$\|\mathcal{G}(.,t;\phi,u)\|_{L^2(\Omega)} \le C_4 + C_5\|\phi\|_{L^p(\Omega)}^{p/2} + C_6\|u\|_{\mathbb{H}}, \tag{2.13}$$

where p' *is such that* $\frac{1}{p} + \frac{1}{p'} = 1$.

In the sequel we will always denote C some positive constant which may be different at each occurrence.

2.2. Variational formulation and preliminary results

We now define the following forms

$$A_i(\psi, v) = \int_\Omega \mathcal{K}_i \nabla\psi \cdot \nabla v d\mathbf{x}, \quad A_e(\psi, v) = \int_\Omega \mathcal{K}_e \nabla\psi \cdot \nabla v d\mathbf{x}. \tag{2.14}$$

Proposition 2.1. *(i) A_i and A_e are symmetric bilinear continuous forms on $\mathbb{V}$ and $\mathbb{U}$, respectively.*
(ii) A_i and A_e are coercive on $\mathbb{V}$ and $\mathbb{U}$, respectively (we denote by α_i and α_e their coercivity coefficients).

Proof. (i) and (ii) are easily obtained providing that properties of tensors $\mathcal{K}_i$ and $\mathcal{K}_e$ and (2.2) are satisfied. □

We can now write the weak formulation of problem (1.1) (for all $v \in \mathbb{V},\ v_e \in \mathbb{U}$ and $\rho \in \mathbb{H}$)

$$\begin{aligned}
&\langle \mathfrak{c}_m \frac{\partial \phi}{\partial t}, v\rangle_{\mathbb{V}',\mathbb{V}} + \int_\Omega \mathcal{I}(.;\phi,u) v d\mathbf{x} + A_i(\phi + \varphi_e, v) = \langle I_i, v\rangle_{\mathbb{V}',\mathbb{V}} + \int_\Omega \mathcal{H}(.,\phi_\tau,u_\tau) v d\mathbf{x},\\
&A_i(\phi + \varphi_e, v_e) + A_e(\varphi_e, v_e) = \langle I, v_e\rangle_{\mathbb{V}',\mathbb{V}},\\
&(\frac{\partial u}{\partial t}, \rho)_{\mathbb{H}} + \int_\Omega \mathcal{G}(.;\phi,u)\rho d\mathbf{x} = \int_\Omega \mathcal{E}(.;\phi_\tau,u_\tau)\rho d\mathbf{x},\\
&\phi(.,t=0) = \phi_0, \quad u(.t=0) = u_0,\\
&\phi(.,t') = \phi_{past}(.,t'), \quad u(.,t') = u_{past}(.,t'), \quad t' \in [-\delta(0), 0[.
\end{aligned} \tag{2.15}$$

Theorem 2.1. *([18]) Let $g \in \mathbb{V}'$ and $\varphi \in \mathbb{U}$ be given. The variational equations*

$$(A_i + A_e)(\underline{\varphi}_e, v_e) + A_i(\varphi, v_e) = 0, \quad \forall v_e \in \mathbb{U} \tag{2.16}$$

and

$$(A_i + A_e)(\overline{\varphi}_e, v_e) = \langle g, v_e\rangle_{\mathbb{V}',\mathbb{V}}, \quad \forall v_e \in \mathbb{U} \tag{2.17}$$

have unique solutions $\underline{\varphi}_e, \overline{\varphi}_e \in \mathbb{U}$. Moreover we have that the operator $\underline{A}_i : (\varphi, v) \in (\mathbb{U})^2 \longrightarrow \underline{A}_i(\varphi, v) = A_i(\varphi, v) + A_i(\underline{\varphi}_e, v)$ is symmetric bilinear continuous forms on $\mathbb{U}$.

Introduce the following spaces for $S_0 < S_f$ be fixed real values (where $Q_S = \Omega \times (S_0, S_f)$, $p \geq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$)

$$\begin{aligned}
&\mathbb{D}_p(S_0, S_f) = L^{p'}(Q_S) + L^2(S_0, S_f; \mathbb{V}') \subset L^{p'}(S_0, S_f; \mathbb{V}'),\\
&\mathbb{W}_p(S_0, S_f) = \{u \in L^p(Q_S) \cap L^2(S_0, S_f; \mathbb{V}) \text{ such that } \frac{\partial u}{\partial t} \in \mathbb{D}_p(S_0, S_f)\}.
\end{aligned}$$

Lemma 2.4. *([6, 18]) Let π_m be a sequence converging toward π in $\mathbb{W}_p(S_0, S_f)$ weakly and in $L^2(Q_S)$ strongly and V_m be a sequence converging toward V in $L^2(Q_S) \cap H^1(S_0, S_f; \mathbb{H})$ weakly. Then we have the following convergence results:*

(i) $\mathcal{I}_0(.;\pi_m) \rightharpoonup \mathcal{I}_0(.;\pi)$ weakly in $L^{p'}(Q_S)$

(ii) $\mathcal{I}_2(.;\pi_m) \rightharpoonup \mathcal{I}_2(.;\pi)$ weakly in $L^2(Q_S)$

(iii) $\mathcal{I}_1(.;\pi_m)V_m \rightharpoonup \mathcal{I}_1(.;\pi)V$ weakly in $L^2(Q_S)$.

The considered functions $\mathcal{I}_i$, in this paper, include the three classical type models in which these assumptions are satisfied (for the proof, we use similar arguments as in [18]) namely the Rogers-McCulloch [51] (RM), Fitz-Hugh-Nagumo [37] (FHN) and Aliev-Panfilov [54](LAP) models as follows. The function $\mathcal{I}_0$ is defined by a cubic reaction term of the form $\mathcal{I}_0(.;v) = b_1(.)v(v-r)(v-1)$, and the functions $\mathcal{I}_1$ and $\mathcal{I}_2$ are given by

(a) for RM type model : $\mathcal{I}_1(.;v) = b_2(.)v, \;\; \mathcal{I}_2(;,v) = -b_3(.)v,$

(b) for FHN type model : $\mathcal{I}_1(.;v) = b_2(.), \;\; \mathcal{I}_2(;,v) = -b_3(.)v,$

(c) for LAP type model : $\mathcal{I}_1(.;v) = b_2(.)v, \;\; \mathcal{I}_2(;,v) = b_3(.)v(r+1-v),$

where $b_i \in W^{1,\infty}(Q)$, $i = 1,3$, are sufficiently regular functions from Q into $\mathbb{R}^{+,*}$ and $r \in [0,1]$. We obtain easily the following Lemma.

Lemma 2.5. *The following properties hold:*

1. *For all v_1, v_2 in $\mathbb{R}$ we have*

$$\mathcal{I}_0(.;v_1) - \mathcal{I}_0(.;v_2) = b_1(v_1-v_2)\left(v_1^2+v_2^2+v_1v_2-(r+1)(v_1+v_2)+r\right) \text{ and}$$

(a) for RM type model : $\mathcal{I}_1(.;v_1) - \mathcal{I}_1(.;v_2) = b_2(v_1-v_2), \;\; \mathcal{I}_2(.;v_1) - \mathcal{I}_2(.;v_2) = -b_3(v_1-v_2),$

(b) for FHN type model : $\mathcal{I}_1(.;v_1) - \mathcal{I}_1(.;v_2) = 0, \;\; \mathcal{I}_2(.;v_1) - \mathcal{I}_2(.;v_2) = -b_3(v_1-v_2),$

(c) for LAP type model : $\mathcal{I}_1(.;v_1) - \mathcal{I}_1(.;v_2) = b_2(v_1-v_2),$

$$\mathcal{I}_2(.;v_1) - \mathcal{I}_2(.;v_2) = b_3(v_1-v_2)((r+1)-v_1-v_2).$$

2. *The partial derivative of the function $\mathcal{I}_0$ is given by $\dfrac{\partial \mathcal{I}_0}{\partial v}(.;v) = b_1\left(3v^2-2(r+1)v+r\right)$ and these of the functions $\mathcal{I}_1$ and $\mathcal{I}_2$ are given by*

(a) for RM type model : $\dfrac{\partial \mathcal{I}_1}{\partial v}(.;v) = b_2, \;\; \dfrac{\partial \mathcal{I}_2}{\partial v}(.;v) = -b_3,$

(b) for FHN type model : $\dfrac{\partial \mathcal{I}_1}{\partial v}(.;v) = 0, \;\; \dfrac{\partial \mathcal{I}_2}{\partial v}(.;v) = -b_3,$

(c) for LAP type model : $\dfrac{\partial \mathcal{I}_1}{\partial v}(.;v) = b_2, \;\; \dfrac{\partial \mathcal{I}_2}{\partial v}(.;v) = b_3(r+1-2v).$

Remark 2.8. *According to Lemma 2.5, the partial derivatives of $\mathcal{I}$ and $\mathcal{G}$ are given by*
(a) for RM type model :

$$\frac{\partial \mathcal{I}}{\partial \phi} = b_1(3\phi^2-2(1+r)\phi+r)+b_2u, \;\; \frac{\partial \mathcal{I}}{\partial u} = b_2\phi, \;\; \frac{\partial \mathcal{G}}{\partial \phi} = -b_3, \;\; \frac{\partial \mathcal{G}}{\partial u} = \hbar, \tag{2.18}$$

(b) for FHN type model :

$$\frac{\partial \mathcal{I}}{\partial \phi} = b_1(3\phi^2-2(1+r)\phi+r), \;\; \frac{\partial \mathcal{I}}{\partial u} = b_2, \;\; \frac{\partial \mathcal{G}}{\partial \phi} = -b_3, \;\; \frac{\partial \mathcal{G}}{\partial u} = \hbar, \tag{2.19}$$

(c) for LAP type model :

$$\frac{\partial \mathcal{I}}{\partial \phi} = b_1(3\phi^2-2(1+r)\phi+r)+b_2u, \;\; \frac{\partial \mathcal{I}}{\partial u} = b_2\phi, \;\; \frac{\partial \mathcal{G}}{\partial \phi} = b_3(r+1-2\phi), \;\; \frac{\partial \mathcal{G}}{\partial u} = \hbar. \tag{2.20}$$

Consequently, $\mathcal{I}_i$ (for $i = 0, 2$) and the partial derivatives of $\mathcal{I}$ and $\mathcal{G}$ for this three models are of the form

$$\begin{aligned} &\mathcal{I}_0 = \hbar_{11}\phi^3 - \hbar_{12}\phi^2 + \hbar_{13}\phi, \;\; \mathcal{I}_1 = \epsilon_1\hbar_{21}\phi + \hbar_{22}, \mathcal{I}_2 = -\epsilon_2\hbar_{31}\phi^2 + (2\epsilon_2 - 1)\hbar_{32}\phi, \\ &\frac{\partial \mathcal{I}}{\partial \phi} = \frac{\partial \mathcal{I}_0}{\partial \phi} + u\frac{\partial \mathcal{I}_1}{\partial \phi} = 3\hbar_{11}\phi^2 - 2\hbar_{12}\phi + \hbar_{13} + \epsilon_1\hbar_{21}u, \quad \frac{\partial \mathcal{I}}{\partial u} = \mathcal{I}_1 = \epsilon_1\hbar_{21}\phi + \hbar_{22}, \\ &\frac{\partial \mathcal{G}}{\partial \phi} = \frac{\partial \mathcal{I}_2}{\partial \phi} = -2\epsilon_2\hbar_{31}\phi + (2\epsilon_2 - 1)\hbar_{32}, \quad \frac{\partial \mathcal{G}}{\partial u} = \hbar, \end{aligned} \tag{2.21}$$

where $\hbar_{ij}$ and $\hbar$ are sufficiently regular and bounded functions from Q into $[h_0, +\infty[$, with $h_0 \in I\!R^{+,*}$ and $(\epsilon_1, \epsilon_2) \in \{(1, 0), (0, 0), (1, 1)\}$. □

For delay operators we have the following estimates.

Lemma 2.6. *Let (v, ρ) be in $(L^q(0, T; L^\sigma(\Omega)))^2$, with $\sigma, q \in [1, \infty[$, such that on the domain Q_0, $(v, \rho) = (v_{past}, \rho_{past}) \in (L^q(-\delta(0), 0; L^\sigma(\Omega)))^2$. Then the following estimates hold.*

(i) There exists a constant $C_{\infty,0} > 0$ (depending on $\|a_k\|_\infty, \|c_k\|_\infty$, $\|b_l\|_\infty, \|d_l\|_\infty$, $1 \le k \le n_1$, $1 \le l \le n_2$) such that

$$\begin{aligned} \| \mathcal{H}(v_\tau, \rho_\tau) \|_{L^\sigma(\Omega)} &\le C_{\infty,0}\left(\sum_{k=1}^{n_1} \|v(., r_k(t))\|_{L^\sigma(\Omega)} + \sum_{l=1}^{n_2} \|\rho(., p_l(t))\|_{L^\sigma(\Omega)}\right), \\ \| \mathcal{E}(v_\tau, \rho_\tau) \|_{L^\sigma(\Omega)} &\le C_{\infty,0}\left(\sum_{k=1}^{n_1} \|v(., r_k(t))\|_{L^\sigma(\Omega)} + \sum_{l=1}^{n_2} \|\rho(., p_l(t))\|_{L^\sigma(\Omega)}\right). \end{aligned} \tag{2.22}$$

(ii) There exists a constant $C_{\infty,1} > 0$ (depending on $\|a_k\|_\infty, \|c_k\|_\infty$, $\|a'_k\|_\infty, \|c'_k\|_\infty$, $\|b_l\|_\infty, \|d_l\|_\infty$, $\|b'_l\|_\infty, \|d'_l\|_\infty$, $1 \le k \le n_1$, $1 \le l \le n_2$) such that

$$\begin{aligned} \| \mathcal{H}(v_\tau, \rho_\tau) \|_{L^q(0,t;L^\sigma(\Omega))} &\le C_{\infty,1}\Big(\|v\|_{L^q(0,t;L^\sigma(\Omega))} + \|\rho\|_{L^q(0,t;L^\sigma(\Omega))} \\ &\quad + \|v_{past}\|_{L^q(-\delta(0),0;L^\sigma(\Omega))} + \|\rho_{past}\|_{L^q(-\delta(0),0;L^\sigma(\Omega))}\Big), \\ \| \mathcal{E}(v_\tau, \rho_\tau) \|_{L^q(0,t;L^\sigma(\Omega))} &\le C_{\infty,1}\Big(\|v\|_{L^q(0,t;L^\sigma(\Omega))} + \|\rho\|_{L^q(0,t;L^\sigma(\Omega))} \\ &\quad + \|v_{past}\|_{L^q(-\delta(0),0;L^\sigma(\Omega))} + \|\rho_{past}\|_{L^q(-\delta(0),0;L^\sigma(\Omega))}\Big), \end{aligned} \tag{2.23}$$

Proof. (i) According to regularity of $(a_k)_{1\le k\le n_1}$, $(c_k)_{1\le k\le n_1}$, $(b_l)_{1\le l\le n_2}$, $(c_l)_{1\le l\le n_2}$and to Remark 2.6, we obtain (for $1 \le k \le n_1$, $1 \le l \le n_2$ and $T \ge t \ge 0$)

$$\begin{aligned} &\int_\Omega |\, \tilde{a}_k(x, t)v(\mathbf{x}, r_k(t)) \,|^\sigma \, d\mathbf{x} \le \|\tilde{a}_k\|_\infty^\sigma \|v(., r_k(t))\|_{L^\sigma(\Omega)}^\sigma, \;\; (\text{for } \tilde{a}_k = a_k \text{ or } c_k) \\ &\int_\Omega |\, \tilde{b}_l(x, t)\rho(\mathbf{x}, p_l(t)) \,|^\sigma \, d\mathbf{x} \le \|\tilde{b}_l\|_\infty^\sigma \|\rho(., p_l(t))\|_{L^\sigma(\Omega)}^\sigma, \;\; (\text{for } \tilde{b}_l = b_l \text{ or } d_l) \end{aligned} \tag{2.24}$$

Then, from the expression of $\mathcal{H}$ and $\mathcal{E}$, we can deduce that

$$\begin{aligned} \| \mathcal{H}(v_\tau(., t), \rho_\tau(., t)) \|_{L^\sigma(\Omega)} &\le D_{1,0}\left(\sum_{k=1}^{n_1} \| v(., r_k(t)) \|_{L^\sigma(\Omega)} + \sum_{l=1}^{n_2} \| \rho(., p_l(t)) \|_{L^\sigma(\Omega)}\right), \\ \| \mathcal{E}(v_\tau(., t), \rho_\tau(., t)) \|_{L^\sigma(\Omega)} &\le D_{2,0}\left(\sum_{k=1}^{n_1} \| v(., r_k(t)) \|_{L^\sigma(\Omega)} + \sum_{l=1}^{n_2} \| \rho(., p_l(t)) \|_{L^\sigma(\Omega)}\right), \end{aligned} \tag{2.25}$$

where $D_{1,0} = \max(\max_{1\le k\le n_1} \|a_k\|_\infty, \max_{1\le l\le n_2} \|b_l\|_\infty), D_{2,0} = \max(\max_{1\le k\le n_1} \|c_k\|_\infty, \max_{1\le l\le n_2} \|d_l\|_\infty)$.

(ii) Setting $\theta = r_k(s)$ (resp. $\theta = p_l(s)$), we have $s = e_k(\theta)$ (resp. $s = q_l(\theta)$) and then $ds = e_k'(\theta)d\theta$ (resp. $ds = q_l'(\theta)d\theta$). So

$$\begin{aligned}
\|v(.,r_k(.))\|^q_{L^q(0,t;L^\sigma(\Omega))} &= \int_0^t \|v(.,r_k(s))\|^q_{L^\sigma(\Omega)}ds \le \|e_k'\|_\infty\left(\int_{-\xi_k(0)}^{t-\xi_k(t)} \|v(.,\theta)\|^q_{L^\sigma(\Omega)}d\theta\right),\\
\|\rho(.,r_k(.))\|^q_{L^q(0,t;L^\sigma(\Omega))} &= \int_0^t \|\rho(.,p_l(s))\|^q_{L^\sigma(\Omega)}ds \le \|q_l'\|_\infty\left(\int_{-\eta_l(0)}^{t-\eta_l(t)} \|\rho(.,\theta)\|^q_{L^\sigma(\Omega)}d\theta\right).
\end{aligned} \tag{2.26}$$

Since $-\delta(0) \le -\xi_k(0), -\delta(0) \le -\eta_k(0)$, $t - \xi_k(t) \le t$ and $t - \eta_k(t) \le t$ we can deduce that (since $v = v_{past}$, $\rho = \rho_{past}$, on Q_0)

$$\begin{aligned}
\|v(.,r_k(.))\|^q_{L^q(0,t;L^\sigma(\Omega))} &\le \|e_k'\|_\infty\left(\int_0^t \|v(.,\theta)\|^q_{L^\sigma(\Omega)}d\theta + \int_{-\delta(0)}^0 \|v_{past}(.,s)\|^q_{L^\sigma(\Omega)}ds\right)\\
&\le \max_{1\le k\le n_1}(\|e_k'\|_\infty)\left(\|v\|^q_{L^q(0,t;L^\sigma(\Omega))} + \|v_{past}\|^q_{L^q(-\delta(0),0;L^\sigma(\Omega))}\right),\\
\|\rho(.,r_k(.))\|^q_{L^q(0,t;L^\sigma(\Omega))} &\le \|q_l'\|_\infty\left(\int_0^t \|\rho(.,\theta)\|^q_{L^\sigma(\Omega)}d\theta + \int_{-\delta(0)}^0 \|\rho_{past}(.,s)\|^q_{L^\sigma(\Omega)}ds\right)\\
&\le \max_{1\le l\le n_2}(\|q_l'\|_\infty)\left(\|\rho\|^q_{L^q(0,t;L^\sigma(\Omega))} + \|\rho_{past}\|^q_{L^q(-\delta(0),0;L^\sigma(\Omega))}\right),
\end{aligned} \tag{2.27}$$

and then, from (2.25) and Jensen inequality, we can deduce the result (ii) of Lemma. This completes the proof. □

For the sake of simplicity, we shall write $\mathcal{I}_i(\psi)$, $\mathcal{I}(\psi, v)$ and $\mathcal{G}(\psi, v)$ in place of $\mathcal{I}_i(x,t;\psi)$, $\mathcal{I}(x,t;\psi,v)$ and $\mathcal{G}(x,t;\psi,v)$, respectively (for $i = 0, 2$).

2.3. Existence, uniqueness and regularity results

The results of this section concern the existence, uniqueness and regularity of solution of (1.1).

Theorem 2.2. *([6]) Let assumptions (H1)-(H3) and (RC) be fulfilled. Let be given* $(\phi_0, u_0) \in (L^2(\Omega))^2$, $(\phi_{past}, u_{past}) \in (L^2(Q_0))^2$ *and* $(I_i, I) \in (L^2(0,T;\mathbb{V}'))^2$. *Then there exists a solution* (ϕ, φ_e, u) *of* (2.15) *verifying :* $\phi \in L^2(0,T;\mathbb{V}) \cap L^p(Q) \cap L^\infty(0,T;\mathbb{H})$, $\varphi_e \in L^2(0,T;\mathbb{U})$ *and* $u \in C^0([0,T];\mathbb{H})$ *with the following a priori estimate*

$$\begin{aligned}
\|\phi\|^2_{L^2(0,T;\mathbb{V})\cap L^\infty(0,T;\mathbb{H})} + \|u\|^2_{L^\infty(0,T;\mathbb{H})} + \|\varphi_e\|^2_{L^2(0,T;\mathbb{U})} &\le C\Big(1 + \|I_i\|^2_{L^2(0,T;\mathbb{V}')} + \|I\|^2_{L^2(0,T;\mathbb{V}')}\\
&+\|\phi_{past}\|^2_{L^2(Q_0)} + \|u_{past}\|^2_{L^2(Q_0)} + \|\phi_0\|^2_{\mathbb{H}} + \|u_0\|^2_{\mathbb{H}}\Big).
\end{aligned} \tag{2.28}$$

Moreover the Lipschitz continuity relation is satisfied, i.e., for any element $(\phi_{0j}, u_{0j}) \in (L^2(\Omega))^2$, $(I_i^{(j)}, I^{(j)}) \in (L^2(0,T;\mathbb{V}'))^2$ *and* $(\phi_{j,past}, u_{j,past}) \in (L^2(Q_0))^2$, *for* $j = 1, 2$, *we have*

$$\begin{aligned}
&\|\phi_1 - \phi_2\|^2_{L^2(0,T;\mathbb{V})\cap L^\infty(0,T;\mathbb{H})} + \|u_1 - u_2\|^2_{L^\infty(0,T;\mathbb{H})} + \|\varphi_{e,1} - \varphi_{e,2}\|^2_{L^2(0,T;\mathbb{U})}\\
&\le C(\|I_i^{(1)} - I_i^{(2)}\|^2_{L^2(0,T;\mathbb{V}')} + \|I^{(1)} - I^{(2)}\|^2_{L^2(0,T;\mathbb{V}')} + \|\phi_{1,past} - \phi_{2,past}\|^2_{L^2(Q_0)}\\
&+\|u_{1,past} - u_{2,past}\|^2_{L^2(Q_0)} + \|\phi_{01} - \phi_{02}\|^2_{\mathbb{H}} + \|u_{01} - u_{02}\|^2_{\mathbb{H}}),
\end{aligned} \tag{2.29}$$

where $(\phi_j, u_j, \varphi_{e,j})$ *is solution of* (2.15)*, which corresponds to data* (ϕ_{0j}, u_{0j})*,* $(\phi_{j,past}, u_{j,past})$ *and* $(I_i^{(j)}, I^{(j)})$*.*

Theorem 2.3. *Consider the case of* $p = 4$*. Assume that* $(u_0, u_{past}, \phi_{past}, \phi_0)$ *is given such that* $(\phi_{past}, u_{past}) \in \big(L^2(-\delta(0), 0; L^3(\Omega))\big)^2$*,* $u_0 \in L^3(\Omega)$ *and* $\phi(t{=}0) = \varphi_i(t{=}0) - \varphi_e(t{=}0) = \varphi_i^{(0)} - \varphi_e^{(0)} = \phi_0$ *with* $(\phi_0, \varphi_e^{(0)}) \in (L^2(\Omega))^2$*.*

(i) If $I_i \in L^2(Q)$ *and* $I \in L^2(Q)$*, we have* u *belongs even to* $C^0([0,T], L^3(\Omega))$ *and it holds that*

$$\| u(.,t) \|_{L^3(\Omega)} \leq C\Big(1 + \| \phi_{past} \|_{L^2(-\delta(0),0;L^3(\Omega))} + \| u_{past} \|_{L^2(-\delta(0),0;L^3(\Omega))} + \|I_i\|_{L^2(Q)} + \|I\|_{L^2(Q)} + \| u_0 \|_{L^3(\Omega)} + \|\phi_0\|_{L^2(\Omega)}\Big). \tag{2.30}$$

(ii) Moreover if $\mathcal{I}_2$ *satisfies the following assumption*
*(***H4***) there exist constants* $\beta_i \geq 0$ *(*$i = 7, ..., 9$*) such that, for any* $(v, w) \in I\!R^2$*,*

$$| \mathcal{I}_2(.;v) - \mathcal{I}_2(.;w) | \leq | v - w | \, (\beta_7 + \beta_8 | v | + \beta_9 | w |),$$

we have for any element $(I_i^{(j)}, I^{(j)}) \in (L^2(Q))^2$*, for* $j = 1, 2$*,*

$$\| u(.,t) \|_{L^3(\Omega)} \leq C \left(\|I_i\|_{L^2(Q)} + \|I\|_{L^2(Q)} \right), \tag{2.31}$$

where $(\phi_j, u_j, \varphi_{e,j})$ *is solution of* (2.15)*, which corresponds to data* (ϕ_0, u_0)*,* (ϕ_{past}, u_{past}) *and* $(I_i^{(j)}, I^{(j)})$*, and* $\phi = \phi_1 - \phi_2$*,* $u = u_1 - u_2$*,* $\varphi_e = \varphi_{e,1} - \varphi_{e,2}$*,* $I = I^{(1)} - I^{(2)}$*,* $I_i = I_i^{(1)} - I_i^{(2)}$*.*

(iii) Assume now that $(\phi_0, \varphi_e^{(0)}) \in (H^1(\Omega))^2$ *and the primitive* $\tilde{\mathcal{I}}_0$ *of* $\mathcal{I}_0$ *satisfies the assumptions*
*(***H5***) there exist constants* $\beta_i \geq 0$ *(*$i = 10, ..., 15$*) such that, for any* $v \in I\!R$*,*

$$\tilde{\mathcal{I}}_0(v) \geq \beta_{10} | v |^4 - \beta_{11} | v |^2, \quad \frac{\partial \tilde{\mathcal{I}}_0}{\partial t}(v) \geq \beta_{12} | v |^4 - \beta_{13} | v |^2,$$
$$| \tilde{\mathcal{I}}_0(v) | \leq \beta_{14} + \beta_{15} | v |^4 .$$

Then

(a) if $I_i \in L^2(Q)$ *and* I *is in the space*

$$U_c = \{v \in L^2(Q) \text{ such that } \frac{\partial v}{\partial t} \in L^2(Q)\} \subset C^0([0,T]; L^2(\Omega)) \text{ (see Remark 2.3)},$$

then $(\phi, \varphi_e) \in \big(L^\infty(0,T; H^1(\Omega))\big)^2$*,* $\dfrac{\partial \phi}{\partial t} \in L^2(Q)$ *and* $u \in C^0([0,T]; L^3(\Omega))$*.*

(b) Moreover if $\mathcal{I}_0$ *and* $\mathcal{I}_1$ *satisfy the following assumption*
*(***H6***) there exist constants* $\beta_i \geq 0$ *(*$i = 16, ..., 19$*) such that, for any* $(v, w) \in I\!R^2$*,*

$$| \mathcal{I}_0(.;v) - \mathcal{I}_0(.;w) | \leq | v - w | \left(\beta_{16} + \beta_{17} | v |^2 + \beta_{18} | w |^2\right),$$
$$| \mathcal{I}_1(.;v) - \mathcal{I}_1(.;w) | \leq \beta_{19} | v - w |,$$

we have, for any element $(I_i^{(j)}, I^{(j)}) \in L^2(Q) \times U_c$ (for $j = 1, 2$),

$$\| \frac{\partial \phi}{\partial t} \|^2_{L^2(Q)} + \| \frac{\partial u}{\partial t} \|^2_{L^2(0,T;L^3(\Omega))} + \| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} + \| \varphi_e \|^2_{L^\infty(0,T;H^1(\Omega))} \leq C\Big(\| I_i \|^2_{L^2(Q)} + \| I \|^2_{U_c}\Big). \tag{2.32}$$

where $(\phi_j, u_j, \varphi_{e,j})$ is solution of (2.15)*, which corresponds to data (ϕ_0, u_0), (ϕ_{past}, u_{past}) and $(I_i^{(j)}, I^{(j)})$, and $\phi = \phi_1 - \phi_2$, $u = u_1 - u_2$, $\varphi_e = \varphi_{e,1} - \varphi_{e,2}$, $I = I^{(1)} - I^{(2)}$, $I_i = I_i^{(1)} - I_i^{(2)}$.*

Proof. (i) Since u satisfies the equation

$$\frac{\partial u}{\partial t} = -\mathcal{I}_2(.;\phi) - \hbar u + \mathcal{E}(\phi_\tau, u_\tau), \text{ in } Q \tag{2.33}$$

where $\mathcal{E}(\phi_\tau, u_\tau) = \sum_{k=1}^{n_1} c_k(\mathbf{x}, t)\phi(\mathbf{x}, t - \xi_k(t)) + \sum_{l=1}^{n_2} d_l(\mathbf{x}, t)u(\mathbf{x}, t - \eta_l(t))$, then we have (for all t)

$$u(\mathbf{x}, t) = u_0 - \int_0^t \mathcal{I}_2(\mathbf{x}, s;\phi)ds - \int_0^t \hbar u(\mathbf{x}, s)ds + \sum_{k=1}^{n_1}\int_0^t c_k(\mathbf{x}, s)\phi(\mathbf{x}, s - \xi_k(s))ds + \sum_{l=1}^{n_2}\int_0^t d_l(\mathbf{x}, s)u(\mathbf{x}, s - \eta_l(s))ds. \tag{2.34}$$

Consequently (as in (2.26))

$$u(\mathbf{x}, t) = u_0 - \int_0^t \mathcal{I}_2(\mathbf{x}, s;\phi)ds - \int_0^t \hbar u(\mathbf{x}, s)ds + \sum_{k=1}^{n_1}\int_{-\xi_k(0)}^{t-\xi_k(t)} e_k'(\theta)c_k(\mathbf{x}, e_k(\theta))\phi(\mathbf{x}, \theta)d\theta + \sum_{l=1}^{n_2}\int_{-\eta_l(0)}^{t-\eta_l(t)} q_l'(\theta)d_l(\mathbf{x}, q_l(\theta))u(\mathbf{x}, \theta)d\theta. \tag{2.35}$$

Since $-\delta(0) \leq -\xi_k(0)$, $-\delta(0) \leq -\eta_k(0)$, $t - \xi_k(t) \leq t$ and $t - \eta_k(t) \leq t$ we can deduce that (according to the regularity of c_k, d_l and $\hbar$)

$$| u(\mathbf{x}, t) | \leq C\Big(| u_0 | + \int_0^t | \mathcal{I}_2(\mathbf{x}, s;\phi) | ds + \int_0^t | u(\mathbf{x}, s) | ds + \int_{-\delta(0)}^t | \phi(\mathbf{x}, \theta) | d\theta + \int_{-\delta(0)}^t | u(\mathbf{x}, \theta) | d\theta\Big) \tag{2.36}$$

and then (since from the assumption (2.6) we have $| \mathcal{I}_2(.;\phi) | \leq \beta_5 + \beta_6 | \phi |^2$)

$$| u(\mathbf{x}, t) | \leq C\Big(1 + \int_0^t | \phi(\mathbf{x}, s) |^2 ds + \int_0^t | u(\mathbf{x}, s) | ds + \int_0^t | \phi(\mathbf{x}, \theta) | d\theta + | u_0 | + \int_{-\delta(0)}^0 | \phi_{past}(\mathbf{x}, \theta) | d\theta + \int_{-\delta(0)}^0 | u_{past}(\mathbf{x}, \theta) | d\theta\Big). \tag{2.37}$$

This implies

$$\left(\int_\Omega \mid u(\mathbf{x},t)\mid^3 d\mathbf{x}\right)^{1/3} \le C\Big(1+\parallel u_0 \parallel_{L^3(\Omega)} +\Big[\int_\Omega (\int_0^t \mid u(\mathbf{x},s)\mid ds)^3 d\mathbf{x}\Big]^{1/3}$$
$$+\Big[\int_\Omega (\int_0^t \mid \phi(\mathbf{x},s)\mid^2 ds)^3 d\mathbf{x}\Big]^{1/3} + \Big[\int_\Omega (\int_0^t \mid \phi(\mathbf{x},\theta)\mid d\theta)^3 d\mathbf{x}\Big]^{1/3} \tag{2.38}$$
$$+\Big[\int_\Omega (\int_{-\delta(0)}^0 \mid \phi_{past}(\mathbf{x},\theta)\mid d\theta)^3 d\mathbf{x}\Big]^{1/3} + \Big[\int_\Omega (\int_{-\delta(0)}^0 \mid u_{past}(\mathbf{x},\theta)\mid d\theta)^3 d\mathbf{x}\Big]^{1/3}\Big)$$

and then (using Minkowski inequality)

$$\left(\int_\Omega \mid u(\mathbf{x},t)\mid^3 d\mathbf{x}\right)^{1/3} \le C\Big(1+\parallel u_0 \parallel_{L^3(\Omega)} + \int_0^t (\int_\Omega \mid u(\mathbf{x},s)\mid^3 d\mathbf{x})^{1/3} ds$$
$$+\int_0^t (\int_\Omega \mid \phi(\mathbf{x},s)\mid^6 d\mathbf{x})^{1/3} ds + \int_0^t (\int_\Omega \mid \phi(\mathbf{x},\theta)\mid^3 d\mathbf{x})^{1/3} d\theta \tag{2.39}$$
$$+\int_{-\delta(0)}^0 (\int_\Omega \mid \phi_{past}(\mathbf{x},\theta)\mid^3 d\mathbf{x})^{1/3} d\theta + \int_{-\delta(0)}^0 (\int_\Omega \mid u_{past}(\mathbf{x},\theta)\mid^3 d\mathbf{x})^{1/3} d\theta\Big).$$

Since $\phi \in L^2(0,T,H^1(\Omega)) \subset L^2(0,T,L^r(\Omega))$ ($r \in [1,6]$), then

$$\parallel u(.,t)\parallel_{L^3(\Omega)} \le C_1\left(1+\parallel u_0 \parallel_{L^3(\Omega)} + \parallel \phi_{past}\parallel_{L^2(-\delta(0),0;L^3(\Omega))} + \parallel u_{past}\parallel_{L^2(-\delta(0),0;L^3(\Omega))}\right)$$
$$+C_2 \parallel \phi \parallel_{L^2(0,T;H^1(\Omega))} + C_3 \parallel \phi \parallel^2_{L^2(0,T;H^1(\Omega))} + C_4 \int_0^t \parallel u(.,s)\parallel_{L^3(\Omega)} ds. \tag{2.40}$$

According to (2.28), we can deduce that

$$\parallel u(.,t)\parallel_{L^3(\Omega)} \le C_5\Big(1+\parallel u_0 \parallel_{L^3(\Omega)} + \parallel \phi_{past}\parallel_{L^2(-\delta(0),0;L^3(\Omega))} + \parallel u_{past}\parallel_{L^2(-\delta(0),0;L^3(\Omega))}$$
$$+\|I_i\|_{L^2(Q)} + \|I\|_{L^2(Q)} + \|\phi_0\|_{\mathbb{H}}\Big) + C_6 \int_0^t \parallel u(.,s)\parallel_{L^3(\Omega)} ds.$$

Consequently (by using Gronwall lemma)

$$\parallel u(.,t)\parallel_{L^3(\Omega)} \le C\Big(1+\parallel u_0 \parallel_{L^3(\Omega)} + \parallel \phi_{past}\parallel_{L^2(-\delta(0),0;L^3(\Omega))} + \parallel u_{past}\parallel_{L^2(-\delta(0),0;L^3(\Omega))}$$
$$+\|I_i\|_{L^2(Q)} + \|I\|_{L^2(Q)} + \|\phi_0\|_{\mathbb{H}}\Big).$$

Since
$(u_0,\phi_0,I,I_i,u_{past},\phi_{past}) \in L^3(\Omega)\times L^2(\Omega)\times L^2(Q)\times L^2(Q)\times L^2(-\delta(0),0;L^3(\Omega))\times L^2(-\delta(0),0;L^3(\Omega))$,
then, $u(.,t) \in L^3(\Omega)$ (for all $t \in (0,T)$).

(ii) From (2.33), we have (for all t)

$$u(\mathbf{x},t) = u_0 - \int_0^t (\mathcal{I}_2(\mathbf{x},s;\phi_1) - \mathcal{I}_2(\mathbf{x},s;\phi_2))ds - \int_0^t \hbar u(\mathbf{x},s)ds$$
$$+\sum_{k=1}^{n_1}\int_0^t c_k(\mathbf{x},s)\phi(\mathbf{x},s-\xi_k(s))ds + \sum_{l=1}^{n_2}\int_0^t d_l(\mathbf{x},s)u(\mathbf{x},s-\eta_l(s))ds. \tag{2.41}$$

Consequently,

$$u(\mathbf{x},t)=u_0-\int_0^t(\mathcal{I}_2(\mathbf{x},s;\phi_1)-\mathcal{I}_2(\mathbf{x},s;\phi_2))\,ds-\int_0^t \hbar u(\mathbf{x},s)ds \\ +\sum_{k=1}^{n_1}\int_{-\xi_k(0)}^{t-\xi_k(t)} c_k(\mathbf{x},e_k(\theta))e_k'(\theta)\phi(\mathbf{x},\theta)d\theta+\sum_{l=1}^{n_2}\int_{-\eta_l(0)}^{t-\eta_l(t)} d_l(\mathbf{x},q_l(\theta))q_l'(\theta)u(\mathbf{x},\theta)d\theta. \tag{2.42}$$

Since $-\delta(0)\le-\xi_k(0)$, $-\delta(0)\le-\eta_k(0)$, $t-\xi_k(t)\le t$ and $t-\eta_k(t)\le t$ we can deduce that (according to the regularity of c_k, d_l and $\hbar$)

$$\mid u(\mathbf{x},t)\mid\le\int_0^t\mid\mathcal{I}_2(\mathbf{x},s;\phi_1)-\mathcal{I}_2(\mathbf{x},s;\phi_2)\mid ds+\mid u_0\mid \\ +C\Big(\int_0^t\mid u(\mathbf{x},s)\mid ds+\int_0^t\mid\phi(\mathbf{x},\theta)\mid d\theta \\ +\int_{-\delta(0)}^0\mid\phi(\mathbf{x},\theta)\mid d\theta+\int_{-\delta(0)}^0\mid u(\mathbf{x},\theta)\mid d\theta\Big) \tag{2.43}$$

and then (since from (H4) we have $\mid\mathcal{I}_2(.;\phi_1)-\mathcal{I}_2(.;\phi_2)\mid\le C\mid\phi\mid(1+\mid\phi_1\mid+\mid\phi_2\mid)$)

$$\mid u(\mathbf{x},t)\mid\le C\Big(\int_0^t\mid\phi(\mathbf{x},s)\mid\mid\phi_1(\mathbf{x},s)\mid ds+\int_0^t\mid\phi(\mathbf{x},s)\mid\mid\phi_2(\mathbf{x},s)\mid ds \\ +\int_0^t\mid u(\mathbf{x},s)\mid ds+\int_0^t\mid\phi(\mathbf{x},s)\mid ds+\mid u_0\mid \\ +\int_{-\delta(0)}^0\mid\phi_{past}(\mathbf{x},\theta)\mid d\theta+\int_{-\delta(0)}^0\mid u_{past}(\mathbf{x},\theta)\mid d\theta\Big). \tag{2.44}$$

Consequently

$$\Big(\int_\Omega\mid u(\mathbf{x},t)\mid^3 d\mathbf{x}\Big)^{1/3} \\ \le C\Bigg(\parallel u_0\parallel_{L^3(\Omega)}+\Big[\int_\Omega\Big(\int_0^t\mid u(\mathbf{x},s)\mid ds\Big)^3 d\mathbf{x}\Big]^{1/3}+\Big[\int_\Omega\Big(\int_0^t\mid\phi(\mathbf{x},s)\mid ds\Big)^3 d\mathbf{x}\Big]^{1/3} \\ +\Big[\int_\Omega\Big(\int_0^t\mid\phi(\mathbf{x},t)\mid\ \mid\phi_2(\mathbf{x},s)\mid ds\Big)^3 d\mathbf{x}\Big]^{1/3}+\Big[\int_\Omega\Big(\int_0^t\mid\phi(\mathbf{x},s)\mid\ \mid\phi_1(\mathbf{x},s)\mid ds\Big)^3 d\mathbf{x}\Big]^{1/3} \\ +\Big[\int_\Omega\Big(\int_{-\delta(0)}^0\mid\phi_{past}(\mathbf{x},\theta)\mid d\theta\Big)^3 d\mathbf{x}\Big]^{1/3}+\Big[\int_\Omega\Big(\int_{-\delta(0)}^0\mid u_{past}(\mathbf{x},\theta)\mid d\theta\Big)^3 d\mathbf{x}\Big]^{1/3}\Bigg). \tag{2.45}$$

This implies (by using Hölder's and Minkowski inequalities)

$$\parallel u(.,t)\parallel_{L^3(\Omega)}\le C\Bigg(\int_0^t\Big(\int_\Omega\mid u(\mathbf{x},s)\mid^3 d\mathbf{x}\Big)^{1/3}ds+\int_0^t\Big(\int_\Omega\mid\phi(\mathbf{x},\theta)\mid^3 d\mathbf{x}\Big)^{1/3}d\theta \\ +\int_0^t\Big(\int_\Omega\mid\phi(\mathbf{x},s)\mid^6 d\mathbf{x}\Big)^{1/6}\Big(\int_\Omega\mid\phi_1(\mathbf{x},s)\mid^6 d\mathbf{x}\Big)^{1/6}ds \\ +\int_0^t\Big(\int_\Omega\mid\phi(\mathbf{x},s)\mid^6 d\mathbf{x}\Big)^{1/6}\Big(\int_\Omega\mid\phi_2(\mathbf{x},s)\mid^6 d\mathbf{x}\Big)^{1/6}ds+\parallel u_0\parallel_{L^3(\Omega)} \\ +\int_{-\delta(0)}^0\Big(\int_\Omega\mid\phi_{past}(\mathbf{x},\theta)\mid^3 d\mathbf{x}\Big)^{1/3}d\theta+\int_{-\delta(0)}^0\Big(\int_\Omega\mid u_{past}(\mathbf{x},\theta)\mid^3 d\mathbf{x}\Big)^{1/3}d\theta\Bigg). \tag{2.46}$$

Since $\phi \in L^2(0,T,H^1(\Omega)) \subset L^2(0,T,L^r(\Omega))$ $(r \in [1,6])$, then

$$\begin{aligned} \| u(.,t) \|_{L^3(\Omega)} \leq C_1 \Big(\| u_0 \|_{L^3(\Omega)} + \| \phi_{past} \|_{L^2(-\delta(0),0;L^3(\Omega))} + \| u_{past} \|_{L^2(-\delta(0),0;L^3(\Omega))} \Big) \\ + C_2 \| \phi \|_{L^2(0,T;H^1(\Omega))} \Big(1 + \| \phi_1 \|_{L^2(0,T;H^1(\Omega))} + \| \phi_2 \|_{L^2(0,T;H^1(\Omega))} \Big) \\ + C_3 \int_0^t \| u(.,s) \|_{L^3(\Omega)} \, ds. \end{aligned} \tag{2.47}$$

According to (2.29), we can deduce that

$$\begin{aligned} \| u(.,t) \|_{L^3(\Omega)} \leq C_4 \Big(\| u_0 \|_{L^3(\Omega)} + \| \phi_{past} \|_{L^2(-\delta(0),0;L^3(\Omega))} + \| u_{past} \|_{L^2(-\delta(0),0;L^3(\Omega))} \Big) \\ + C_5 \Big(\|I_i\|_{L^2(Q)} + \|I\|_{L^2(Q)} + \|\phi_0\|_{\mathbb{H}} \Big) + C_6 \int_0^t \| u(.,s) \|_{L^3(\Omega)} \, ds. \end{aligned}$$

Consequently (by using Gronwall lemma)

$$\begin{aligned} \| u(.,t) \|_{L^3(\Omega)} \leq C \Big(\| u_0 \|_{L^3(\Omega)} + \| \phi_{past} \|_{L^2(-\delta(0),0;L^3(\Omega))} + \| u_{past} \|_{L^2(-\delta(0),0;L^3(\Omega))} \\ + \|I_i\|_{L^2(Q)} + \|I\|_{L^2(Q)} + \|\phi_0\|_{\mathbb{H}} \Big). \end{aligned} \tag{2.48}$$

(iii).a. Put $S_1 = \mathcal{H}(.,\phi_\tau,u_\tau)$, then from Lemma 2.6, we can deduce that

$$\begin{aligned} \| S_1 \|_{L^2(0,t;L^2(\Omega))} \leq C \Big(\|\phi\|_{L^2(0,t;L^2(\Omega))} + \|u\|_{L^2(0,t;L^2(\Omega))} \\ + \|\phi_{past}\|_{L^2(-\delta(0),0;L^2(\Omega))} + \|u_{past}\|_{L^2(-\delta(0),0;L^2(\Omega))} \Big). \end{aligned}$$

Since $(\phi,u) \in (L^2(0,T;L^2(\Omega)))^2$ and $(\phi_{past},u_{past}) \in (L^2(-\delta(0),0;L^2(\Omega)))^2$, we can deduce that

$$S_1 \in L^2(Q).$$

From (2.15), we can deduce that (ϕ,u) satisfies (for all $v \in \mathbb{V},\ v_e \in \mathbb{U}$)

$$\begin{aligned} & \langle \mathfrak{c}_m \frac{\partial \phi}{\partial t}, v \rangle_{\mathbb{V}',\mathbb{V}} + \int_\Omega \mathcal{I}(.;\phi,u) v d\mathbf{x} + A_i(\phi+\varphi_e, v) = \int_\Omega (I_i + S_1) v d\mathbf{x}, \\ & A_i(\phi+\varphi_e, v_e) + A_e(\varphi_e, v_e) = \int_\Omega I v_e d\mathbf{x}. \end{aligned} \tag{2.49}$$

In order to derive the result of (a), we will just sketch the proof based on suitable a priori estimates. From (2.49) with $(v,v_e) = (\frac{\partial \phi}{\partial t}, \frac{\partial \varphi_e}{\partial t})$ (since $\mathfrak{c}_m = \mathfrak{b}_m^2$)

$$\begin{aligned} & \| \mathfrak{b}_m \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} + \int_\Omega \mathcal{I}_0(\phi) \frac{\partial \phi}{\partial t} d\mathbf{x} + A_i(\phi+\varphi_e, \frac{\partial \phi}{\partial t}) + \int_\Omega \mathcal{I}_1(\phi) u \frac{\partial \phi}{\partial t} d\mathbf{x} \\ & = \int_\Omega I_i \frac{\partial \phi}{\partial t} d\mathbf{x} + \int_\Omega S_1 \frac{\partial \phi}{\partial t} d\mathbf{x}, \\ & A_i(\phi+\varphi_e, \frac{\partial \varphi_e}{\partial t}) + A_e(\varphi_e, \frac{\partial \varphi_e}{\partial t}) = \int_\Omega I \frac{\partial \varphi_e}{\partial t} d\mathbf{x}. \end{aligned} \tag{2.50}$$

Then

$$\begin{aligned}
&\| \mathfrak{b}_m \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} + \frac{d}{dt}(\int_\Omega \tilde{\mathcal{I}}_0(\phi) d\mathbf{x}) + \int_\Omega \frac{\partial \tilde{\mathcal{I}}_0}{\partial t}(\phi) d\mathbf{x} + \int_\Omega \mathcal{I}_1(\phi) u \frac{\partial \phi}{\partial t} d\mathbf{x} \\
&+ \frac{d}{2dt} A_i(\phi + \varphi_e, \phi + \varphi_e) + \frac{d}{2dt} A_i(\varphi_e, \varphi_e) \\
&= - \int_\Omega \varphi_e \frac{\partial I}{\partial t} d\mathbf{x} + \frac{d}{dt}(\int_\Omega I \varphi_e d\mathbf{x}) + \int_\Omega I_i \frac{\partial \phi}{\partial t} d\mathbf{x} + \int_\Omega S_1 \frac{\partial \phi}{\partial t} d\mathbf{x}.
\end{aligned} \tag{2.51}$$

Since $I \in U_c \subset C^0([0,T]; L^2(\Omega))$, then $I(0) \in L^2(\Omega)$ and we have $\| I(0) \|^2_{L^2(\Omega)} \leq C \| I \|^2_{U_c}$. Consequently, by integrating (2.51) by time we can deduce (from (2.4), (2.5), (H5), the boundedness of $\mathfrak{b}_m$ and, the coercivity and continuity of A_i and A_e)

$$\begin{aligned}
&2\underline{c}_m \int_0^t \| \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} ds + \alpha_i \| \phi(t) \|^2_{H^1(\Omega)} + (\alpha_e + \alpha_i) \| \varphi_e(t) \|^2_{H^1(\Omega)} \\
&+2\beta_{12} \int_0^t \| \phi \|^4_{L^4(\Omega)} ds + 2\beta_{10} \| \phi(t) \|^4_{L^4(\Omega)} \leq \delta_0 \int_0^t \| \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} ds + \delta_1 \| \varphi_e(t) \|^2_{H^1(\Omega)} \\
&\quad + C_0 \int_0^t \| u \|^2_{L^3(\Omega)} \| \phi \|^2_{H^1(\Omega)} ds + C_1 \left(\| I_i \|^2_{L^2(Q)} + \| S_1 \|^2_{L^2(Q)} + \| I \|^2_{U_c} \right) \\
&\quad + C_2 \left(1 + \| \phi_0 \|^2_{H^1(\Omega)} + \| \varphi_e^{(0)} \|^2_{H^1(\Omega)} + \| \phi_0 \|^2_{L^2(\Omega)} \right) + \int_\Omega \tilde{\mathcal{I}}_0(\phi_0) d\mathbf{x} \\
&\quad + C_3 \left(\| \phi \|^4_{L^4(Q)} + \| \phi \|^2_{L^\infty(0,T;L^2(\Omega))} + \| \varphi_e \|^2_{L^2(Q)} \right).
\end{aligned} \tag{2.52}$$

From (H5) we can deduce that $\int_\Omega \tilde{\mathcal{I}}_0(\phi_0) d\mathbf{x} \leq C(1 + \| \phi_0 \|^4_{L^4(\Omega)}) \leq C(1 + \| \phi_0 \|^4_{H^1(\Omega)})$ and then (by choosing $\delta_0 = \underline{c}_m$ and $\delta_1 = (\alpha_e + \alpha_i)/2$)

$$\begin{aligned}
&\underline{c}_m \int_0^t \| \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} ds + \alpha_i \| \phi(t) \|^2_{H^1(\Omega)} + (\alpha_e + \alpha_i)/2 \| \varphi_e(t) \|^2_{H^1(\Omega)} \\
&\quad +2\beta_{12} \int_0^t \| \phi \|^4_{L^4(\Omega)} ds + 2\beta_{10} \| \phi(t) \|^4_{L^4(\Omega)} \leq C_4 \int_0^t \| u \|^2_{L^3(\Omega)} \| \phi \|^2_{H^1(\Omega)} ds \\
&\qquad + C_5 \left(\| I_i \|^2_{L^2(Q)} + \| S_1 \|^2_{L^2(Q)} + \| I \|^2_{U_c} \right) \\
&\qquad + C_6 \left(1 + \| \phi_0 \|^2_{H^1(\Omega)} + \| \varphi_e^{(0)} \|^2_{H^1(\Omega)} + \| \phi_0 \|^4_{H^1(\Omega)} \right) \\
&\qquad + C_7 \left(\| \phi \|^4_{L^4(Q)} + \| \phi \|^2_{L^\infty(0,T;L^2(\Omega))} + \| \varphi_e \|^2_{L^2(Q)} \right).
\end{aligned} \tag{2.53}$$

Consequently (since $I \in U_c$, $\phi \in L^4(Q) \cap L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$, $\varphi_e \in L^2(Q)$, $u \in L^\infty(0,T; L^3(\Omega))$, $(\phi_{past}, u_{past}) \in (L^2(Q_0))^2$, $(I_i, S_1) \in (L^2(Q))^2$ and $(\phi_0, \varphi_e^{(0)}) \in (H^1(\Omega))^2$),

$$\int_0^t \| \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} ds + \| \phi(t) \|^2_{H^1(\Omega)} + \| \varphi_e(t) \|^2_{H^1(\Omega)} \leq C \tag{2.54}$$

and then $\frac{\partial \phi}{\partial t} \in L^2(Q)$ and $(\phi, \varphi_e) \in L^\infty(0,T; H^1(\Omega))$.

The proof of (a) can be completed by implementing the classical Faedo-Galerkin method and by taking advantage of the above estimates. So we omit the details.

Prove now that $u \in C^0([0,T]; L^3(\Omega))$. Let $S = -\mathcal{I}_2(.;\phi) - \hbar u + \mathcal{E}(\phi_\tau, u_\tau)$ be the right hand side of (2.33). According to the expression of $\mathcal{I}_2$, we can deduce that (since $\phi \in L^\infty(0,T; H^1(\Omega))$ and $u \in L^\infty(0,T; L^3(\Omega))$)

$$\| S \|_{L^3(\Omega)} \leq C\left(1 + \| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} + \| u \|_{L^\infty(0,T;L^3(\Omega))} + \| \mathcal{E}(\phi_\tau, u_\tau) \|_{L^3(\Omega)}\right)$$

and then (from Lemma 2.6)

$$\begin{aligned}\| S \|_{L^2(0,T;L^3(\Omega))} \leq C\Big(1 + \| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} + \| u \|_{L^\infty(0,T;L^3(\Omega))} + \|\phi\|_{L^2(0,T;L^3(\Omega))} \\ + \|u\|_{L^2(0,T;L^3(\Omega))} + \|\phi_{past}\|_{L^2(-\delta(0),0;L^3(\Omega))} + \|u_{past}\|_{L^2(-\delta(0),0;L^3(\Omega))}\Big).\end{aligned} \tag{2.55}$$

Consequently, since $(\phi_{past}, u_{past}) \in (L^2(-\delta(0), 0; L^3(\Omega)))^2$, we can deduce that $S \in L^2(0,T; L^3(\Omega))$ and then $\dfrac{\partial u}{\partial t} \in L^2(0,T; L^3(\Omega))$. Since $u \in L^2(0,T; L^3(\Omega))$ and $\dfrac{\partial u}{\partial t} \in L^2(0,T; L^3(\Omega))$ then, from Remark 2.3, $u \in C^0([0,T]; L^3(\Omega))$.

(iii).b. Prove now the estimate relation. Let $(\phi_j, \varphi_{e,j}, u_j)$ (for $j = 1,2$) be two solutions corresponding to $(I_i^{(j)}, I^{(j)}) \in L^2(Q) \times U_c$ with $(\phi_1 - \phi_2, \varphi_{e,1} - \varphi_{e,2}, u_1 - u_2)(t = 0) = (0,0,0)$ and $(\phi_{past}, u_{past}) = (\phi_{past,1} - \phi_{past,2}, u_{past,1} - u_{past,2}) = (0,0)$. Then $(\phi, \varphi_e, u) = (\phi_1 - \phi_2, \varphi_{e,1} - \varphi_{e,2}, u_1 - u_2)$ satisfies, from (2.15) with $(v, v_e) = (\frac{\partial \phi}{\partial t}, \frac{\partial \varphi_e}{\partial t})$

$$\begin{aligned}&\| \mathfrak{b}_m \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} + \int_\Omega (\mathcal{I}_0(.;\phi_1) - \mathcal{I}_0(.;\phi_2)) \frac{\partial \phi}{\partial t} d\mathbf{x} + A_i(\phi + \varphi_e, \frac{\partial \phi}{\partial t}) \\ &\qquad + \int_\Omega (u_1 \mathcal{I}_1(.;\phi_1) - u_2 \mathcal{I}_2(.;\phi_2)) \frac{\partial \phi}{\partial t} d\mathbf{x} = \int_\Omega I_i \frac{\partial \phi}{\partial t} d\mathbf{x} + \int_\Omega \mathcal{H}(.,\phi_\tau, u_\tau) \frac{\partial \phi}{\partial t} d\mathbf{x}, \\ &A_i(\phi + \varphi_e, \frac{\partial \varphi_e}{\partial t}) + A_e(\varphi_e, \frac{\partial \varphi_e}{\partial t}) = \int_\Omega I \frac{\partial \varphi_e}{\partial t} d\mathbf{x}, \\ &\frac{\partial u}{\partial t} = -(\mathcal{I}_2(.;\phi_1) - \mathcal{I}_2(.;\phi_2)) - \hbar u + \mathcal{E}(\phi_\tau, u_\tau).\end{aligned} \tag{2.56}$$

Then

$$\begin{aligned}&\| \mathfrak{b}_m \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} + \frac{d}{2dt} A_i(\phi + \varphi_e, \phi + \varphi_e) + \frac{d}{2dt} A_i(\varphi_e, \varphi_e) \\ &\qquad = -\int_\Omega (\mathcal{I}_0(.;\phi_1) - \mathcal{I}_0(.;\phi_2)) \frac{\partial \phi}{\partial t} d\mathbf{x} - \int_\Omega u \mathcal{I}_1(.;\phi_1) \frac{\partial \phi}{\partial t} d\mathbf{x} \\ &\qquad - \int_\Omega u_2 (\mathcal{I}_1(.;\phi_1) - \mathcal{I}_1(.;\phi_2)) \frac{\partial \phi}{\partial t} d\mathbf{x} - \int_\Omega \varphi_e \frac{\partial I}{\partial t} d\mathbf{x} \\ &\qquad + \frac{d}{dt}\left(\int_\Omega I \varphi_e d\mathbf{x}\right) + \int_\Omega I_i \frac{\partial \phi}{\partial t} d\mathbf{x} + \int_\Omega \mathcal{H}(.,\phi_\tau, u_\tau) \frac{\partial \phi}{\partial t} d\mathbf{x}, \\ &\frac{\partial u}{\partial t} = -(\mathcal{I}_2(.;\phi_1) - \mathcal{I}_2(.;\phi_2)) - \hbar u + \mathcal{E}(\phi_\tau, u_\tau).\end{aligned} \tag{2.57}$$

According to assumptions (H2),(H4) and (H6), we can deduce that (from the boundedness of $\mathfrak{b}_m$)

$$\begin{aligned}
&\underline{c}_m \| \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} + \frac{d}{2dt} A_i(\phi + \varphi_e, \phi + \varphi_e) + \frac{d}{2dt} A_i(\varphi_e, \varphi_e) \\
&\qquad \le C_1 \left(\int_\Omega | \phi | (| \phi_1 |^2 + | \phi_2 |^2 + 1) | \frac{\partial \phi}{\partial t} | d\mathbf{x} + \int_\Omega | u | (1 + | \phi_1 |) | \frac{\partial \phi}{\partial t} | d\mathbf{x} \right) \\
&\qquad + C_2 \int_\Omega | \phi \| u_2 \| \frac{\partial \phi}{\partial t} | d\mathbf{x} + \int_\Omega | \varphi_e \| \frac{\partial I}{\partial t} | d\mathbf{x} \\
&\qquad + \frac{d}{dt} \left(\int_\Omega I \varphi_e d\mathbf{x} \right) + \int_\Omega | I_i \| \frac{\partial \phi}{\partial t} | d\mathbf{x} + \int_\Omega | \mathcal{H}(., \phi_\tau, u_\tau) \| \frac{\partial \phi}{\partial t} | d\mathbf{x}, \\
&\| \frac{\partial u}{\partial t} \|_{L^3(\Omega)} \le C_3 \left(\| \phi \|_{L^6(\Omega)} \| \phi_1 \|_{L^6(\Omega)} + \| \phi \|_{L^6(\Omega)} \| \phi_2 \|_{L^6(\Omega)} + \| \phi \|_{L^3(\Omega)} + \| u \|_{L^3(\Omega)} \right) \\
&\qquad\qquad + C_4 \| \mathcal{E}(\phi_\tau, u_\tau) \|_{L^3(\Omega)} .
\end{aligned} \tag{2.58}$$

Integrating by time we obtain (according to the coercivity and continuity of A_i and A_e, and to the estimate of $\mathcal{H}(.; \phi_\tau, u_\tau)$ and $\mathcal{E}(.; \phi_\tau, u_\tau)$ given by Lemma 2.6)

$$\begin{aligned}
&2\underline{c}_m \int_0^t \| \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} ds + \alpha_i \| \phi(t) \|^2_{H^1(\Omega)} + (\alpha_e + \alpha_i) \| \varphi_e(t) \|^2_{H^1(\Omega)} \\
&\qquad \le C_5 \| \phi \|^2_{L^2(0,T;H^1(\Omega)} \left(\| \phi_1 \|^2_{L^\infty(0,T;H^1(\Omega))} + \| \phi_2 \|^2_{L^\infty(0,T;H^1(\Omega))} + \| u_2 \|^2_{L^\infty(0,T;L^3(\Omega))} \right) \\
&\qquad + C_6 \| u \|^2_{L^\infty(0,T;L^3(\Omega))} \| \phi_1 \|^2_{L^\infty(0,T;H^1(\Omega))} + \delta_0 \int_0^t \| \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} ds + \delta_1 \| \varphi_e(t) \|^2_{H^1(\Omega)} \\
&\qquad + C_7 \left(\| \phi \|^2_{L^2(Q)} + \| u \|^2_{L^2(Q)} + \| I_i \|^2_{L^2(Q)} + \| I \|^2_{U_c} + \| \varphi_e \|^2_{L^2(Q)} \right), \\
&\| \frac{\partial u}{\partial t} \|_{L^2(0,T;L^3(\Omega))} \le C_8 \| \phi \|_{L^\infty(0,T;H^1(\Omega))} \left(\| \phi_1 \|_{L^\infty(0,T;H^1(\Omega))} + \| \phi_2 \|_{L^\infty(0,T;H^1(\Omega))} \right) \\
&\qquad\qquad + C_9 \left(\| \phi \|_{L^2(0,T;L^3(\Omega))} + \| u \|_{L^2(0,T;L^3(\Omega))} \right).
\end{aligned}$$

Since $(\phi_j, u_j) \in L^\infty(0, T; H^1(\Omega)) \times L^\infty(0, T; L^3(\Omega))$ (for $j = 1, 2$) we can deduce that (by choosing $\delta_0 = \underline{c}_m$ and $\delta_1 = (\alpha_e + \alpha_i)/2$)

$$\begin{aligned}
&\underline{c}_m \int_0^t \| \frac{\partial \phi}{\partial t} \|^2_{L^2(\Omega)} ds + \alpha_i \| \phi(t) \|^2_{H^1(\Omega)} + \frac{\alpha_e + \alpha_i}{2} \| \varphi_e(t) \|^2_{H^1(\Omega)} \\
&\qquad \le C_{10} \left(\| u \|^2_{L^\infty(0,T;L^3(\Omega))} + \| \phi \|^2_{L^2(0,T;H^1(\Omega))} + \| I_i \|^2_{L^2(Q)} + \| \varphi_e \|^2_{L^2(Q)} + \| I \|^2_{U_c} \right) \\
&\| \frac{\partial u}{\partial t} \|^2_{L^2(0,T;L^3(\Omega))} \le C_{11} \left(\| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} + \| \phi \|^2_{L^2(0,T;L^3(\Omega))} + \| u \|^2_{L^2(0,T;L^3(\Omega))} \right).
\end{aligned} \tag{2.59}$$

Consequently (from the estimates (2.29)–(2.48))

$$\begin{aligned}
&\| \frac{\partial \phi}{\partial t} \|^2_{L^2(Q)} + \| \frac{\partial u}{\partial t} \|^2_{L^2(0,T;L^3(\Omega))} + \| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} + \| \varphi_e \|^2_{L^\infty(0,T;H^1(\Omega))} \\
&\qquad \le C \left(\| I_i \|^2_{L^2(Q)} + \| I \|^2_{U_c} \right).
\end{aligned} \tag{2.60}$$

This completes the proof. □

According to previous Theorems, we can derive the following results.

Theorem 2.4. *Consider the case of* $p = 4$. *Let assumptions (H1)-(H6) and (RC) be fulfilled. For* $(u_0, u_{past}, \phi_{past}, \phi_0)$ *and* (I, I_i) *given such that* $(\phi_{past}, u_{past}) \in (L^2(-\delta(0), 0; L^3(\Omega)))^2$, $u_0 \in L^3(\Omega)$, $\varphi_i^{(0)} - \varphi_e^{(0)} = \phi_0$ *with* $(\phi_0, \varphi_e^{(0)}) \in (H^1(\Omega))^2$, *and* $(I_i, I) \in L^2(Q) \times U_c$, *there exists a unique solution* (ϕ, φ_e, u) *of problem* (2.15) *verifying* $(\phi, \varphi_e, u) \in \mathbb{D}$, *with*

$$\begin{aligned} &\mathbb{D} = \mathbb{A} \times L^\infty(0, T; \mathbb{U}) \times \mathbb{B} \text{ such that } \mathbb{A} = L^\infty(0, T; \mathbb{V}) \cap H^1(0, T; \mathbb{H}) \cap C^0([0, T]; \mathbb{H}) \\ &\text{and } \mathbb{B} = C^0([0, T]; L^3(\Omega)) \cap H^1(0, T; L^3(\Omega)), \text{ equipped with } (\forall (v, v_e, \rho) \in \mathbb{D}) \\ &\| (v, v_e, \rho) \|_{\mathbb{D}}^2 = \| v \|_{\mathbb{A}}^2 + \| v_e \|_{L^\infty(0,T;\mathbb{U})}^2 + \| \rho \|_{\mathbb{B}}^2, \end{aligned} \tag{2.61}$$

and it holds that

$$\| (\phi, \varphi_e, u) \|_{\mathbb{D}}^2 \leq C \left(1 + \|I_i\|_{L^2(Q)}^2 + \|I\|_{U_c}^2 \right). \tag{2.62}$$

Moreover the Lipschitz continuity relation is satisfied, i.e., for any element $(I_i^{(j)}, I^{(j)}) \in L^2(Q) \times U_c$ *for* $j = 1, 2$, *we have*

$$\| (\phi_1 - \phi_2, \varphi_{e,1} - \varphi_{e,2}, u_1 - u_2) \|_{\mathbb{D}}^2 \leq C \left(\|I_i^{(1)} - I_i^{(2)}\|_{L^2(Q)}^2 + \|I^{(1)} - I^{(2)}\|_{U_c}^2 \right), \tag{2.63}$$

where $(\phi_j, u_j, \varphi_{e,j})$ *is the solution of* (2.15)*, which corresponds to data* (ϕ_0, u_0)*,* (ϕ_{past}, u_{past}) *and* $(I_i^{(j)}, I^{(j)})$ *(for* $j = 1, 2$*).*

3. Minimax control problems

In this section, we formulate the minimax control problem and discuss the existence and necessary optimality conditions for an optimal solution.

3.1. Formulation of control problem

Our problem in this section is to find the best admissible source function ξ in presence of the admissible disturbance in the function π. In order to illustrate our minimax control problem, we assume here that the control ξ is in I and the disturbance π is in I_i (which act on control domain Ω_c and disturbance domain Ω_d, respectively), i.e., $I = \mathcal{B}_1\xi$ and $I_i = \mathcal{B}_2\pi + f$, where $f \in L^2(Q)$, $support(\xi) \subset \Omega_c \times (0, T)$ and $support(\pi) \subset \Omega_d \times (0, T)$, and the operators $\mathcal{B}_i \in \mathfrak{L}(L^2(\Omega); L^2(\Omega))$, with $\| \mathcal{B}_i\theta \|_{L^2(\Omega)} \leq C \| \theta \|_{L^2(\Omega)}$, for all θ (for $i = 1, 2$). Therefore, the function (ϕ, φ_e, u) is assumed to be related to the disturbance π and control ξ through the problem (under conditions (1.3) and (1.2) for φ_e and $\mathcal{B}_1\xi$, respectively)

$$\begin{aligned} &\mathfrak{c}_m \frac{\partial \phi}{\partial t} + \mathcal{I}(\phi, u) - div(\mathcal{K}_i \nabla(\phi + \varphi_e)) = \mathcal{H}(\phi_\tau, u_\tau) + \mathcal{B}_2\pi + f, \text{ in } Q \\ &-div(\mathcal{K}_i \nabla\phi) - div((\mathcal{K}_i + \mathcal{K}_e)\varphi_e) = \mathcal{B}_1\xi, \text{ in } Q \\ &\frac{\partial u}{\partial t} + \mathcal{G}(\phi, u) = \mathcal{E}(\phi_\tau, u_\tau), \text{ in } Q \\ &(\mathcal{K}_i \nabla\phi).\mathbf{n} + (K_i \nabla\varphi_e).\mathbf{n} = 0, \text{ on } \Sigma \\ &(\mathcal{K}_e \nabla\varphi_e).\mathbf{n} = 0, \text{ on } \Sigma \\ &\phi(., t = 0) = \phi_0, \quad u(., t = 0) = u_0, \text{ in } \Omega \\ &\phi = \phi_{past}, \quad u = u_{past}, \text{ in } Q_0 \end{aligned} \tag{3.1}$$

under pointwise constraints

$$
\begin{aligned}
&| \xi | \le \tau_1 \ \ a.e. \text{ in } Q, \\
&| \pi | \le \tau_2 \ \ a.e. \text{ in } Q.
\end{aligned}
\tag{3.2}
$$

Let $\mathcal{U}_{ad}$ and $\mathcal{V}_{ad}$ be convex, closed, non-empty and bounded subsets of U_c and $L^2(Q)$, respectively, and describing constraints (3.2) and compatibility condition (1.2) such that

$$
\begin{aligned}
\mathcal{U}_{ad} &= \left\{ \xi \in U_c : \ support(\xi) \subset \Omega_c \times (0,T), \ \ \int_\Omega \mathcal{B}_1 \xi d\mathbf{x} = 0, \ \ | \xi | \le \tau_1 \ a.e. \ in \ Q \right\}, \\
\mathcal{V}_{ad} &= \left\{ \pi \in L^2(Q) : \ support(\pi) \subset \Omega_d \times (0,T), \ \ | \pi | \le \tau_2 \ \ a.e. \ in \ Q \right\}.
\end{aligned}
$$

Although $\mathcal{U}_{ad}$ and $\mathcal{V}_{ad}$ are subsets of $L^\infty(Q)$, we prefer to use standard norms of space $L^2(Q)$. the reason is that we would like to take advantages of differentiability of the latter norms away from the origin to perform our variational analysis. Moreover the spaces $\mathcal{U}_{ad}$ and $\mathcal{V}_{ad}$ form closed, convex, weak-star sequentially compact subsets of spaces $L^\infty(0,T;L^2(\Omega))$ (for weak-star sequential compactness see e.g. [59] and for similar result see e.g. [45]). For operator $\mathcal{B}_1$, we can consider, for example, the operator

$$
\theta \in L^2(\Omega) \longrightarrow \mathcal{B}_1 \theta = \theta - \left(\frac{1}{mes(\Omega_c)} \int_\Omega \theta d\mathbf{x} \right) \chi_{\Omega_c}. \tag{3.3}
$$

Then $\int_\Omega \mathcal{B}_1 \theta d\mathbf{x} = \mathbf{0}$ and if $support(\theta) \subset \Omega_c$ we have $\int_{\Omega_c} \mathcal{B}_1 \theta d\mathbf{x} = \mathbf{0}$. Moreover the operator $\mathcal{B}_1$ is autoadjoint on the domain $\{\theta \in L^2(\Omega) : \ \ support(\theta) \subset \Omega_c\}$. The studied control problem is to find a saddle point of cost function $\mathcal{J}$ which measures the distance between known observations (a nominal desired states) and the prognostic variables ϕ. We assume that we have observations on some domain Ω_{obs} at certain times t_k, $k = 1; ...; N_{obs}$ or at final time T. Precisely we will study the following control problem (**SP**).

Find an admissible control-disturbance $(\xi^*, \pi^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ such that cost functional (in the reduced form)

$$
\begin{aligned}
\mathcal{J}(\xi, \pi) = \frac{m_1}{2} \int_0^T \Upsilon(t) \parallel \phi - \phi_{obs} \parallel^2_{L^2(\Omega_{obs})} dt + \frac{m_2}{2} \parallel \phi(T) - \psi_{obs} \parallel^2_{L^2(\Omega_{obs})} \\
+ \frac{\alpha}{2} \parallel \xi \parallel^2_{U_c} - \frac{\beta}{2} \int_0^T \parallel \pi \parallel^2_{L^2(\Omega_d)} dt,
\end{aligned}
\tag{3.4}
$$

is minimized with respect to ξ and maximized with respect to π subject to problem (3.1), where the weight function Υ is given by (with $\varpi > 0$ large enough)

$$
\Upsilon(t) = \sum_{k=1}^{N_{obs}} exp(-\varpi (t - t_k)^2), \tag{3.5}
$$

$\alpha, \beta > 0$ and $m_i \ge 0$ $(i = 1, 2)$ are fixed such that $m_1 + m_2 > 0$, the functions $\phi_{obs} \in L^2(0,T;\Omega_{obs})$ and $\psi_{obs} \in L^2(\Omega_{obs})$ are the observations (given), $\Omega_{obs} \subset \Omega$ is the observation domain, $\Omega_c \subset \Omega$ is the control domain and $\Omega_d \subset \Omega$ is the disturbance domain. Clearly, find $(\xi^*, \pi^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ (a saddle point for the functional $\mathcal{J}$) such that $(\forall (\xi, \pi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad})$

$$
\mathcal{J}(\xi^*, \pi) \le \mathcal{J}(\xi^*, \pi^*) \le \mathcal{J}(\xi, \pi^*). \tag{3.6}
$$

Remark 3.1. *(i) The coefficient $\alpha > 0$ can be interpreted as the measure of price of control (that the engineer can afford) and the coefficient $\beta > 0$ can be interpreted as the measure of price of disturbance (that the environment can afford).*
(ii) Operators $\mathcal{B}_i$, $i = 1, 2$, also include the quantification of source profiles, inside the considered area, which results of change in disturbance and control variables. □

In the sequel of this paper, we restrict our analysis to a generalized form of the three models mentioned at the beginning of article, namely : Rogers-McCulloch model, Fitzhugh-Nagumo model and Aliev-Panfilov model. More precisely:

(HMC) $p = 4$ and the operators $\mathcal{I}$ and $\mathcal{G}$ are supposed to be of the form given in (2.21).

Remark 3.2. *According to expression (2.21) of operators $\mathcal{I}$ and $\mathcal{G}$, we verify easily that hypotheses (H5)-(H6) are satisfied.* □

3.2. Solution operator and its Fréchet differentiability

In this section, we study the Fréchet differentiability of the nonlinear operator solution and derive some necessary estimates. For a given $f \in L^2(Q)$, initial condition $(\phi_0, \varphi_e^{(0)}, u_0)$ in $(H^1(\Omega))^2 \times L^3(\Omega)$ and past condition $(\phi_{past}, u_{past}) \in (L^2(-\delta_0, 0; L^3(\Omega)))^2$, let us introduce the following mapping $\mathcal{F}$: $\mathcal{U}_{ad} \times \mathcal{V}_{ad} \longrightarrow \mathbb{D}$, which maps the source term $(\xi, \pi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ of (3.1) into the corresponding solution $X = (\phi, \varphi_e, u)$ in $\mathbb{D}$, where $\mathbb{D}$ is defined by (2.61).

Before proceeding with investigation of Fréchet differentiability of operator $\mathcal{F}$, we study the following linear parabolic problem, for $(h_1, h_2) \in U_c \times L^2(\Omega_d)$,

$$\begin{aligned}
&c_m \frac{\partial \Psi}{\partial t} + \frac{\partial \mathcal{I}}{\partial \phi}(\phi, u).\Psi + \frac{\partial \mathcal{I}}{\partial u}(\phi, u).w - div(\mathcal{K}_i \nabla(\Psi + \psi_e)) = \mathcal{H}(\Psi_\tau, w_\tau) + \mathcal{B}_2 h_2, \text{ in } Q\\
&-div(\mathcal{K}_i \nabla \Psi) - div((\mathcal{K}_i + \mathcal{K}_e)\psi_e) = \mathcal{B}_1 h_1, \text{ in } Q\\
&\frac{\partial w}{\partial t} + \frac{\partial \mathcal{G}}{\partial \phi}(\phi, u).\Psi + \frac{\partial \mathcal{G}}{\partial u}(\phi, u).w = \mathcal{E}(\Psi_\tau, w_\tau), \text{ in } Q\\
&(\mathcal{K}_i \nabla \Psi).\mathbf{n} + (\mathcal{K}_i \nabla \psi_e).\mathbf{n} = 0, \text{ on } \Sigma\\
&(\mathcal{K}_e \nabla \psi_e).\mathbf{n} = 0, \text{ on } \Sigma\\
&\Psi(., t = 0) = 0, \quad w(., t = 0) = 0, \text{ in } \Omega\\
&\Psi = 0, \quad w = 0, \text{ in } Q_0
\end{aligned} \tag{3.7}$$

and under conditions (1.3) and (1.2) for ψ_e and $\mathcal{B}_1 h_1$, respectively.

The weak formulation of Problem (3.7) can be written as follows $(\forall (v, v_e, \rho) \in \mathbb{V} \times \mathbb{U} \times \mathbb{H}$ and *a.e.* in $(0, T))$

$$\begin{aligned}
&\langle \mathfrak{c}_m \frac{\partial \Psi}{\partial t}, v\rangle_{\mathbb{V}',\mathbb{V}} + \int_\Omega \mathcal{K}_i \nabla\Psi\nabla v d\mathbf{x} + \int_\Omega \mathcal{K}_i \nabla\psi_e \nabla v d\mathbf{x} + \int_\Omega \frac{\partial \mathcal{I}}{\partial \phi}\Psi v d\mathbf{x} \\
&\qquad + \int_\Omega \frac{\partial \mathcal{I}}{\partial u} w v d\mathbf{x} = \int_\Omega \mathcal{H}(\Psi_\tau, w_\tau) v d\mathbf{x} + \int_\Omega \mathcal{B}_2 h_2 v d\mathbf{x}, \\
&\int_\Omega (\mathcal{K}_i + \mathcal{K}_e)\nabla\psi_e \nabla v_e d\mathbf{x} + \int_\Omega \mathcal{K}_i \nabla\Psi\nabla v_e d\mathbf{x} = \int_\Omega \mathcal{B}_1 h_1 v_e d\mathbf{x}, \\
&(\frac{\partial w}{\partial t}, \rho)_{\mathbb{H}} + \int_\Omega \frac{\partial \mathcal{G}}{\partial \phi}.\Psi\rho d\mathbf{x} + \int_\Omega \hbar w\rho d\mathbf{x} = \int_\Omega \mathcal{E}(\Psi_\tau, w_\tau)\rho d\mathbf{x}.
\end{aligned} \tag{3.8}$$

We are now going to prove the existence and uniqueness result of problem (3.8). To this end, we begin by proving the following necessary Lemma, which correspond to the existence, uniqueness and some regularity properties for the following nondelayed problem on $Q_S = \Omega \times (S_0, S_f)$ (with $0 \le S_0 < S_f \le T$)

$$\begin{aligned}
&\mathfrak{c}_m \frac{\partial \Pi}{\partial t} + \frac{\partial \mathcal{I}}{\partial \phi}(\phi, u).\Pi + \frac{\partial \mathcal{I}}{\partial u}(\phi, u).V - div(\mathcal{K}_i \nabla(\Pi + \pi_e)) = g_1, \\
&-div\,((\mathcal{K}_e + \mathcal{K}_i)\nabla\pi_e) - div(\mathcal{K}_i \nabla\Pi) = \mathcal{B}_1 h_1 \in U_c, \\
&\frac{\partial V}{\partial t} + \frac{\partial \mathcal{G}}{\partial \phi}(\phi, u).\Pi + \frac{\partial \mathcal{G}}{\partial u}(\phi, u).V = g_2, \\
&\text{with initial and boundary conditions} \\
&(\mathcal{K}_i \nabla(\Pi + \pi_e)) \cdot \mathbf{n} = 0, \text{ on } \Sigma_S = \partial\Omega \times (S_0, S_f) \\
&(\mathcal{K}_e \nabla\pi_e) \cdot \mathbf{n} = 0, \text{ on } \Sigma_S \\
&\Pi(t = S_0) = \Pi_0, V(t = S_0) = V_0, \text{ on } \Omega.
\end{aligned} \tag{3.9}$$

Lemma 3.1. *Let assumptions (H1)-(H4) and (HMC) be fulfilled. Suppose that $(\phi, \varphi_e, u) \in \mathbb{D}$ and $h_1 \in U_c$, then the following results hold.*

1. *For $(V_0, \Pi_0) \in (L^2(\Omega))^2$ and $(g_1, g_2) \in (L^2(Q_S))^2$ given, there exists a unique solution (Π, π_e, V) of problem (3.9) verifying the following regularity*

$$(\Pi, \pi_e, V) \in \mathcal{W}(Q_S), (\frac{\partial \Pi}{\partial t}, \frac{\partial V}{\partial t}) \in L^{4/3}(S_0, S_f; \mathbb{V}') \times L^2(Q_S), (\Pi, V) \in (C^0([S_0, S_f]; L^2(\Omega)))^2,$$

   where $\mathcal{W}(Q_S) = (L^2(S_0, S_f; \mathbb{V}) \cap L^\infty(S_0, S_f; \mathbb{H})) \times L^2(S_0, S_f; \mathbb{U}) \times L^\infty(S_0, S_f; L^2(\Omega))$.
2. *For (V_0, Π_0) and (g_1, g_2) given such that $V_0 \in L^3(\Omega)$, $\pi_i^{(0)} - \pi_e^{(0)} = \Pi_0$ with $(\Pi_0, \pi_e^{(0)}) \in (H^1(\Omega))^2$ and $(g_1, g_2) \in L^2(Q_S) \times L^2(S_0, S_f; L^3(\Omega))$, the solution (Π, π_e, V) of (3.9) is in $\mathbb{D}(Q_S)$, where $\mathbb{D}(Q_S) = \mathbb{A}_S \times L^\infty(S_0, S_f; \mathbb{U}) \times \mathbb{B}_S$, with $\mathbb{A}_S = L^\infty(S_0, S_f; \mathbb{V}) \cap H^1(S_0, S_f; \mathbb{H}) \cap C^0([S_0, S_f]; \mathbb{H})$ and $\mathbb{B}_S = C^0([S_0, S_f]; L^3(\Omega)) \cap H^1(S_0, S_f; L^3(\Omega))$.*

Proof. We will just sketch the proof based on suitable a priori estimates. The weak formulation of

Problem (3.9) can be written as follows ($\forall(v, v_e, \rho) \in \mathbb{V} \times \mathbb{U} \times \mathbb{H}$ and *a.e.* in (S_0, S_f))

$$\begin{aligned}
&\langle \mathfrak{c}_m \frac{\partial \Pi}{\partial t}, v\rangle_{\mathbb{V}',\mathbb{V}} + \int_\Omega \mathcal{K}_i \nabla\Pi\nabla v d\mathbf{x} + \int_\Omega \mathcal{K}_i \nabla\pi_e \nabla v d\mathbf{x} + \int_\Omega \frac{\partial \mathcal{I}}{\partial \phi}\Pi v d\mathbf{x} \\
&\quad + \int_\Omega \frac{\partial \mathcal{I}}{\partial u} V v d\mathbf{x} = \int_\Omega g_1 v d\mathbf{x}, \\
&\int_\Omega (\mathcal{K}_i + \mathcal{K}_e)\nabla\pi_e\nabla v_e d\mathbf{x} + \int_\Omega \mathcal{K}_i\nabla\Pi\nabla v_e d\mathbf{x} = \int_\Omega B_1 h_1 v_e d\mathbf{x}, \\
&(\frac{\partial V}{\partial t}, \rho)_{\mathbb{H}} + \int_\Omega \frac{\partial \mathcal{G}}{\partial \phi}.\Pi\rho d\mathbf{x} + \int_\Omega \hbar V\rho d\mathbf{x} = \int_\Omega g_2\rho d\mathbf{x}.
\end{aligned} \tag{3.10}$$

According to Theorem 2.1, we have then

$$\begin{aligned}
&\langle \mathfrak{c}_m \frac{\partial \Pi}{\partial t}, v\rangle_{\mathbb{V}',\mathbb{V}} + \underline{A}_i(\Pi, v) + \int_\Omega \frac{\partial \mathcal{I}}{\partial \phi}\Pi v d\mathbf{x} \\
&\quad + \int_\Omega \frac{\partial \mathcal{I}}{\partial u} V v d\mathbf{x} = \int_\Omega g_1 v d\mathbf{x} - \int_\Omega \mathcal{K}_i \nabla\overline{\pi}_e\nabla v d\mathbf{x}, \\
&(\frac{\partial V}{\partial t}, \rho)_{\mathbb{H}} + \int_\Omega \frac{\partial \mathcal{G}}{\partial \phi}.\Pi\rho d\mathbf{x} + \int_\Omega \hbar V\rho d\mathbf{x} = \int_\Omega g_2\rho d\mathbf{x},
\end{aligned} \tag{3.11}$$

where $\underline{\pi}_e$ and $\overline{\pi}_e$ are the unique solutions of

$$(A_i + A_e)(\underline{\pi}_e, v_e) + A_i(\Pi, v_e) = 0, \quad \forall v_e \in \mathbb{U} \tag{3.12}$$

and

$$(A_i + A_e)(\overline{\pi}_e, v_e) = \langle \mathcal{B}_1 h_1, v_e\rangle_{\mathbb{V}',\mathbb{V}}, \quad \forall v_e \in \mathbb{U}. \tag{3.13}$$

From (3.13), we can deduce that (by taking $v_e = \overline{\pi}_e$)

$$\| \overline{\pi}_e \|_{\mathbb{V}} \leq C \| h_1 \|_{L^2(\Omega)} \tag{3.14}$$

and then

$$\begin{aligned}
| \int_\Omega g_1 v d\mathbf{x} - \int_\Omega \mathcal{K}_i\nabla\overline{\pi}_e\nabla v d\mathbf{x} | &\leq C_1 \| h_1 \|_{L^2(\Omega)} \| v \|_{\mathbb{V}} + C_2 \| g_1 \|_{L^2(\Omega)} \| v \|_{\mathbb{V}} \\
&\leq C_3(\| h_1 \|^2_{L^2(\Omega)} + \| g_1 \|^2_{L^2(\Omega)}) + \epsilon_0 \| v \|^2_{\mathbb{V}}
\end{aligned} \tag{3.15}$$

where ϵ_0 can be chosen. For $(v, \rho) = (\Pi, V)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \| \mathfrak{b}_m \Pi \|^2_{L^2(\Omega)} + \underline{A}_i(\Pi, \Pi) + \int_\Omega \frac{\partial \mathcal{I}}{\partial \phi}\Pi^2 d\mathbf{x} \\
&\quad + \int_\Omega \frac{\partial \mathcal{I}}{\partial u} V\Pi d\mathbf{x} = \int_\Omega g_1 \Pi d\mathbf{x} - \int_\Omega \mathcal{K}_i\nabla\overline{\pi}_e\nabla\Pi d\mathbf{x}, \\
&\frac{1}{2}\frac{d}{dt} \| V \|^2_{L^2(\Omega)} + \int_\Omega \frac{\partial \mathcal{G}}{\partial \phi}.\Pi V d\mathbf{x} + \int_\Omega \hbar V^2 d\mathbf{x} = \int_\Omega g_2 V d\mathbf{x}.
\end{aligned} \tag{3.16}$$

According to the partial derivatives of $\mathcal{I}$ and $\mathcal{G}$ given by (2.21), we can deduce from (3.16) that (with $\hbar_{11} > 0$)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \parallel \mathfrak{b}_m\Pi \parallel^2_{L^2(\Omega)} + \underline{\alpha}_i \parallel \Pi \parallel^2_{\mathbb{V}} + \int_\Omega 3\hbar_{11}\Pi^2\phi^2 d\mathbf{x} \\
&\quad \le C_4(\parallel h_1 \parallel^2_{L^2(\Omega)} + \parallel g_1 \parallel^2_{L^2(\Omega)}) + \epsilon_0 \parallel \Pi \parallel^2_{\mathbb{V}} \\
&\quad + C_5\left(\int_\Omega \Pi^2 \mid \phi \mid d\mathbf{x} + \int_\Omega \Pi^2 \mid u \mid d\mathbf{x} + \int_\Omega \Pi^2 d\mathbf{x} + \int_\Omega V^2 d\mathbf{x} + \int_\Omega \mid u \parallel V \parallel \Pi \mid d\mathbf{x}\right), \\
&\frac{1}{2}\frac{d}{dt} \parallel V \parallel^2_{L^2(\Omega)} \le C_6\left(\int_\Omega \mid \Pi \parallel \phi \parallel V \mid d\mathbf{x} + \int_\Omega \mid \Pi \parallel V \mid d\mathbf{x}\right) \\
&\qquad + C_7(\parallel V \parallel^2_{L^2(\Omega)} + \parallel g_2 \parallel^2_{L^2(\Omega)}).
\end{aligned} \tag{3.17}$$

Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \parallel \mathfrak{b}_m\Pi \parallel^2_{L^2(\Omega)} + \underline{\alpha}_i \parallel \Pi \parallel^2_{\mathbb{V}} + \int_\Omega 3\hbar_{11}\Pi^2\phi^2 d\mathbf{x} \\
&\quad \le C_4(\parallel h_1 \parallel^2_{L^2(\Omega)} + \parallel g_1 \parallel^2_{L^2(\Omega)}) + \epsilon_0 \parallel \Pi \parallel^2_{\mathbb{V}} \\
&\quad + C_8 \parallel \Pi \parallel_{L^2(\Omega)} (\parallel \phi \parallel_{L^4(\Omega)} \parallel \Pi \parallel_{L^4(\Omega)} + \parallel u \parallel_{L^3(\Omega)} \parallel \Pi \parallel_{L^6(\Omega)}) \\
&\quad + C_9(\parallel \Pi \parallel^2_{L^2(\Omega)} + \parallel u \parallel_{L^3(\Omega)} \parallel \Pi \parallel_{L^6(\Omega)} \parallel V \parallel_{L^2(\Omega)} + \parallel V \parallel^2_{L^2(\Omega)}), \\
&\frac{1}{2}\frac{d}{dt} \parallel V \parallel^2_{L^2(\Omega)} \le C_{10} \parallel \phi \parallel_{L^4(\Omega)} \parallel \Pi \parallel_{L^4(\Omega)} \parallel V \parallel_{L^2(\Omega)} \\
&\qquad + C_{11}(\parallel \Pi \parallel^2_{L^2(\Omega)} + \parallel V \parallel^2_{L^2(\Omega)} + \parallel g_2 \parallel^2_{L^2(\Omega)}).
\end{aligned} \tag{3.18}$$

Since $L^r(\Omega) \subset H^1(\Omega)$ for $r \le 6$, then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \parallel \mathfrak{b}_m\Pi \parallel^2_{L^2(\Omega)} + \underline{\alpha}_i \parallel \Pi \parallel^2_{\mathbb{V}} + \int_\Omega 3\hbar_{11}\Pi^2\phi^2 d\mathbf{x} \\
&\quad \le C_4(\parallel h_1 \parallel^2_{\mathbb{V}'} + \parallel g_1 \parallel^2_{L^2(\Omega)}) + (\epsilon_0 + \epsilon_1) \parallel \Pi \parallel^2_{\mathbb{V}} \\
&\quad + C_{12} \parallel \Pi \parallel^2_{L^2(\Omega)} (1 + \parallel \phi \parallel^2_{L^4(\Omega)} + \parallel u \parallel^2_{L^3(\Omega)}) + C_{13}(1 + \parallel u \parallel^2_{L^3(\Omega)}) \parallel V \parallel^2_{L^2(\Omega)}, \\
&\frac{1}{2}\frac{d}{dt} \parallel V \parallel^2_{L^2(\Omega)} \le \epsilon_2 \parallel \Pi \parallel^2_{\mathbb{V}} + C_{14} \parallel V \parallel^2_{L^2(\Omega)} (1 + \parallel \phi \parallel^2_{L^4(\Omega)}) \\
&\qquad + C_{15} \parallel \Pi \parallel^2_{L^2(\Omega)} + C_{16} \parallel g_2 \parallel^2_{L^2(\Omega)} .
\end{aligned} \tag{3.19}$$

By adding the first and second equation of previous system we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\frac{d}{dt} \parallel \mathfrak{b}_m\Pi \parallel^2_{L^2(\Omega)} + \frac{d}{dt} \parallel V \parallel^2_{L^2(\Omega)}\right) + \underline{\alpha}_i \parallel \Pi \parallel^2_{\mathbb{V}} + \int_\Omega 3\hbar_{11}\Pi^2\phi^2 d\mathbf{x} \\
&\quad \le (\epsilon_0 + \epsilon_1 + \epsilon_2) \parallel \Pi \parallel^2_{\mathbb{V}} + C_{17}(\parallel h_1 \parallel^2_{L^2(\Omega)} + \parallel g_1 \parallel^2_{L^2(\Omega)} + \parallel g_2 \parallel^2_{L^2(\Omega)}) \\
&\qquad + C_{18}(\parallel V \parallel^2_{L^2(\Omega)} + \parallel \Pi \parallel^2_{L^2(\Omega)})(1 + \parallel \phi \parallel^2_{L^4(\Omega)} + \parallel u \parallel^2_{L^3(\Omega)}).
\end{aligned} \tag{3.20}$$

By choosing ϵ_i such that $(\epsilon_0+\epsilon_1+\epsilon_2)=\frac{1}{2}\underline{\alpha}_i$, we can deduce that

$$\begin{aligned}\left(\frac{d}{dt}\parallel \mathfrak{b}_m\Pi\parallel^2_{L^2(\Omega)}+\frac{d}{dt}\parallel V\parallel^2_{L^2(\Omega)}\right)+\parallel \Pi\parallel^2_{\mathbb{V}}+\int_\Omega \Pi^2\phi^2 d\mathbf{x}\\ \leq C_{19}(\parallel h_1\parallel^2_{L^2(\Omega)}+\parallel g_1\parallel^2_{L^2(\Omega)}+\parallel g_2\parallel^2_{L^2(\Omega)})\\ +C_{20}(\parallel V\parallel^2_{L^2(\Omega)}+\parallel \mathfrak{b}_m\Pi\parallel^2_{L^2(\Omega)})(1+\parallel\phi\parallel^2_{L^4(\Omega)}+\parallel u\parallel^2_{L^3(\Omega)}).\end{aligned}\tag{3.21}$$

Then, Gronwall's lemma yields (for all $t\in(S_0,S_f)$)

$$\parallel V(t)\parallel^2_{L^2(\Omega)}+\parallel \mathfrak{b}_m\Pi(t)\parallel^2_{L^2(\Omega)}\leq\Big(\parallel V_0\parallel^2_{L^2(\Omega)}+\parallel \Pi_0\parallel^2_{L^2(\Omega)}\Big)e^{G(t)}+e^{G(t)}\int_{S_0}^{S_f}e^{-G(s)}F(s)ds,$$

where the functions

$$\begin{aligned}G(s)&=\int_0^s C_{20}(1+\parallel\phi\parallel^2_{L^4(\Omega)}+\parallel u\parallel^2_{L^3(\Omega)})d\tau,\\ F(s)&=C_{19}(\parallel h_1\parallel^2_{L^2(\Omega)}+\parallel g_1\parallel^2_{L^2(\Omega)}+\parallel g_2\parallel^2_{L^2(\Omega)}),\end{aligned}$$

exist (since $(h_1,g_1,g_2,u,\phi)\in L^2(Q_S)\times L^2(Q_S)\times L^2(Q_S)\times L^2(S_0,S_f;L^3(\Omega))\times L^4(Q_S)$). Consequently, according to the boundedness of $\mathfrak{b}_m$ (for all $t\in(S_0,S_f)$)

$$\begin{aligned}\parallel V(t)\parallel^2_{L^2(\Omega)}+\parallel \Pi(t)\parallel^2_{L^2(\Omega)}\leq C\Big(\parallel V_0\parallel^2_{L^2(\Omega)}+\parallel\Pi_0\parallel^2_{L^2(\Omega)}\\ +\parallel h_1\parallel^2_{L^2(Q_S)}+\parallel g_1\parallel^2_{L^2(Q_S)}+\parallel g\parallel^2_{L^2(Q_S)}\Big).\end{aligned}\tag{3.22}$$

Then according to (3.21), we can deduce that

$$\begin{aligned}\parallel\Pi\parallel^2_{L^2(S_0,S_f;\mathbb{V})}\leq C\Big(\parallel V_0\parallel^2_{L^2(\Omega)}+\parallel\Pi_0\parallel^2_{L^2(\Omega)}\\ +\parallel h_1\parallel^2_{L^2(Q_S)}+\parallel g_1\parallel^2_{L^2(Q_S)}+\parallel g_2\parallel^2_{L^2(Q_S)}\Big),\\ \int_{S_0}^{S_f}\int_\Omega\Pi^2\phi^2 d\mathbf{x}dt=\parallel\Pi\phi\parallel^2_{L^2(Q_S)}\leq C\Big(\parallel V_0\parallel^2_{L^2(\Omega)}+\parallel\Pi_0\parallel^2_{L^2(\Omega)}\\ +\parallel h_1\parallel^2_{L^2(Q_S)}+\parallel g_1\parallel^2_{L^2(Q_S)}+\parallel g_2\parallel^2_{L^2(Q_S)}\Big).\end{aligned}\tag{3.23}$$

From the second equation of (3.10) and relation (3.23), we can deduce that

$$\parallel\pi_e\parallel^2_{L^2(S_0,S_f;\mathbb{V})}\leq C\Big(\parallel V_0\parallel^2_{L^2(\Omega)}+\parallel\Pi_0\parallel^2_{L^2(\Omega)}+\parallel h_1\parallel^2_{L^2(Q_S)}+\parallel g_1\parallel^2_{L^2(Q_S)}+\parallel g_2\parallel^2_{L^2(Q_S)}\Big).\tag{3.24}$$

According to (3.22), (3.23) and (3.24), we can conclude that

$$(\Pi,V)\in(L^\infty(S_0,S_f;L^2(\Omega)))^2,\quad(\Pi,\pi_e)\in(L^2(S_0,S_f;\mathbb{V}))^2\ \text{ and }\ \Pi\phi\in L^2(Q_S).\tag{3.25}$$

From (3.10), we can deduce that

$$
\begin{aligned}
| \langle \mathfrak{c}_m \frac{\partial \Pi}{\partial t}, v \rangle_{\mathbb{V}',\mathbb{V}} | &\leq C_1 \| v \|_{\mathbb{V}} \left(\| \Pi \|_{\mathbb{V}} + \| \pi_e \|_{\mathbb{V}} + \| g_1 \|_{L^2(\Omega)} \right) \\
&\quad + C_2 \int_\Omega | \phi \| \phi\Pi \| v | \, d\mathbf{x} \\
&\quad + C_3 \Big(\int_\Omega | \phi \| \Pi \| v | \, d\mathbf{x} + \int_\Omega | u \| \Pi \| v | \, d\mathbf{x} + \int_\Omega | \phi \| V \| v | \, d\mathbf{x} \Big), \\
| (\frac{\partial V}{\partial t}, \rho)_{\mathbb{H}} | &\leq C_4 \| \rho \|_{L^2(\Omega)} \left(\| \Pi \|_{L^2(\Omega)} + \| V \|_{L^2(\Omega)} + \| g_2 \|_{L^2(\Omega)} \right) \\
&\quad + C_5 \int_\Omega | \phi \| \Pi \| \rho | \, d\mathbf{x},
\end{aligned}
\tag{3.26}
$$

and then

$$
\begin{aligned}
\| \frac{\partial \Pi}{\partial t} \|_{\mathbb{V}'} &\leq C_6 \left(\| \Pi \|_{\mathbb{V}} + \| \pi_e \|_{\mathbb{V}} + \| g_1 \|_{L^2(\Omega)} \right) + C_7 \| V \|_{L^2(\Omega)} \| \phi \|_{L^4(\Omega)} \\
&\quad + C_8 \left(\| \phi\Pi \|_{L^2(\Omega)} \| \phi \|_{L^4(\Omega)} + \| \phi \|_{L^2(\Omega)} \| \Pi \|_{\mathbb{V}} + \| u \|_{L^2(\Omega)} \| \Pi \|_{\mathbb{V}} \right), \\
\| \frac{\partial V}{\partial t} \|_{L^2(\Omega)} &\leq C_9 \left(\| V \|_{L^2(\Omega)} + \| \Pi \|_{L^2(\Omega)} + \| \phi\Pi \|_{L^2(\Omega)} + \| g_2 \|_{L^2(\Omega)} \right).
\end{aligned}
\tag{3.27}
$$

Consequently

$$
\begin{aligned}
\int_{S_0}^{S_f} \| \frac{\partial \Pi}{\partial t} \|_{\mathbb{V}'}^{4/3} dt &\leq C_{10} \left(\| \Pi \|_{L^2(S_0,S_f;\mathbb{V})}^{4/3} + \| \pi_2 \|_{L^2(S_0,S_f;\mathbb{V})}^{4/3} + \| g_1 \|_{L^2(Q_S)}^{4/3} \right) \\
&\quad + C_{11} \left(\| \phi\Pi \|_{L^2(Q_S)}^{4/3} \| \phi \|_{L^4(S_0,S_f;L^4(\Omega))}^{4/3} + \| V \|_{L^2(Q_S)}^{4/3} \| \phi \|_{L^4(S_0,S_f;L^4(\Omega))}^{4/3} \right) \\
&\quad + C_{12} \left(\| \Pi \|_{L^2(S_0,S_f;\mathbb{V})}^{4/3} \| u \|_{L^4(S_0,S_f;L^2(\Omega))}^{4/3} + \| \Pi \|_{L^2(S_0,S_f;\mathbb{V})}^{4/3} \| \phi \|_{L^4(S_0,S_f;L^2(\Omega))}^{4/3} \right), \\
\int_{S_0}^{S_f} \| \frac{\partial V}{\partial t} \|_{L^2(\Omega)}^2 dt &\leq C_{13} \left(\| \Pi \|_{L^2(Q_S)}^2 + \| V \|_{L^2(Q_S)}^2 + \| \phi\Pi \|_{L^2(Q_S)}^2 + \| g \|_{L^2(Q_S)}^2 \right)
\end{aligned}
$$

and then (according to (3.22), (3.23) and (3.24))

$$
\begin{aligned}
&\| \frac{\partial \Pi}{\partial t} \|_{L^{4/3}(S_0,S_f;\mathbb{V}')} + \| \frac{\partial V}{\partial t} \|_{L^2(S_0,S_f;L^2(\Omega))} \\
&\quad \leq C \Big(\| V_0 \|_{L^2(\Omega)} + \| \Pi_0 \|_{L^2(\Omega)} + \| h_1 \|_{L^2(Q_S)} + \| g_1 \|_{L^2(Q_S)} + \| g_2 \|_{L^2(Q_S)} \Big).
\end{aligned}
\tag{3.28}
$$

We can conclude that $\frac{\partial \Pi}{\partial t} \in L^{4/3}(S_0, S_f; \mathbb{V}')$ and $\frac{\partial V}{\partial t} \in L^2(Q_S)$. The proof of Theorem can be completed by implementing the Galerkin method and by taking advantage of the above estimates, (3.22)–(3.24) and (3.28), and Lemma 2.4 and Remark 2.1. So we omit the details. Since the problem is linear, then from the estimates (3.22)–(3.24) and (3.28) we can deduce the Lipschitz continuity result and then the uniqueness of solution.
(ii) Prove first that $V \in L^2(S_0, S_f; L^3(\Omega))$. Since V satisfies the equation

$$
\frac{\partial V}{\partial t} = -\frac{\partial \mathcal{I}_2}{\partial \phi}(.;\phi)\Pi - \hbar V + g_2.
\tag{3.29}
$$

Since (3.29) is similar as (3.39), then by using similar argument to derive (3.43) we obtain

$$\begin{aligned} \| V(.,t) \|_{L^3(\Omega)} \leq C_1 \int_{S_0}^{t} \| V(.,s) \|_{L^3(\Omega)} \, ds \\ + C_2 \Big(\| g_2 \|_{L^2(S_0,S_f;L^3(\Omega))} + \| \phi \|_{L^2(S_0,S_f;L^6(\Omega))} \| \Pi \|_{L^2(S_0,S_f;L^6(\Omega))} \Big). \end{aligned} \tag{3.30}$$

Since Π and ϕ are in $L^2(0,T,H^1) \subset L^2(0,T,L^r)$ $(r \in [1,6])$ and $g_2 \in L^2(S_0,S_f;L^3(\Omega))$, then

$$\| V(.,t) \|_{L^3(\Omega)} \leq C_3 + C_4 \int_{S_0}^{t} \| V(.,s) \|_{L^3(\Omega)} \, ds. \tag{3.31}$$

Consequently, by using Gronwall lemma, $\| V(.,t) \|_{L^3(\Omega)} \leq C$ and then $V \in L^\infty(S_0,S_f;L^3(\Omega))$.

Prove now that $(\Pi, \pi_e) \in (L^\infty(S_0,S_f;H^1(\Omega)))^2$. For this we will just sketch the proof based on suitable a priori estimates. From (3.10) with $(v, v_e) = (\frac{\partial \Pi}{\partial t}, \frac{\partial \pi_e}{\partial t})$ (put $\tilde{h}_1 = B_1 h_1$)

$$\begin{aligned} &\| \mathfrak{b}_m \frac{\partial \Pi}{\partial t} \|^2_{L^2(\Omega)} + A_i(\Pi + \pi_e, \frac{\partial \Pi}{\partial t}) + \int_\Omega \frac{\partial \mathcal{I}}{\partial \phi} \Pi \frac{\partial \Pi}{\partial t} d\mathbf{x} + \int_\Omega \frac{\partial \mathcal{I}}{\partial u} V \frac{\partial \Pi}{\partial t} d\mathbf{x} = \int_\Omega g_1 \frac{\partial \Pi}{\partial t} d\mathbf{x}, \\ &A_i(\Pi + \pi_e, \frac{\partial \pi_e}{\partial t}) + A_e(\pi_e, \frac{\partial \pi_e}{\partial t}) = \int_\Omega \tilde{h}_1 \frac{\partial \pi_e}{\partial t} d\mathbf{x}. \end{aligned} \tag{3.32}$$

Then

$$\begin{aligned} \| \mathfrak{b}_m \frac{\partial \Pi}{\partial t} \|^2_{L^2(\Omega)} &+ \frac{d}{2dt} A_i(\Pi + \pi_e, \Pi + \pi_e) + \frac{d}{2dt} A_i(\pi_e, \pi_e) \\ &= - \int_\Omega \frac{\partial \mathcal{I}}{\partial \phi} \Pi \frac{\partial \Pi}{\partial t} d\mathbf{x} - \int_\Omega \frac{\partial \mathcal{I}}{\partial u} V \frac{\partial \Pi}{\partial t} d\mathbf{x} \\ &\quad + \int_\Omega g_1 \frac{\partial \Pi}{\partial t} d\mathbf{x} + \int_\Omega \pi_e \frac{\partial \tilde{h}_1}{\partial t} d\mathbf{x} + \frac{d}{dt} \Big(\int_\Omega \tilde{h}_1 \pi_e d\mathbf{x} \Big). \end{aligned} \tag{3.33}$$

Since $h_1 \in U_c$ then $\tilde{h}_1 \in U_c \subset C^0([S_0,S_f];L^2(\Omega))$. So, $\tilde{h}_1(S_0) \in L^2(\Omega)$ and we have $\| \tilde{h}_1(S_0) \|^2_{L^2(\Omega)} \leq C_1 \| \tilde{h}_1 \|^2_{U_c}$. Consequently, by integrating (3.33) by time, we can deduce (from (2.21), the boundedness of $\mathfrak{b}_m$, coercivity and continuity of A_i and A_e, and regularity H^1 of $(\Pi_0, \pi_e^{(0)})$)

$$\begin{aligned} 2\underline{c}_m \int_{S_0}^{t} \| \frac{\partial \Pi}{\partial t} \|^2_{L^2(\Omega)} ds &+ \alpha_i \| \Pi(t) \|^2_{H^1(\Omega)} + (\alpha_e + \alpha_i) \| \pi_e(t) \|^2_{H^1(\Omega)} \\ &\leq C_0 \Big(\int_{S_0}^{t} \int_\Omega | \phi |^4 | \Pi |^2 \, d\mathbf{x} ds + \int_{S_0}^{t} \int_\Omega | \phi |^2 | \Pi |^2 \, d\mathbf{x} ds \Big) \\ &\quad + C_1 \Big(\int_{S_0}^{t} \int_\Omega | u |^2 | \Pi |^2 \, d\mathbf{x} ds + \int_{S_0}^{t} \int_\Omega | \phi |^2 | V |^2 \, d\mathbf{x} ds \Big) \\ &\quad + C_2 \Big(\int_{S_0}^{t} \int_\Omega | \Pi |^2 \, d\mathbf{x} ds + \int_{S_0}^{t} \int_\Omega | V |^2 \, d\mathbf{x} ds + \int_{S_0}^{t} \int_\Omega | \pi_e |^2 \, d\mathbf{x} ds \Big) \\ &\quad + \delta_0 \int_{S_0}^{t} \| \frac{\partial \Pi}{\partial t} \|^2_{L^2(\Omega)} ds + \delta_1 \| \pi_e(t) \|^2_{H^1(\Omega)} + C_3 (1 + \| g_1 \|^2_{L^2(Q_S)} + \| \tilde{h}_1 \|^2_{U_c}). \end{aligned} \tag{3.34}$$

Since ϕ is in $L^\infty(0,T,H^1) \subset L^\infty(0,T,L^r)$ ($r \in [1,6]$), then (by choosing $\delta_0 = \underline{c}_m$, $\delta_1 = (\alpha_e + \alpha_i)/2$)

$$\begin{aligned}
&\int_{S_0}^t \| \frac{\partial \Pi}{\partial t} \|^2_{L^2(\Omega)} ds + \| \Pi(t) \|^2_{H^1(\Omega)} + \| \pi_e(t) \|^2_{H^1(\Omega)} \\
&\leq C_4(\| \phi \|^4_{L^\infty(0,T;H^1(\Omega))} + \| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} + \| u \|^2_{L^\infty(0,T;L^3(\Omega))}) \int_{S_0}^t \| \Pi \|^2_{H^1(\Omega)} ds \\
&\quad + C_5(\| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} \| V \|^2_{L^\infty(S_0,S_f;L^3(\Omega))} + \| \Pi \|^2_{L^2(Q_S)} + \| \pi_e \|^2_{L^2(Q_S)} + \| V \|^2_{L^2(Q_S)}) \\
&\quad + C_6(1 + \| g_1 \|^2_{L^2(Q_S)} + \| \tilde{h}_1 \|^2_{U_c}).
\end{aligned} \tag{3.35}$$

From the regularity of $(\Pi, \pi_e, V, g_1, \tilde{h}_1)$, estimation of (Π, π_e) in $L^\infty(S_0, S_f; H^1(\Omega))$ and and estimation of V in $L^\infty(S_0, S_f; L^3(\Omega))$, respectively, we can deduce that

$$\int_{S_0}^t \| \frac{\partial \Pi}{\partial t} \|^2_{L^2(\Omega)} ds + \| \Pi(t) \|^2_{H^1(\Omega)} + \| \pi_e(t) \|^2_{H^1(\Omega)} \leq C \tag{3.36}$$

and then $\frac{\partial \Pi}{\partial t} \in L^2(Q_S)$ and $(\Pi, \pi_e) \in L^\infty(S_0, S_f; H^1(\Omega))$.

The proof of the result can be completed by implementing the classical Faedo-Galerkin method and by taking advantage of above estimates. So we omit the details.

Finally we prove that $V \in C^0([S_0, S_f], L^3(\Omega))$. Let $R = -\dfrac{\partial \mathcal{I}_2}{\partial \phi} - \hbar V + g_2$ be the right hand side of equation (3.29). Since $\dfrac{\partial \mathcal{I}_2}{\partial \phi}$ is a polynomial of degree 1 on ϕ, we can deduce that (since $(\phi, \Pi) \in (L^\infty(S_0, S_f; H^1(\Omega)))^2$ and $V \in L^\infty(S_0, S_f; L^3(\Omega))$)

$$\| R \|_{L^3(\Omega)} \leq C \Big(\| \Pi \|_{L^\infty(S_0,S_f;H^1(\Omega))} (1 + \| \phi \|_{L^\infty(S_0,S_f;H^1(\Omega))}) + \| V \|_{L^\infty(S_0,S_f;L^3(\Omega))} + \| g_2 \|_{L^3(\Omega)} \Big).$$

Since $g_2 \in L^2(S_0, S_f; L^3(\Omega))$, then $\| R \|_{L^2(S_0,S_f;L^3(\Omega))} \leq C$. Consequently, $R \in L^2(S_0, S_f; L^3(\Omega))$ and then, from (3.29), $\dfrac{\partial V}{\partial t} \in L^2(S_0, S_f; L^3(\Omega))$. As $V \in L^2(S_0, S_f; L^3(\Omega))$ and $\dfrac{\partial V}{\partial t} \in L^2(S_0, S_f; L^3(\Omega))$ we can conclude, from Remark 2.3, that $V \in C^0([S_0, S_f]; L^3(\Omega))$. This completes the proof. □

We can now prove the well-posedness of problem (3.7).

Theorem 3.1. *Let assumptions (H1)-(H4), (HMC) and (RC) be fulfilled. Suppose that* $(\phi, \varphi_e, u) \in \mathbb{D}$, *then the following results hold.*

(i) For any $(h_1, h_2) \in U_c \times L^2(Q)$ *under the compatibility condition (1.2), there exists a unique weak solution* $(\Psi, \psi_e, w) \in \mathbb{D}$, *of the linear problem (3.7).*

(ii) Let $(h_{1,i}, h_{2,i})$, $i = 1, 2$ *be given in* $U_c \times L^2(Q)$. *If* $(\Psi_i, \psi_{e,i}, w_i)$ *is the solution of (3.7) corresponding to data* $(h_{1,i}, h_{2,i})$, *for* $i = 1, 2$, *then*

$$\| (\Psi, \psi_e, w) \|^2_{\mathbb{D}} \leq C \Big(\|h_1\|^2_{U_c} + \|h_2\|^2_{L^2(Q)} \Big), \tag{3.37}$$

where $(\Psi, \psi_e, w) = (\Psi_1 - \Psi_2, \psi_{e,1} - \psi_{e,2}, w_1 - w_2)$ *and* $(h_1, h_2) = (h_{1,1} - h_{1,2}, h_{2,1} - h_{2,2})$.

Proof. To prove the existence of a unique solution on Q, we first establish the existence of a unique solution on $\tilde{Q}_j = \Omega \times (s_0, s_j)$, $j \geq 1$ and obtain some estimations.

We solve the problem on $\tilde{Q}_1$ and obtain the existence of a unique solution on $\tilde{Q}_1$. Then, the existence of a unique solution on $\tilde{Q}_2$ is proved by using the solution on $\tilde{Q}_1$ to generate the initial data at s_1. This advancing process is repeated for $\tilde{Q}_3, \tilde{Q}_4, ...$ until the final set is reached. Hereafter, the solution on $\tilde{Q}_j$ will be denoted by $(\Psi_j, \psi_{e,j}, w_j)$ for $j = 1, ...$
Now we introduce the following problems $(\mathcal{P}_j)$ for $j \in \mathbb{N} - \{0\}$ (for $(\mathbf{x}, t) \in \Omega \times (s_{j-1}, s_j)$)

$$\begin{aligned}
&c_m \frac{\partial \Pi_j}{\partial t} + \frac{\partial \mathcal{I}}{\partial \phi}(\phi, u).\pi_j + \frac{\partial \mathcal{I}}{\partial u}(\phi, u).V_j - div(\mathcal{K}_i \nabla(\Pi_j + \pi_{e,j})) = g_{1,j},\\
&-div\left((\mathcal{K}_e + \mathcal{K}_i)\nabla \pi_{e,j}\right) - div(\mathcal{K}_i \nabla \Pi_j) = \mathcal{B}_1 h_1,\\
&\frac{\partial V_j}{\partial t} + \frac{\partial \mathcal{G}}{\partial \phi}(\phi, u).\Pi_j + \frac{\partial \mathcal{G}}{\partial u}(\phi, u).V_j = g_{2,j},\\
&(\mathcal{K}_i \nabla(\Pi_j + \pi_{e,j})) \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \times (s_{j-1}, s_j)\\
&(\mathcal{K}_e \nabla \pi_{e,j}) \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \times (s_{j-1}, s_j)\\
&\Pi_j(\mathbf{x}, s_{j-1}) = \Psi_{j-1}(\mathbf{x}, s_{j-1}) \in L^2(\Omega),\\
&V_j(\mathbf{x}, s_{j-1}) = w_{j-1}(\mathbf{x}, s_{j-1}) \in L^2(\Omega)
\end{aligned} \tag{3.38}$$

where $(\Psi_{j-1}, w_{j-1}) \in (L^2(\tilde{Q}_{j-1}))^2$ and

$$g_{1,j}(\mathbf{x}, t) = \mathcal{B}_2 h_2 + \sum_{k=1}^{n_1} a_k(\mathbf{x}, t)\Psi_{j-1}(\mathbf{x}, t - \xi_k(t)) + \sum_{l=1}^{n_2} b_l(\mathbf{x}, t) w_{j-1}(\mathbf{x}, t - \eta_l(t)),$$
$$g_{2,j}(\mathbf{x}, t) = \sum_{k=1}^{n_1} c_k(\mathbf{x}, t)\Psi_{j-1}(\mathbf{x}, t - \xi_k(t)) + \sum_{l=1}^{n_2} d_l(\mathbf{x}, t) w_{j-1}(\mathbf{x}, t - \eta_l(t)).$$

Since $B_2 h_2 \in L^2(Q)$, $(\Psi_{j-1}, w_{j-1}) \in L^2(\tilde{Q}_{j-1})$ and a_k, c_k, b_l, d_l (for $1 \le k \le n_1$ and $1 \le l \le n_2$) are in $C^\infty(\overline{Q})$, then, according to Lemma 2.6, we have that $g_{1,j}$ and $g_{2,j}$ are in $L^2(\tilde{Q}_j)$. Then using Lemma 3.1 we have that the problem $(\mathcal{P}_j)$ admits a unique solution $(\Pi_j, \pi_{e,j}, V_j) \in \mathcal{W}(\Omega \times (s_{j-1}, s_j))$ and verifying $(\frac{\partial \Pi_j}{\partial t}, \frac{\partial V_j}{\partial t}) \in L^{4/3}(s_{j-1}, s_j; \mathbb{V}') \times L^2(s_{j-1}, s_j; L^2(\Omega)))$ and $(\Pi_j, V_j) \in (C^0([s_{j-1}, s_j]; L^2(\Omega)))^2$. Then, we can extend the result to the cylinder set $\tilde{Q}_{j+1}$ by taking $(\Psi_j, \psi_{e,j}, w_j) = (\Psi_{j-1}, \psi_{e,j-1}, w_{j-1})$ on $\tilde{Q}_{j-1}$ and $(\Psi_j, \psi_{e,j}, w_j) = (\Pi_j, \pi_{e,j}, V_j)$ on $\Omega \times (s_{j-1}, s_j)$.

We observe that for j=1, we have (according to the initial condition in (3.7))

$$g_{1,1}(\mathbf{x}, t) = \mathcal{B}_2 h_2, \text{ and } g_{2,1}(\mathbf{x}, t) = 0$$

and then $g_{1,1}, g_{2,1} \in L^2(\tilde{Q}_1)$. By using the previous result, the problem $(\mathcal{P}_1)$ admits a unique solution $(\Pi_1, \pi_{e,1}, V_1)$ and then the solution $(\Psi_1, \psi_{e,1}, w_1)$. We inject now $(\Psi_1, \psi_{e,1}, w_1)$ in the problem $(\mathcal{P}_2)$ and by using the same approach, we obtain the existence and uniqueness of $(\Pi_2, \pi_{e,2}, V_2)$ (solution of $(\mathcal{P}_2)$).
We can now iterate the process for any domain $\tilde{Q}_j, for\ j \ge 1$ and we obtain the existence and uniqueness of $(\Pi_j, \pi_{e,j}, V_j)$ solution of $(\mathcal{P}_j)$.
We deduce then the existence and uniqueness of the solution $(\Psi, \psi_e, w) \in \mathcal{W}(Q)$ of (3.8) (verifying $(\frac{\partial \Psi}{\partial t}, \frac{\partial w}{\partial t}) \in L^{4/3}(0, T; \mathbb{V}') \times L^2(0, T; L^2(\Omega)))$ and $(\Psi, w) \in (C^0([0, T]; L^2(\Omega)))^2)$ such that $(\Psi, \psi_e, w)|_{\tilde{Q}_j} = (\Psi_j, \psi_{e,j}, w_j), j \ge 1$.

Prove now that the unique solution (Ψ, ψ_e, w) satisfies the following regularity: $(\Psi, \psi_e, w) \in (L^{\infty}(0, T, ; H^1(\Omega)))^2 \times C^0([0, T]; L^3(\Omega))$ and $(\frac{\partial \Psi}{\partial t}, \frac{\partial w}{\partial t}) \in L^2(Q) \times L^2(0, T; L^3(\Omega)))$.

Step1. Prove first that $w \in L^2(0, T; L^3(\Omega))$. Since w satisfies the equation

$$\frac{\partial w}{\partial t} = -\frac{\partial \mathcal{I}_2}{\partial \phi}(.; \phi)\Psi - \hbar w + \mathcal{E}(\Psi_\tau, w_\tau). \tag{3.39}$$

Then for all t we have (since $w(t = 0) = 0$)

$$w(\mathbf{x}, t) = -\int_0^t \frac{\partial \mathcal{I}_2}{\partial \phi}(\mathbf{x}, s; \phi)\Psi ds - \int_0^t \hbar w(\mathbf{x}, s) ds + \int_0^t \mathcal{E}(\Psi_\tau, w_\tau) ds. \tag{3.40}$$

Consequently,

$$| w(\mathbf{x}, t) | \leq C\left(\int_0^t | \frac{\partial \mathcal{I}_2}{\partial \phi}(\mathbf{x}, s; \phi) \| \Psi | ds + \int_0^t | w(\mathbf{x}, s) | ds\right) + \int_0^t | \mathcal{E}(\Psi_\tau, w_\tau) | ds \tag{3.41}$$

and then (since $\frac{\partial \mathcal{I}_2}{\partial \phi}(\mathbf{x}, s; \phi)$ is linear operator)

$$| w(\mathbf{x}, t) | \leq C\left(\int_0^t | w(\mathbf{x}, s) | ds + \int_0^t | \Psi(\mathbf{x}, s) \| \phi(\mathbf{x}, s) | ds\right) + \int_0^t | \mathcal{E}(\Psi_\tau, w_\tau) | ds. \tag{3.42}$$

This implies

$$\begin{aligned}\left(\int_\Omega | w(\mathbf{x}, t) |^3 d\mathbf{x}\right)^{1/3} \leq C\Big(& (\int_\Omega (\int_0^t | w(\mathbf{x}, s) | ds)^3 d\mathbf{x})^{1/3} \\ & +(\int_\Omega (\int_0^t | \mathcal{E}(\Psi_\tau, w_\tau) | ds)^3 d\mathbf{x})^{1/3} \\ & +(\int_\Omega (\int_0^t | \Psi(\mathbf{x}, s) \| \phi(\mathbf{x}, s) | ds)^3 d\mathbf{x})^{1/3}\Big)\end{aligned}$$

and then (using Minkowski inequality)

$$\begin{aligned}\left(\int_\Omega | w(\mathbf{x}, t) |^3 d\mathbf{x}\right)^{1/3} \leq C\Big(& \int_0^t (\int_\Omega | w(\mathbf{x}, s) |^3 d\mathbf{x})^{1/3} ds \\ & + \int_0^t (\int_\Omega | \mathcal{E}(\Psi_\tau, w_\tau) |^3 d\mathbf{x})^{1/3} ds \\ & + \int_0^t (\int_\Omega | \Psi(\mathbf{x}, s) |^3 | \phi(\mathbf{x}, s) |^3 d\mathbf{x})^{1/3} ds\Big).\end{aligned} \tag{3.43}$$

According to Lemma 2.6 and the fact that $(\Psi_{past}, w_{past}) = (0, 0)$, we can deduce that

$$\begin{aligned}\| w(., t) \|_{L^3(\Omega)} \leq C_1 & \int_0^t \| w(., s) \|_{L^3(\Omega)} ds \\ & + C_2\left(\| \Psi \|_{L^2(0,T;L^3(\Omega))} + \| \phi \|_{L^2(0,T;L^6(\Omega))} \| \Psi \|_{L^2(0,T;L^6(\Omega))}\right).\end{aligned} \tag{3.44}$$

Since Ψ and ϕ are in $L^2(0, T, H^1(\Omega)) \subset L^2(0, T, L^r(\Omega))$ $(r \in [1, 6])$, then

$$\| w(., t) \|_{L^3(\Omega)} \leq C_3 + C_4 \int_0^t \| w(., s) \|_{L^3(\Omega)} ds. \tag{3.45}$$

Consequently, by using Gronwall lemma, $\| w(.,t) \|_{L^3(\Omega)} \leq C$ and then $w \in L^\infty(0,T;L^3(\Omega))$.

Step2. Prove now that $(\Psi,\psi_e) \in (L^\infty(0,T;H^1(\Omega)))^2$ and $w \in C^0([0,T];L^3(\Omega))$.
Put $g_1 = \mathcal{H}(\Psi_\tau,w_\tau) + \mathcal{B}_2 h_2$ and $g_2 = \mathcal{E}(\Psi_\tau,w_\tau)$. Since $w \in L^\infty(0,T;L^3(\Omega))$ and $\Psi \in L^2(0,T;H^1(\Omega))$ then, according to Lemma 2.6, $g_1 \in L^2(Q)$ and $g_2 \in L^2(0,T,L^3(\Omega))$. Since (Ψ,ψ_e,w) is a solution of (3.9) which correspond to data (g_1,g_2,h_1) and $(\Psi,w)(t=0) = (0,0)$ with $(S_0,S_f) = (0,T)$, we can deduce from Lemma 3.1 that $(\Psi,\psi_e) \in (L^\infty(0,T;H^1(\Omega)))^2$ and $w \in C^0([0,T];L^3(\Omega))$.

We are now going to prove the estimation given in theorem.
Let $(\Psi_i,\psi_{e,i},w_i)_{i=1,2}$ be two solutions of (3.8), corresponding to data $(h_{i,1},h_{i,2})_{i=1,2}$, respectively. We denote $\Psi = \Psi_1 - \Psi_2$, $w = w_1 - w_2$, $\psi_e = \psi_{e,1} - \psi_{e,2}$, $h_1 = h_{1,1} - h_{2,1}$, $h_2 = h_{1,2} - h_{2,2}$. Then according to (3.8) and setting $(v,u,v_e) = (\Psi,w,\psi_e)$ we can deduce that (since delay operators are linear)

$$\begin{aligned}
&\frac{d}{2dt}\|\mathfrak{b}_m\Psi\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{K}_i\nabla\Psi\nabla\Psi d\mathbf{x} + \int_\Omega \mathcal{K}_i\nabla\psi_e\nabla\Psi d\mathbf{x} + \int_\Omega \frac{\partial\mathcal{I}}{\partial\phi}(\phi,u)\Psi^2 d\mathbf{x}\\
&\quad + \int_\Omega \frac{\partial\mathcal{I}}{\partial u}(\phi,u)w\Psi d\mathbf{x} = \int_\Omega \mathcal{H}(\Psi_\tau,w_\tau)\Psi d\mathbf{x} + \int_\Omega \mathcal{B}_2 h_2\Psi d\mathbf{x},\\
&\int_\Omega (\mathcal{K}_i + \mathcal{K}_e)\nabla\psi_e\nabla\psi_e d\mathbf{x} + \int_\Omega \mathcal{K}_i\nabla\Psi\nabla\psi_e d\mathbf{x} = \int_\Omega \mathcal{B}_1 h_1\psi_e d\mathbf{x},\\
&\frac{d}{2dt}\|w\|^2_{L^2(\Omega)} + \int_\Omega \frac{\partial\mathcal{G}}{\partial\phi}(\phi,u)\Psi w d\mathbf{x} + \int_\Omega \hbar w^2 d\mathbf{x} = \int_\Omega \mathcal{E}(\Psi_\tau,w_\tau)w d\mathbf{x}\\
&\Psi(.,t=0) = 0, \quad w(.,t=0) = 0, \text{ in } \Omega\\
&\Psi = 0, \quad w = 0, \text{ in } Q_0.
\end{aligned} \tag{3.46}$$

Consequently (from the expression of the derivatives of $\mathcal{I}$ and $\mathcal{G}$)

$$\begin{aligned}
&\frac{d}{2dt}(\|\mathfrak{b}_m\Psi\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}) + A_i(\Psi+\psi_e,\Psi+\psi_e) + A_e(\psi_e,\psi_e)\\
&+ \int_\Omega (\hbar_{11}\phi^2 - \hbar_{12}\phi + \hbar_{13} + \epsilon_1\hbar_{21}u)\Psi^2 d\mathbf{x} + \int_\Omega \hbar w^2 d\mathbf{x}\\
&+ \int_\Omega (\epsilon_1\hbar_{21}\phi + h_{22})w\Psi d\mathbf{x} + \int_\Omega (-\epsilon_2\hbar_{31}\phi + (2\epsilon_2 - 1)h_{32})\Psi w d\mathbf{x}\\
&= \int_\Omega \mathcal{H}(\Psi_\tau,w_\tau)\Psi d\mathbf{x} + \int_\Omega \mathcal{E}(\Psi_\tau,w_\tau)w d\mathbf{x} + \int_\Omega \mathcal{B}_1 h_1\psi_e d\mathbf{x} + \int_\Omega \mathcal{B}_2 h_2\Psi d\mathbf{x},\\
&\Psi(.,t=0) = 0, \quad w(.,t=0) = 0, \text{ in } \Omega\\
&\Psi = 0, \quad w = 0, \text{ in } Q_0.
\end{aligned} \tag{3.47}$$

Using the boundedness of the function $\hbar_{ij}$ and $\hbar$, coercivity of A_i and A_e, and the assumption concerning the operators $\mathcal{B}_i$, $i = 1,2$), we obtain

$$\begin{aligned}
&\frac{d}{2dt}(\|\mathfrak{b}_m\Psi\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}) + \alpha_i\|\Psi+\psi_e\|^2_{\mathbb{V}} + \alpha_e\|\psi_e\|^2_{\mathbb{V}} + \int_\Omega \hbar_{11}\phi^2\Psi^2 d\mathbf{x}\\
&\quad \leq C_1\left(\int_\Omega (|\phi| + 1 + |u|)\Psi^2 d\mathbf{x} + \int_\Omega w^2 d\mathbf{x} + \int_\Omega (|\phi| + 1)|w||\Psi| d\mathbf{x}\right)\\
&\quad + \|\mathcal{H}(\Psi_\tau,w_\tau)\|_{L^2(\Omega)}\|\Psi\|_{L^2(\Omega)} + \|\mathcal{E}(\Psi_\tau,w_\tau)\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}\\
&\quad + C_0\left(\|h_2\|_{L^2(\Omega)}\|\Psi\|_{L^2(\Omega)} + \|h_1\|_{L^2(\Omega)}\|\psi_e\|_{L^2(\Omega)}\right)
\end{aligned} \tag{3.48}$$

and then

$$\frac{d}{2dt}(\|\mathfrak{b}_m\Psi\|^2_{L^2(\Omega)}+\|w\|^2_{L^2(\Omega)})+\alpha_i\|\Psi+\psi_e\|^2_{\mathbb{V}}+\alpha_e\|\psi_e\|^2_{\mathbb{V}}+\int_\Omega \hbar_{11}\phi^2\Psi^2 d\mathbf{x}$$
$$\leq C_2\parallel\Psi\parallel_{L^2(\Omega)}(\parallel\phi\parallel_{L^4(\Omega)}\parallel\Psi\parallel_{L^4(\Omega)}+\parallel u\parallel_{L^3(\Omega)}\parallel\Psi\parallel_{L^6(\Omega)})$$
$$+C_3\parallel\phi\parallel_{L^4(\Omega)}\parallel\Psi\parallel_{L^4(\Omega)}\parallel w\parallel_{L^2(\Omega)}$$
$$+C_4(\parallel\mathcal{H}(\Psi_\tau,w_\tau)\parallel^2_{L^2(\Omega)}+\parallel\mathcal{E}(\Psi_\tau,w_\tau)\parallel^2_{L^2(\Omega)})$$
$$+C_5(\parallel h_2\parallel^2_{L^2(\Omega)}+\parallel h_1\parallel^2_{L^2(\Omega)})+C_6(\parallel\Psi\parallel^2_{L^2(\Omega)}+\parallel w\parallel^2_{L^2(\Omega)}).$$

Consequently (since $\|\Psi\|_{\mathbb{V}}\leq\|\Psi+\psi_e\|_{\mathbb{V}}+\|\psi_e\|_{\mathbb{V}}$)

$$\begin{aligned}&\frac{d}{2dt}(\|\mathfrak{b}_m\Psi\|^2_{L^2(\Omega)}+\|w\|^2_{L^2(\Omega)})+\min(\alpha_i,\alpha_e)(\|\Psi+\psi_e\|^2_{\mathbb{V}}+\|\psi_e\|^2_{\mathbb{V}})\\&+\int_\Omega\hbar_{11}\phi^2\Psi^2d\mathbf{x}\leq\frac{1}{2}\min(\alpha_i,\alpha_e)(\|\Psi+\psi_e\|^2_{\mathbb{V}}+\|\psi_e\|^2_{\mathbb{V}})\\&\quad+C_6(\parallel\mathcal{H}(\Psi_\tau,w_\tau)\parallel^2_{L^2(\Omega)}+\parallel\mathcal{E}(\Psi_\tau,w_\tau)\parallel^2_{L^2(\Omega)})\\&\quad+C_7(\parallel h_2\parallel^2_{L^2(\Omega)}+\parallel h_1\parallel^2_{L^2(\Omega)})\\&\quad+C_8(\parallel\Psi\parallel^2_{L^2(\Omega)}+\parallel w\parallel^2_{L^2(\Omega)})(1+\parallel\phi\parallel^2_{L^4(\Omega)}+\parallel u\parallel^2_{L^3(\Omega)}).\end{aligned}\tag{3.49}$$

By integrating in time between 0 and t (since $\Psi(0)=w(0)=0$), we can deduce that (according to the boundedness of $\mathfrak{b}_m$)

$$\begin{aligned}&\|\Psi(.,t)\|^2_{L^2(\Omega)}+\|w(.,t)\|^2_{L^2(\Omega)}+\int_0^t(\|\Psi+\psi_e\|^2_{\mathbb{V}}+\|\psi_e\|^2_{\mathbb{V}})ds+\int_0^T\int_\Omega\phi^2\Psi^2d\mathbf{x}ds\\&\leq C_9\left(\int_0^t\parallel\mathcal{H}(\Psi_\tau,w_\tau)\parallel^2_{L^2(\Omega)}ds+\int_0^t\parallel\mathcal{E}(\Psi_\tau,w_\tau)\parallel^2_{L^2(\Omega)}ds\right)\\&\quad+C_{10}\left(\int_0^T\parallel h_2\parallel^2_{L^2(\Omega)}ds+\int_0^T\parallel h_1\parallel^2_{L^2(\Omega)}ds\right)\\&\quad+C_{11}\int_0^t(\parallel\Psi\parallel^2_{L^2(\Omega)}+\parallel w\parallel^2_{L^2(\Omega)})(1+\parallel\phi\parallel^2_{L^4(\Omega)}+\parallel u\parallel^2_{L^3(\Omega)})ds.\end{aligned}\tag{3.50}$$

According to Lemma 2.6, we can deduce that

$$\int_0^t\left(\parallel\mathcal{H}(\Psi_\tau,w_\tau)\parallel^2_{L^2(\Omega)}+\parallel\mathcal{E}(\Psi_\tau,w_\tau)\parallel^2_{L^2(\Omega)}\right)ds\leq C\int_0^t\left(\parallel\Psi\parallel^2_{L^2(\Omega)}+\parallel w\parallel^2_{L^2(\Omega)}\right)ds.\tag{3.51}$$

From (3.51), (3.50) becomes (since $\|\Psi\|_{\mathbb{V}}\leq\|\Psi+\psi_e\|_{\mathbb{V}}+\|\psi_e\|_{\mathbb{V}}$)

$$\begin{aligned}&(\|\Psi(.,t)\|^2_{L^2(\Omega)}+\|w(.,t)\|^2_{L^2(\Omega)})+\int_0^t(\|\Psi\|^2_{\mathbb{V}}+\|\psi_e\|^2_{\mathbb{V}})ds+\int_0^T\int_\Omega\phi^2\Psi^2d\mathbf{x}ds\\&\leq C_{12}(\int_0^T\parallel h_2\parallel^2_{L^2(\Omega)}ds+\int_0^T\parallel h_1\parallel^2_{L^2(\Omega)}ds)\\&\quad+C_{13}\int_0^t(\parallel\Psi\parallel^2_{L^2(\Omega)}+\parallel w\parallel^2_{L^2(\Omega)})(1+\parallel\phi\parallel^2_{L^4(\Omega)}+\parallel u\parallel^2_{L^3(\Omega)})ds\end{aligned}\tag{3.52}$$

and then

$$(\|\Psi(.,t)\|^2_{L^2(\Omega)} + \|w(.,t)\|^2_{L^2(\Omega)}) \leq C_{12}(\int_0^T \| h_2 \|^2_{L^2(\Omega)} ds + \int_0^T \| h_1 \|^2_{L^2(\Omega)} ds) \\ +C_{13}\int_0^t (\| \Psi \|^2_{L^2(\Omega)} + \| w \|^2_{L^2(\Omega)})(1+ \| \phi \|^2_{L^4(\Omega)} + \| u \|^2_{L^3(\Omega)})ds. \tag{3.53}$$

By first using Gronwall lemma in (3.53), we can deduce that

$$\|\Psi(.,t)\|^2_{L^2(\Omega)} + \|w(.,t)\|^2_{L^2(\Omega)} \\ \leq C(\int_0^T \| h_2 \|^2_{L^2(\Omega)} ds + \int_0^T \| h_1 \|^2_{L^2(\Omega)} ds)\exp(\| \phi \|^2_{L^2(0,T;L^4(\Omega))} + \| u \|^2_{L^2(0,T;L^3(\Omega))}),$$

and then from (3.52), we easily deduce the following estimates

$$\begin{aligned} &\|\Psi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|w\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C(\| h_2 \|^2_{L^2(Q)} + \| h_1 \|^2_{L^2(Q)}), \\ &\|\Psi\|^2_{L^2(0,T;\mathbb{V})} + \|\psi_e\|^2_{L^2(0,T;\mathbb{V})} \leq C(\| h_2 \|^2_{L^2(Q)} + \| h_1 \|^2_{L^2(Q)}), \\ &\| \phi\Psi\|^2_{L^2(Q)} \leq C(\| h_2 \|^2_{L^2(Q)} + \| h_1 \|^2_{L^2(Q)}). \end{aligned} \tag{3.54}$$

Derive now the estimate of results, for the solution (Ψ, ψ_e, w), in the following space: $(L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))) \times L^\infty(0,T;H^1(\Omega)) \times (L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^3(\Omega)))$.

Since w satisfies: $\dfrac{\partial w}{\partial t} = -\dfrac{\partial \mathcal{I}_2}{\partial \phi}(.;\phi)\Psi - \hbar w + \mathcal{E}(\Psi_\tau, w_\tau)$, then for all t we have (since $w(t=0) = 0$)

$$w(\mathbf{x},t) = -\int_0^t \frac{\partial \mathcal{I}_2}{\partial \phi}(\mathbf{x},s;\phi)\Psi ds - \int_0^t \hbar w(\mathbf{x},s)ds + \int_0^t \mathcal{E}(\Psi_\tau, w_\tau)ds. \tag{3.55}$$

Since relation (3.55) is similar to (3.40), then we can use similar argument as to derive (3.44) and we obtain (since ψ and ϕ are in $L^2(0,T,H^1) \subset L^2(0,T,L^r)$, for $r \in [1,6]$)

$$\| w(.,t) \|_{L^3(\Omega)} \leq C_1 \int_0^t \| w(.,s) \|_{L^3(\Omega)} ds \\ +C_2\left(\| \Psi \|_{L^2(0,T;L^3(\Omega))} + \| \phi \|_{L^2(0,T;H^1(\Omega))}\| \Psi \|_{L^2(0,T;H^1(\Omega))}\right). \tag{3.56}$$

According to (3.54), we can deduce that

$$\| w(.,t) \|_{L^3(\Omega)} \leq C_3 \int_0^t \| w(.,s) \|_{L^3(\Omega)} ds + C_4(\|h_1\|_{L^2(Q)} + \|h_2\|_{L^2(Q)}).$$

Consequently (by using Gronwall lemma)

$$\| w(.,t) \|_{L^3(\Omega)} \leq C\left(\|h_1\|_{L^2(Q)} + \|h_2\|_{L^2(Q)}\right). \tag{3.57}$$

Finally, from (3.10) with $(v, v_e) = (\frac{\partial\Psi}{\partial t}, \frac{\partial\psi_e}{\partial t})$ (put $\tilde{h}_i = \mathcal{B}_i h_i$)

$$\begin{aligned} &\| \mathfrak{b}_m \frac{\partial \Psi}{\partial t} \|^2_{L^2(\Omega)} + A_i(\Psi + \psi_e, \frac{\partial \Psi}{\partial t}) + \int_\Omega \frac{\partial \mathcal{I}}{\partial \phi}\Psi\frac{\partial \Psi}{\partial t}d\mathbf{x} \\ &\quad + \int_\Omega \frac{\partial \mathcal{I}}{\partial u}w\frac{\partial \Psi}{\partial t}d\mathbf{x} = \int_\Omega \mathcal{H}(.,\phi_\tau,u_\tau)\frac{\partial \phi}{\partial t}d\mathbf{x} + \int_\Omega \tilde{h}_2\frac{\partial \Psi}{\partial t}d\mathbf{x}, \\ &A_i(\Psi + \psi_e, \frac{\partial \psi_e}{\partial t}) + A_e(\psi_e, \frac{\partial \psi_e}{\partial t}) = \int_\Omega \tilde{h}_1\frac{\partial \psi_e}{\partial t}d\mathbf{x}, \\ &\frac{\partial w}{\partial t} = -\frac{\partial \mathcal{I}_2}{\partial \phi}(.;\phi)\Psi - \hbar w + \mathcal{E}(\Psi_\tau, w_\tau). \end{aligned} \tag{3.58}$$

Then (since $\frac{\partial \mathcal{I}_2}{\partial \phi}(.;\phi)$ is a polynomial of degree 1 on ϕ)

$$
\begin{aligned}
&\| \mathfrak{b}_m \frac{\partial \Psi}{\partial t} \|^2_{L^2(\Omega)} + \frac{d}{2dt} A_i(\phi+\varphi_e, \phi+\varphi_e) + \frac{d}{2dt} A_i(\varphi_e, \varphi_e) = \\
&\quad - \int_\Omega \frac{\partial \mathcal{I}}{\partial \phi} \Psi \frac{\partial \Psi}{\partial t} d\mathbf{x} - \int_\Omega \frac{\partial \mathcal{I}}{\partial u} w \frac{\partial \Psi}{\partial t} d\mathbf{x} \\
&\quad + \int_\Omega \tilde{h}_2 \frac{\partial \Psi}{\partial t} d\mathbf{x} + \int_\Omega \psi_e \frac{\partial \tilde{h}_1}{\partial t} d\mathbf{x} + \frac{d}{dt}\Big(\int_\Omega \tilde{h}_1 \psi_e d\mathbf{x}\Big) + \int_\Omega \mathcal{H}(.,\Psi_\tau, w_\tau) \frac{\partial \Psi}{\partial t} d\mathbf{x}, \\
&\| \frac{\partial w}{\partial t} \|_{L^3(\Omega)} \le C\big(\| \Psi \|_{L^\infty(0,T;H^1(\Omega))} (1 + \| \phi \|_{L^\infty(0,T;H^1(\Omega))}) + \| w \|_{L^\infty(0,T;L^3(\Omega))} \big) \\
&\qquad\qquad + \| \mathcal{E}(\Psi_\tau, w_\tau) \|_{L^3(\Omega)} .
\end{aligned} \tag{3.59}
$$

Since $\tilde{h}_1 \in U_c \subset C^0([0,T]; L^2(\Omega))$, then $\tilde{h}_1(0) \in L^2(\Omega)$ and we have $\| \tilde{h}_1(0) \|^2_{L^2(\Omega)} \le C \| \tilde{h}_1 \|^2_{U_c}$. Consequently, by integrating (3.59) by time, we can deduce (from (2.21), the boundedness of $\mathfrak{b}_m$, the coercivity and continuity of A_i and A_e, and the estimate of $\mathcal{H}$ and $\mathcal{E}$ given by Lemma 2.6)

$$
\begin{aligned}
&2\underline{c}_m \int_0^t \| \frac{\partial \Psi}{\partial t} \|^2_{L^2(\Omega)} ds + \alpha_i \| \Psi(t) \|^2_{H^1(\Omega)} + (\alpha_e + \alpha_i) \| \psi_e(t) \|^2_{H^1(\Omega)} \\
&\le C_1 \Big(\int_0^t \int_\Omega | \phi |^4 | \Psi |^2 d\mathbf{x} ds + \int_0^t \int_\Omega | \phi |^2 | \Psi |^2 d\mathbf{x} ds + \int_0^t \int_\Omega | u |^2 | \Psi |^2 d\mathbf{x} ds \Big) \\
&\quad + C_2 \Big(\int_0^t \int_\Omega | \Psi |^2 d\mathbf{x} ds + \int_0^t \int_\Omega | w |^2 d\mathbf{x} ds + \int_0^t \int_\Omega | \psi_e |^2 d\mathbf{x} ds \Big) \\
&\quad + C_3 \int_0^t \int_\Omega | \phi |^2 | w |^2 d\mathbf{x} ds + \delta_0 \int_0^t \| \frac{\partial \Psi}{\partial t} \|^2_{L^2(\Omega)} ds + \delta_1 \| \psi_e(t) \|^2_{H^1(\Omega)} \\
&\quad + C_4 \Big(\| \tilde{h}_2 \|^2_{L^2(Q)} + \| \tilde{h}_1 \|^2_{U_c} \Big), \\
&\| \frac{\partial w}{\partial t} \|_{L^2(0,T;L^3(\Omega))} \le C_5 \Big(\| \Psi \|_{L^\infty(0,T;H^1(\Omega))} (1 + \| \phi \|_{L^\infty(0,T;H^1(\Omega))}) + \| w \|_{L^\infty(0,T;L^3(\Omega))} \Big).
\end{aligned} \tag{3.60}
$$

Since ϕ is in $L^\infty(0,T,H^1) \subset L^\infty(0,T,L^r)$ ($r \in [1,6]$), then (by choosing $\delta_0 = \underline{c}_m$, $\delta_1 = (\alpha_e + \alpha_i)/2$)

$$
\begin{aligned}
&\underline{c}_m \int_0^t \| \frac{\partial \Psi}{\partial t} \|^2_{L^2(\Omega)} ds + \alpha_i \| \Psi(t) \|^2_{H^1(\Omega)} + (\alpha_e + \alpha_i)/2 \| \psi_e(t) \|^2_{H^1(\Omega)} \\
&\quad \le C_6 \Big(\| \phi \|^4_{L^\infty(0,T;H^1(\Omega))} + \| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} + \| u \|^2_{L^\infty(0,T;L^3(\Omega))} \Big) \int_0^t \| \Psi \|^2_{H^1(\Omega)} ds \\
&\quad + C_7 \Big(\| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} \| w \|^2_{L^\infty(0,T;L^3(\Omega))} + \| \Psi \|^2_{L^2(Q)} + \| \psi_e \|^2_{L^2(Q)} + \| w \|^2_{L^2(Q)} \Big) \\
&\quad + C_8 \Big(\| \tilde{h}_2 \|^2_{L^2(Q)} + \| \tilde{h}_1 \|^2_{U_c} \Big), \\
&\| \frac{\partial w}{\partial t} \|_{L^2(0,T;L^3(\Omega))} \le C_5 \Big(\| \Psi \|_{L^\infty(0,T;H^1(\Omega))} (1 + \| \phi \|_{L^\infty(0,T;H^1(\Omega))}) + \| w \|_{L^\infty(0,T;L^3(\Omega))} \Big).
\end{aligned} \tag{3.61}
$$

From the regularity of (ϕ, φ_e, u), estimation of (Ψ, ψ_e) in $L^2(Q)$ and estimation of w in $L^\infty(0,T;L^3(\Omega))$ (see the estimates (3.54)–(3.57)), we can deduce that (according to the assumption concerning the operators $\mathcal{B}_i$, $i = 1, 2$)

$$\| \frac{\partial \Psi}{\partial t} \|^2_{L^2(Q)} + \| \Psi(t) \|^2_{L^\infty(0,T;H^1(\Omega))} + \| \psi_e(t) \|^2_{L^\infty(0,T;H^1(\Omega))} + \| \frac{\partial w}{\partial t} \|_{L^2(0,T;L^3(\Omega))} \leq C \left(\| h_2 \|^2_{L^2(Q)} + \| h_1 \|^2_{U_c} \right). \tag{3.62}$$

This completes the proof. □

We are now going to study the Fréchet differentiability of $\mathcal{F}$.

Theorem 3.2. *Let assumptions (H1)-(H4), (HMC) and (RC) be fulfilled. Suppose that $(\phi, \varphi_e, u) \in \mathbb{D}$, then the following results hold (for all $(\xi, \pi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$).*

(i) Let $(\xi, \pi + h_2) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$, with $h_2 \in L^\infty(Q)$ such that $\pi + h_2 \in \mathcal{V}_{ad}$ and, $\mathcal{F}(\xi, \pi)$ and $\mathcal{F}(\xi, \pi + h_2)$ being the corresponding solutions of (3.1). Then

$$\| \mathcal{F}(\xi, \pi + h_2) - \mathcal{F}(\xi, \pi) - \mathcal{F}'_\pi(\xi, \pi) h_2 \|_{\mathbb{D}} \leq C \| h_2 \|^2_{L^2(Q)}, \tag{3.63}$$

where $\mathcal{F}'_\pi(\xi, \pi) : L^\infty(Q) \longrightarrow \mathbb{D}$ is a linear operator, and $(\Psi, \psi_e, w) = \mathcal{F}'_\pi(\xi, \pi) h$ is the solution of problem (3.7) with $h_1 = 0$ (we denote this problem by $(\mathcal{P}_{FP})$). Moreover $\forall (\xi_i, \pi_i) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ ($i = 1, 2$), we have the following estimate

$$\| \mathcal{F}'_\pi(\xi_1, \pi_1) h_2 - \mathcal{F}'_\pi(\xi_2, \pi_2) h_2 \|_{\mathbb{D}} \leq C_e \| h_2 \|_{L^2(Q)} \| X_h \|_{U_c \times L^2(Q)}, \tag{3.64}$$

where $X_h = (\xi, \pi) = (\xi_1 - \xi_2, \pi_1 - \pi_2)$.

(ii) Let $(\xi + h_1, \pi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$, with $h_1 \in L^\infty(Q) \cap U_c$ such that $\xi + h_1 \in \mathcal{U}_{ad}$ and, $\mathcal{F}(\xi, \pi)$ and $\mathcal{F}(\xi + h_1, \pi)$ being the corresponding solutions of (3.1). Then

$$\| \mathcal{F}(\xi + h_1, \pi) - \mathcal{F}(\xi, \pi) - \mathcal{F}'_\xi(\xi, \pi) h_1 \|_{\mathbb{D}} \leq C \| h_1 \|^2_{U_c}, \tag{3.65}$$

where $\mathcal{F}'_\xi(\xi, \pi) : L^\infty(Q) \cap U_c \longrightarrow \mathbb{D}$ is a linear operator, and $(\Psi, \psi_e, w) = \mathcal{F}'_\xi(\xi, \pi) h$ is the solution of the problem (3.7) with $h_2 = 0$ (we denote this problem by $(\mathcal{P}_{FX})$). Moreover $\forall (\xi_i, \pi_i) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ ($i = 1, 2$), we have the following estimate

$$\| \mathcal{F}'_\xi(\xi_1, \pi_1) h_1 - \mathcal{F}'_\xi(\xi_2, \pi_2) h_1 \|_{\mathbb{D}} \leq C_e \| h_1 \|_{U_c} \| X_h \|_{U_c \times L^2(Q)}, \tag{3.66}$$

where $X_h = (\xi, \pi) = (\xi_1 - \xi_2, \pi_1 - \pi_2)$.

Proof. According to Theorem 3.1 the problems $(\mathcal{P}_{FP})$, $(\mathcal{P}_{FX})$ have a unique solution in $\mathbb{D}$.

(i) Let $Y = (\phi, \varphi_e, u) = \mathcal{F}(\xi, \pi)$ and $Y_h = (\phi_h, \varphi_{e,h}, u_h) = \mathcal{F}(\xi, \pi + h_2)$. From the stability estimate in Theorem 2.4, we know that

$$\| Y - Y_h \|^2_{\mathbb{D}} \leq C \| h_2 \|^2_{L^2(Q)} . \tag{3.67}$$

Denote by $\Phi_h = \phi_h - \phi$, $\Pi_{e,h} = \varphi_{e,h} - \varphi_e$, $U_h = u_h - u$, $\phi^* = \Phi_h - \Psi$, $\varphi^*_e = \Pi_{e,h} - \psi_e$ and $u^* = U_h - w$. It is easy to see that $(\phi^*, \varphi_e^*, u^*)$ satisfies the linear problem

$$\begin{aligned}
&\mathfrak{c}_m \frac{\partial \phi^*}{\partial t} - div(\mathcal{K}_i \nabla(\phi^* + \varphi_e^*)) + \frac{\partial \mathcal{I}}{\partial u} u^* + \frac{\partial \mathcal{I}}{\partial \phi} \phi^* = S_1 + \mathcal{H}(\phi_\tau^*, u_\tau^*), \text{ in } Q \\
&-div(\mathcal{K}_i \nabla \phi^*) - div((\mathcal{K}_i + \mathcal{K}_e)\varphi_e^*) = 0, \text{ in } Q \\
&\frac{\partial u^*}{\partial t} + \hbar u^* + \frac{\partial \mathcal{G}}{\partial \phi} \phi^* = S_2 + \mathcal{E}(\phi_\tau^*, u_\tau^*), \text{ in } Q \\
&(\mathcal{K}_i \nabla \phi^*).\mathbf{n} + (\mathcal{K}_i \nabla \varphi_e^*).\mathbf{n} = 0, \text{ on } \Sigma \\
&(\mathcal{K}_e \nabla \varphi_e^*).\mathbf{n} = 0, \text{ on } \Sigma \\
&\phi^*(., t = 0) = 0, \quad u^*(., t = 0) = 0, \text{ in } \Omega \\
&\phi^* = 0, \quad u^* = 0, \text{ in } Q_0
\end{aligned} \tag{3.68}$$

where the condition (1.3) for φ_e^* holds and

$$S_1 = -(g_0 + u g_1) - U_h(\mathcal{I}_1(\phi_h) - \mathcal{I}_1(\phi)), \quad S_2 = -g_2, \tag{3.69}$$

with $g_i = \left(\mathcal{I}_i(\phi_h) - \mathcal{I}_i(\phi) - \mathcal{I}_i'(\phi).\Phi_h\right)$ and $\mathcal{I}_i'(\phi) = \dfrac{\partial \mathcal{I}_i}{\partial \phi}$, for $i = 0, 2$. Now we have to derive some estimates necessary to prove the result of theorem. By using a simple manipulation we obtain that

$$g_i = \int_0^1 (\mathcal{I}_i'(\phi) - \mathcal{I}_i'(\phi + s\Phi_h))\Phi_h ds \text{ (for } i = 0, 1, 2).$$

Since $\mathcal{I}_0'(\phi) = \hbar_{11}\phi^2 - \hbar_{12}\phi + \hbar_{13}$, $\mathcal{I}_1'(\phi) = \epsilon_1 \hbar_{21}$, $\mathcal{I}_2'(\phi) = -\epsilon_2 \hbar_{31}\phi + (\epsilon_2 - 1)\hbar_{32}$ and $\mathcal{I}_1(\phi_h) - \mathcal{I}_1(\phi) = \epsilon_1 \hbar_{21} \Phi_h$ then

$$\begin{aligned}
g_0 &= \int_0^1 -s^2 \Phi_h^2 (\hbar_{11}(2\phi + s\Phi_h) - \hbar_{12}) ds = \frac{-2\hbar_{11}}{3} \Phi_h^2 \phi - \frac{\hbar_{11}}{4} \Phi_h^3 + \frac{\hbar_{12}}{3} \Phi_h^2, \\
g_1 &= 0, \\
g_2 &= \int_0^1 s \Phi_h^2 \epsilon_2 \hbar_{32} ds = \frac{\epsilon_2 \hbar_{32}}{2} \Phi_h^2.
\end{aligned} \tag{3.70}$$

According to the regularity of $\hbar_{ij}$ we can deduce that

$$\begin{aligned}
&| S_1 |^2 \leq C \left(| \Phi_h |^6 + | \phi |^2 | \Phi_h |^4 + | \Phi_h |^4 + | U_h |^2 | \Phi_h |^2 \right), \\
&| S_2 |^3 \leq C \, | \Phi_h |^6
\end{aligned} \tag{3.71}$$

Integrating by space and using Young's formula, we can deduce

$$\begin{aligned}
&\int_\Omega | S_1 |^2 \, d\mathbf{x} \leq C \left(\| \Phi_h \|_{L^6(\Omega)}^6 + \| \Phi_h \|_{L^6(\Omega)}^4 \| \phi \|_{L^6(\Omega)}^2 + \| \Phi_h \|_{L^4(\Omega)}^4 + \| \Phi_h \|_{L^6(\Omega)}^2 \| U_h \|_{L^3(\Omega)}^2 \right), \\
&\int_\Omega | S_2 |^3 \, d\mathbf{x} \leq C \, \| \Phi_h \|_{L^6(\Omega)}^6 .
\end{aligned}$$

From Gagliardo-Nirenberg inequality, we have, for all $v \in H^1(\Omega)$, $\| v \|_{L^r(\Omega)} \leq C \| v \|_{H^1(\Omega)}$, for $r \in [2,6]$, we have (since $\phi, U_h, \Phi_h \in L^\infty(0,T;H^1(\Omega))$)

$$\begin{aligned}
&\int_\Omega | S_1 |^2 \, d\mathbf{x} \leq C \left(\| \Phi_h \|^4_{L^\infty(0,T;H^1(\Omega))} + \| U_h \|^4_{L^\infty(0,T;L^3(\Omega))} \right), \\
&\int_\Omega | S_2 |^3 \, d\mathbf{x} \leq C \| \Phi_h \|^6_{L^\infty(0,T;H^1(\Omega))} .
\end{aligned} \tag{3.72}$$

We can deduce that (according to the estimate (3.67))

$$\begin{aligned}
&\| S_1 \|_{L^\infty(0,T;L^2(\Omega))} \leq C \| h_2 \|^2_{L^2(Q)}, \\
&\| S_2 \|_{L^\infty(0,T;L^3(\Omega))} \leq C \| h_2 \|^2_{L^2(Q)} .
\end{aligned} \tag{3.73}$$

We can conclude that source term (S_1, S_2) in (3.68) is in $L^2(0,T;L^2(\Omega)) \times L^2(0,T;L^3(\Omega))$ and satisfies estimates (3.73). Since (3.68) is similar as system (3.7) with source term (S_1, S_2) then from Theorem 3.1 we can deduce that

$$\| (\phi^*, \varphi_e^*, u^*) \|^2_{\mathbb{D}} \leq C \left(\| S_1 \|^2_{L^2(Q)} + \| S_2 \|^2_{L^2(0,T;L^3(\Omega))} \right)$$

and then according to estimates (3.73), we can deduce estimate (3.63).

Prove now the second part of (i). Let $(\xi_i, \pi_i) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$, i=1,2 be given and $\mathbf{w}_i = (\Psi_i, \psi_{e,i}, w_i) = \mathcal{F}'_\pi(\xi_i, \pi_i).h_2$ solution of $(\mathcal{P}_{FP})$ (we denote by $Y_i = (\phi_i, \varphi_{e,i}, u_i) = \mathcal{F}(\xi_i, \pi_i)$ and by $(\phi, \varphi_e, u) = Y_1 - Y_2$). Set $\mathbf{w} = (\Psi, \psi_e, w) = \mathbf{w}_1 - \mathbf{w}_2$, $(\xi, \pi) = (\xi_1 - \xi_2, \pi_1 - \pi_2)$. According to equations satisfied by $\mathbf{w}_1$ and $\mathbf{w}_2$ we have

$$\begin{aligned}
&\mathfrak{c}_m \frac{\partial \Psi}{\partial t} - div(\mathcal{K}_i \nabla(\Psi + \psi_e)) + \frac{\partial \mathcal{I}}{\partial \phi}(\phi_1, u_1)\Psi + \frac{\partial \mathcal{I}}{\partial u}(\phi_1, u_1) w = S_1 + \mathcal{H}(\Psi_\tau, w_\tau), \\
&-div(K_i \nabla \Psi) - div((\mathcal{K}_i + \mathcal{K}_e)\psi_e) = 0, \\
&\frac{\partial w}{\partial t} + \frac{\partial \mathcal{G}}{\partial \phi}(\phi_1, u_1)\Psi + \hbar w = S_2 + \mathcal{E}(\Psi_\tau, w_\tau), \\
&(\mathcal{K}_i \nabla \Psi).\mathbf{n} + (\mathcal{K}_i \nabla \psi_e).\mathbf{n} = 0, \ \text{ on } \Sigma \\
&(\mathcal{K}_e \nabla \psi_e).\mathbf{n} = 0, \ \text{ on } \Sigma \\
&\Psi(., t = 0) = 0, \quad w(., t = 0) = 0, \ \text{ in } \Omega \\
&\Psi = 0, \quad w = 0, \ \text{ in } Q_0
\end{aligned} \tag{3.74}$$

where the condition (1.3) for ψ_e holds and

$$\begin{aligned}
&S_1 = (\frac{\partial \mathcal{I}}{\partial \phi}(\phi_1, u_1) - \frac{\partial \mathcal{I}}{\partial \phi}(\phi_2, u_2))\Psi_2 + (\frac{\partial \mathcal{I}}{\partial u}(\phi_1, u_1) - \frac{\partial \mathcal{I}}{\partial u}(\phi_2, u_2))w_2, \\
&S_2 = (\frac{\partial \mathcal{G}}{\partial \phi}(\phi_1, u_1) - \frac{\partial \mathcal{G}}{\partial \phi}(\phi_2, u_2))\Psi_2.
\end{aligned} \tag{3.75}$$

From the expression (2.21) of partial derivatives of $\mathcal{I}$ and $\mathcal{G}$ and Hölder's inequality, we can have (according to regularity of $\hbar_{ij}$)

$$\begin{aligned}
&| S_1 | \leq C(| \phi | (1 + | \phi_1 | + | \phi_2 |) + | u |)) | \Psi_2 | + | \phi || w_2 |, \\
&| S_2 | \leq C | \phi || \Psi_2 | .
\end{aligned} \tag{3.76}$$

So

$$\int_\Omega |S_1|^2\, d\mathbf{x} \le C_1(1+ \| \phi_1 \|^2_{L^6(\Omega)} + \| \phi_2 \|^2_{L^6(\Omega)}) \| \phi \|^2_{L^6(\Omega)} \| \Psi_2 \|^2_{L^6(\Omega)}$$
$$+C_2 \| u \|^2_{L^3(\Omega)} \| \Psi_2 \|^2_{L^6(\Omega)} + C_3 \| \phi \|^2_{L^6(\Omega)} \| w_2 \|^2_{L^3(\Omega)}, \qquad (3.77)$$
$$\int_\Omega |S_2|^3\, d\mathbf{x} \le C_4 \| \phi \|^3_{L^6(\Omega)} \| \Psi_2 \|^3_{L^6(\Omega)} .$$

According to the regularity of ϕ_i $(i = 1, 2)$ in $L^\infty(0, T; L^6(\Omega)) \subset L^\infty(0, T; H^1(\Omega))$, we can deduce that

$$\int_\Omega |S_1|^2\, d\mathbf{x} \le C_5(\| \phi \|^2_{L^6(\Omega)} + \| u \|^2_{L^3(\Omega)}) \| \Psi_2 \|^2_{L^6(\Omega)} + C_6 \| \phi \|^2_{L^6(\Omega)} \| w_2 \|^2_{L^3(\Omega)},$$
$$\int_\Omega |S_2|^3\, d\mathbf{x} \le C_4 \| \phi \|^3_{L^6(\Omega)} \| \Psi_2 \|^3_{L^6(\Omega)}$$

and then (from Gagliardo-Nirenberg inequalities)

$$\int_\Omega |S_1|^2\, d\mathbf{x} \le C_5(\| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} + \| u \|^2_{L^\infty(0,T;L^3(\Omega))}) \| \Psi_2 \|^2_{H^1(\Omega)}$$
$$+C_6 \| \phi \|^2_{L^\infty(0,T;H^1(\Omega))} \| w_2 \|^2_{L^3(\Omega)}, \qquad (3.78)$$
$$\int_\Omega |S_2|^3\, d\mathbf{x} \le C_4 \| \phi \|^3_{L^\infty(0,T;H^1(\Omega))} \| \Psi_2 \|^3_{H^1(\Omega)} .$$

According to Theorems 2.4 and 3.1, we can deduce that

$$\| S_1 \|_{L^2(0,T;L^2(\Omega))} \le C \| h_2 \|_{L^2(Q)} \| (\xi, \pi) \|_{U_c \times L^2(Q)},$$
$$\| S_2 \|_{L^2(0,T;L^3(\Omega))} \le C \| h_2 \|_{L^2(Q)} \| (\xi, \pi) \|_{U_c \times L^2(Q)} . \qquad (3.79)$$

We can conclude that source term (S_1, S_2), in (3.74), is in $L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^3(\Omega))$ and satisfies estimates (3.79). Since (3.74) is similar as system (3.7) with source term (S_1, S_2) then from Theorem 3.1 we can deduce that the result (3.64) of the theorem holds.

(ii) By using the same technique as in the proof of results of (i), we have results of (ii). Therefore, we omit the details. □

3.3. Existence and necessary optimality condition of an optimal solution

Theorem 3.3. *Assume that assumptions of Theorem 3.2 are satisfied. Then, for α and β sufficiently large (i.e. there exist (α_l, β_l) such that $\alpha \ge \alpha_l$ and $\beta \ge \beta_l$) there exist $(\xi^*, \pi^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ and $(\phi^*, \varphi_e^*, u^*) \in \mathbb{D}$ such that (ξ^*, π^*) is a saddle point of $\mathcal{J}$ and $(\phi^*, \varphi_e^*, u^*) = \mathcal{F}(\xi^*, \pi^*)$ is the solution of (3.1).*

Proof. Let P_π be the map: $\xi \longrightarrow \mathcal{J}(\xi, \pi)$ and Q_ξ be the map: $\pi \longrightarrow \mathcal{J}(\xi, \pi)$. To obtain the existence of minimax control problem we prove that P_π is convex and lower semicontinuous for all $\pi \in \mathcal{V}_{ad}$, Q_ξ is concave and upper semicontinuous for all $\xi \in \mathcal{U}_{ad}$, and we use the classical minimax theorem in infinite dimensions (see, e.g. [10, 35]).

First we prove, for α and β sufficiently large, the convexity of the map P_π and the concavity of the map Q_ξ. In order to prove the convexity, it is sufficient to show that for all $(\xi_1, \xi_2) \in \mathcal{U}_{ad}$ we have:

$$(P'_\pi(\xi_1) - P'_\pi(\xi_2)).\xi \ge 0,$$

where $\xi = \xi_1 - \xi_2$ (because P_π is Fréchet differentiable).
According to definition of $\mathcal{J}$, we have that

$$
\begin{aligned}
&(P'_\pi(\xi_1) - P'_\pi(\xi_2)).\xi = \alpha \parallel \xi \parallel^2_{U_c} \\
&+m_1 \int_0^T \int_{\Omega_{obs}} \Upsilon(t)(\phi_1 - \phi_2)\Psi_2 d\mathbf{x}dt + m_2 \int_{\Omega_{obs}} (\phi_1(T) - \phi_2(T))\Psi_2(T)d\mathbf{x} \\
&+m_1 \int_0^T \int_{\Omega_{obs}} \Upsilon(t)(\phi_1 - \phi_{obs})(\Psi_1 - \Psi_2)d\mathbf{x}dt + m_2 \int_{\Omega_{obs}} (\phi_1(T) - \psi_{obs})(\Psi_1(T) - \Psi_2(T))d\mathbf{x},
\end{aligned}
\tag{3.80}
$$

where $(\phi_i, \varphi_{e,i}, u_i) = \mathcal{F}(\xi_i, \pi)$ and function $(\Psi_i, \psi_{e,i}, w_i) = \mathcal{F}'(\xi_i, \pi).(\xi, 0)$ is the solution of problem (3.7), for $i = 1, 2$. According to Theorems 2.4 and 3.1 we can deduce that

$$
\begin{aligned}
&\mid m_1 \int_0^T \int_{\Omega_{obs}} \Upsilon(t)(\phi_1 - \phi_2)\Psi_2 d\mathbf{x}dt + m_2 \int_{\Omega_{obs}} (\phi_1(T) - \phi_2(T))\Psi_2(T)d\mathbf{x} \mid \\
&\quad \le C_1\Big(\parallel \phi_1 - \phi_2 \parallel_{L^2(0,T;\Omega_{obs})} \parallel \psi_2 \parallel_{L^2(0,T;\Omega_{obs})} \\
&\qquad + \parallel \phi_1(T) - \phi_2(T) \parallel_{L^2(\Omega_{obs})} \parallel \psi_2(T) \parallel_{L^2(\Omega_{obs})} \Big) \\
&\quad \le M_0 \parallel \xi \parallel^2_{U_c}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mid m_1 \int_0^T \int_{\Omega_{obs}} \Upsilon(t)(\phi_1 - \phi_{obs})(\Psi_1 - \Psi_2)d\mathbf{x}dt + m_2 \int_{\Omega_{obs}} (\phi_1(T) - \psi_{obs})(\Psi_1(T) - \Psi_2(T))d\mathbf{x} \mid \\
&\quad \le C_2\Big(\parallel \phi_1 - \phi_{obs} \parallel_{L^2(0,T;\Omega_{obs})} \parallel \psi_1 - \psi_2 \parallel_{L^2(0,T;\Omega_{obs})} \\
&\qquad + \parallel \phi_1(T) - \psi_{obs} \parallel_{L^2(\Omega_{obs})} \parallel \psi_1(T) - \psi_2(T) \parallel_{L^2(\Omega_{obs})} \Big) \\
&\quad \le M_1 \parallel \xi \parallel^2_{U_c} .
\end{aligned}
$$

From (3.80) and the previous relations we deduce that for $\alpha \ge \alpha_l$ such that $\alpha_l \ge M_0 + M_1$ we have $(P'_\pi(\xi_1) - P'_\pi(\xi_2)).\xi \ge 0$ and then the convexity of P_π. In the same way, we can find β_l such that for $\beta \ge \beta_l$ we have the concavity of Q_ξ.

We prove now that P_π is lower semicontinuous for all $\pi \in \mathcal{V}_{ad}$, and Q_ξ is upper semicontinuous for all $\xi \in \mathcal{U}_{ad}$.

Let $\xi^{(k)}$ be a minimizing sequence of $\mathcal{J}$ i.e. $\liminf_k \mathcal{J}(\xi^{(k)}, \pi) = \min_{\xi \in \mathcal{U}_{ad}} \mathcal{J}(\xi, \pi)$ $(\forall \pi \in \mathcal{V}_{ad})$. Then $\xi^{(k)}$ is uniformly bounded in $\mathcal{U}_{ad}$. Set $(\phi_\pi, \varphi_{e,\pi}, u_\pi) = \mathcal{F}(\xi, \pi)$, and $(\phi^{(k)}, \varphi_e^{(k)}, u^{(k)}) = \mathcal{F}(\xi^{(k)}, \pi)$. In view of Theorem 2.2 and the nature of the operator $\mathcal{B}_1$, we can deduce that the sequence $(\phi^{(k)}, \varphi_e^{(k)}, u^{(k)})$ is uniformly bounded in $\mathcal{W}(Q)$ with respect to k. Therefore, we can extract from $(\xi^{(k)}, \phi^{(k)}, \varphi_e^{(k)}, u^{(k)})$ a subsequence also denoted by $(\xi^{(k)}, \phi^{(k)}, \varphi_e^{(k)}, u^{(k)})$ and such that

$$
\begin{aligned}
&\mathcal{B}_1\xi^{(k)} \rightharpoonup \mathcal{B}_1\xi_\pi \textit{ weakly in } U_c, \\
&(\phi^{(k)}, \varphi_e^{(k)}, u^{(k)}) \rightharpoonup (\underline{\phi}_\pi, \underline{\varphi}_{e,\pi}, \underline{u}_\pi) \textit{ weakly in } \mathcal{W}(Q), \\
&(\phi^{(k)}, \varphi_e^{(k)}, u^{(k)}) \longrightarrow (\underline{\phi}_\pi, \underline{\varphi}_{e,\pi}, \underline{u}_\pi) \textit{ strongly in } L^2(Q).
\end{aligned}
$$

Passing to limit in the corresponding system satisfied by $(\xi^{(k)}, \phi^{(k)}, \varphi_e^{(k)}, u^{(k)})$, we can conclude that $(\underline{\phi}_\pi, \underline{\varphi}_{e,\pi}, \underline{u}_\pi) = \mathcal{F}(\xi, \pi)$ and according to uniqueness of solution of (3.1), we have then $(\underline{\phi}_\pi, \underline{\varphi}_{e,\pi}, \underline{u}_\pi) = (\phi_\pi, \varphi_{e,\pi}, u_\pi)$. Therefore, using the sequential weak lower semicontinuous of $\mathcal{J}$ with respect to convergence and uniform boundedness of sequence $\xi^{(k)}$ (according to structure of $\mathcal{J}$), we conclude that the map $P_\pi : \xi \longrightarrow \mathcal{J}(\xi, \pi)$ is lower semicontinuous for all $\pi \in \mathcal{V}_{ad}$. By using the same technique we obtain then Q_ξ is upper semicontinuous for all $\xi \in \mathcal{K}_1$. □

We now turn to necessary optimality conditions which have be satisfied by each solution of the control problem. In order to simplify the presentation we assume that the functions $(p_l)_{1\leq l\leq n_2}$ and $(r_k)_{1\leq k\leq n_1}$ at final time T satisfy

$$\begin{aligned} &p_1(T) \geq ... \geq p_{n_2}(T) \text{ and } r_1(T) \geq ... \geq r_{n_1}(T), \\ &\text{with the convention } r_{n_1+1}(T) = 0 \text{ and } p_{n_2+1}(T) = 0. \end{aligned} \tag{3.81}$$

Since the cost $\mathcal{J}$ is a composition of F-differentiable maps then $\mathcal{J}$ is F-differentiable and we have

$$\begin{aligned} \mathcal{J}'(\xi, \pi).\mathbf{h} = m_1 \int_0^T \int_{\Omega_{obs}} \Upsilon(t)(\phi - \phi_{obs})\psi d\mathbf{x}dt + m_2 \int_{\Omega_{obs}} (\phi(T) - \psi_{obs})\psi(T) d\mathbf{x} \\ + \alpha(\xi, h_1)_{U_c} - \beta \int_0^T \int_{\Omega_d} \pi h_2 d\mathbf{x}dt \; (\forall \mathbf{h} = (h_1, h_2) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}) \,, \end{aligned} \tag{3.82}$$

where $(\Psi, \psi_e, w) = \mathcal{F}'(\xi, \pi).\mathbf{h}$ is solution of problem (3.7).

Theorem 3.4. *Assume that assumptions of Theorem 3.3 are satisfied and α and β are sufficiently large. Let $(\xi^*, \pi^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ be an optimal solution of (3.6) and $(\phi^*, \varphi_e^*, u^*) = \mathcal{F}(\xi^*, \pi^*) \in \mathbb{D}$ be its corresponding solution. Then $(\forall(\xi, \pi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad})$*

$$\begin{aligned} &\frac{\partial \mathcal{J}}{\partial \xi}(\xi^*, \pi^*).(\xi - \xi^*) = \left(\alpha \xi^* + \chi_{\Omega_c} \aleph_1^* \tilde{\varphi}_e^*, \xi - \xi^*\right)_{U_c} \geq 0, \\ &\frac{\partial \mathcal{J}}{\partial \phi}(\xi^*, \pi^*).(\pi - \pi^*) = \int_0^T \int_{\Omega_d} (-\beta \pi^* + \mathcal{B}_2^* \tilde{\phi}^*)(\pi - \pi^*) d\mathbf{x}dt \leq 0, \end{aligned} \tag{3.83}$$

where $(\tilde{\phi}^*, \tilde{\varphi}_e^*, \tilde{u}^*) := \mathcal{F}^\perp(\xi^*, \pi^*)$ is the solution of the so-called adjoint problem (3.86) where the condition (1.3) for $\tilde{\varphi}_e$ holds (given below), corresponding to $(\phi^*, \varphi_e^*, u^*)$, and with $\aleph_1^* = \Lambda^{-1}\mathcal{B}_1^*$ where Λ is the canonical isomorphism : $U_c \longrightarrow U_c'$ such that

$$\langle g, v \rangle_{U_c, U_c'} = (\Lambda g, v)_{U_c'} = (g, \Lambda^{-1} v)_{U_c}, \quad \forall (g, v) \in U_c \times U_c'. \tag{3.84}$$

Moreover the gradients of $\mathcal{J}$ at any point (ξ, π), in the weak sense, are given by

$$\frac{\partial \mathcal{J}}{\partial \xi}(\xi, \pi) = \alpha \xi + \chi_{\Omega_c} \aleph_1^* \tilde{\varphi}_e, \quad \frac{\partial \mathcal{J}}{\partial \phi}(\xi, \pi) = -\beta \pi + \chi_{\Omega_d} \mathcal{B}_2^* \tilde{\phi}, \tag{3.85}$$

where $(\tilde{\phi},\tilde{\varphi}_e,\tilde{u}) = \mathcal{F}^{\perp}(\xi,\pi)$ *is the solution of adjoint problem (corresponding to* $(\phi,\varphi_e,u) = \mathcal{F}(\xi,\pi)$*)*

$$
\begin{aligned}
&-\mathfrak{c}_m\frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial\mathcal{I}}{\partial\phi}(\phi,u)\tilde{\phi} + \frac{\partial\mathcal{G}}{\partial\phi}(\phi,u)\tilde{u}\\
&\qquad\qquad -div(\mathcal{K}_i\nabla(\tilde{\phi}+\tilde{\varphi}_e)) = m_1(\phi-\phi_{obs})\Upsilon(t)\chi_{\Omega_{obs}}, \;\; in\;\; \Omega\times(r_1(T),T)\\
&-\mathfrak{c}_m\frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial\mathcal{I}}{\partial\phi}(\phi,u)\tilde{\phi} + \frac{\partial\mathcal{G}}{\partial\phi}(\phi,u)\tilde{u} - div(\mathcal{K}_i\nabla(\tilde{\phi}+\tilde{\varphi}_e))\\
&\qquad\qquad + \sum_{i=1}^{k}\Big(a_i(.,e_i(t))\tilde{\phi}(.,e_i(t)) + c_i(.,e_i(t))\tilde{u}(.,e_i(t))\Big)e_i'(t)\\
&\qquad\qquad\quad = m_1(\phi-\phi^{obs})\Upsilon(t)\chi_{\Omega_{obs}}, \;\; in\;\; \Omega\times(r_{k+1}(T),r_k(T)),\, k=n_1,\ldots,1\\
&-div(\mathcal{K}_i\nabla\tilde{\phi}) - div((\mathcal{K}_i+\mathcal{K}_e)\nabla\tilde{\varphi}_e) = 0, \;\; in\;\; Q\\
&-\frac{\partial\tilde{u}}{\partial t} + \frac{\partial\mathcal{G}}{\partial u}(\phi,u)\tilde{u} + \frac{\partial\mathcal{I}}{\partial u}(\phi,u)\tilde{\phi} = 0, \;\; in\;\; \Omega\times(p_1(T),T)\\
&-\frac{\partial\tilde{u}}{\partial t} + \frac{\partial\mathcal{G}}{\partial u}(\phi,u)\tilde{u} + \frac{\partial\mathcal{I}}{\partial u}(\phi,u)\tilde{\phi}\\
&\qquad\qquad + \sum_{i=1}^{l}\Big(b_i(.,q_i(t))\tilde{\phi}(.,q_i(t)) + d_i(.,q_i(t))\tilde{u}(.,q_i(t))\Big)q_i'(t)\\
&\qquad\qquad\quad = 0, \;\; in\;\; \Omega\times(p_{l+1}(T),p_l(T)),\, l=n_2,\ldots,1,\\
&\textit{subject to boundary and final conditions}\\
&(\mathcal{K}_i\nabla(\tilde{\phi}+\tilde{\varphi}_e)).\mathbf{n} = 0, \;\; (\mathcal{K}_e\nabla\tilde{\varphi}_e).\mathbf{n} = 0, \;\; on\;\; \Sigma,\\
&\tilde{\phi}(.,T) = \frac{m_2}{\mathfrak{c}_m}(\phi(T)-\psi_{obs})\chi_{\Omega_{obs}}, \;\; \tilde{u}(.,T) = 0, \;\; in\;\; \Omega.
\end{aligned}
\tag{3.86}
$$

Proof. Let $\widetilde{\mathbf{X}} = (\tilde{\phi},\tilde{\varphi}_e,\tilde{u})$ be sufficiently regular such that $(\tilde{\phi},\tilde{u})(T) = (\frac{m_2}{\mathfrak{c}_m}(\phi(T)-\psi_{obs})\chi_{\Omega_{obs}},0)$.

Now multiplying the system (3.7) by $\widetilde{\mathbf{X}}$ and integrating over Q, we obtain

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega\left(\mathfrak{c}_m\frac{\partial\Psi}{\partial t} + \frac{\partial\mathcal{I}}{\partial\phi}(\phi,u)\Psi + \frac{\partial\mathcal{I}}{\partial u}(\phi,u)w\right)\tilde{\phi}d\mathbf{x}dt - \int_0^T\!\!\int_\Omega div(\mathcal{K}_i\nabla(\Psi+\psi_e))\tilde{\phi}d\mathbf{x}dt\\
&= \int_0^T\!\!\int_\Omega\left(\sum_{k=1}^{n_1}a_k(\mathbf{x},t)\Psi(\mathbf{x},r_k(t)) + \sum_{l=1}^{n_2}b_l(\mathbf{x},t)w(\mathbf{x},p_l(t))\right)\tilde{\phi}d\mathbf{x}dt + \int_0^T\!\!\int_\Omega\tilde{\phi}\mathcal{B}_2h_2d\mathbf{x}dt,\\
&-\int_0^T\!\!\int_\Omega div(\mathcal{K}_i\nabla\Psi + (\mathcal{K}_i+\mathcal{K}_e)\nabla\psi_e)\tilde{\varphi}_ed\mathbf{x}dt = \int_0^T\!\!\int_\Omega\tilde{\varphi}_e\mathcal{B}_1h_1d\mathbf{x}dt,\\
&\int_0^T\!\!\int_\Omega\left(\frac{\partial w}{\partial t} + \frac{\partial\mathcal{G}}{\partial\phi}(\phi,u)\Psi + \frac{\partial\mathcal{G}}{\partial u}(\phi,u)w\right)\tilde{u}d\mathbf{x}dt\\
&= \int_0^T\!\!\int_\Omega\left(\sum_{k=1}^{n_1}c_k(\mathbf{x},t)\Psi(\mathbf{x},r_k(t)) + \sum_{l=1}^{n_2}d_l(\mathbf{x},t)w(\mathbf{x},p_l(t))\right)\tilde{u}d\mathbf{x}dt.
\end{aligned}
\tag{3.87}
$$

Using Green's theorem and integrating by part in time, the above system takes the following form (since $(\Psi, w)(0) = 0$) and according to boundary conditions satisfied by (Ψ, ψ_e))

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(-\mathfrak{c}_m \frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial \mathcal{I}}{\partial\phi}(\phi,u)\tilde{\phi} - div(\mathcal{K}_i\nabla\tilde{\phi})\Big)\Psi d\mathbf{x}dt + \int_{\Omega_{obs}} m_2(\phi(T)-\psi_{obs})\Psi(T)d\mathbf{x}\\
&\qquad - \int_0^T\int_\Omega div(\mathcal{K}_i\nabla\tilde{\phi})\psi_e d\mathbf{x}dt - \int_0^T\int_\Omega \Big(\sum_{k=1}^{n_1} a_k(\mathbf{x},t)\tilde{\phi}(\mathbf{x},t)\Psi(\mathbf{x},r_k(t))\Big)d\mathbf{x}dt\\
&\qquad + \int_0^T\int_\Omega (\frac{\partial \mathcal{I}}{\partial u}(\phi,u)\tilde{\phi})w d\mathbf{x}dt - \int_0^T\int_\Omega \Big(\sum_{l=1}^{n_2} b_l(\mathbf{x},t)\tilde{\phi}(\mathbf{x},t)w(\mathbf{x},p_l(t))\Big)d\mathbf{x}dt\\
&\quad = \int_0^T\int_\Omega \tilde{\phi}\mathcal{B}_2 h_2 d\mathbf{x}dt - \int_0^T\int_\Gamma ((\mathcal{K}_i\nabla\tilde{\phi}).\mathbf{n})(\Psi+\psi_e)d\gamma dt,\\
&-\int_0^T\int_\Omega div(\mathcal{K}_i\nabla\tilde{\varphi}_e)\Psi d\mathbf{x}dt - \int_0^T\int_\Omega div((\mathcal{K}_i+\mathcal{K}_e)\nabla\tilde{\varphi}_e)\psi_e d\mathbf{x}dt\\
&\quad = -\int_0^T\int_\Gamma \Big\{\Big(((\mathcal{K}_i+\mathcal{K}_e)\nabla\tilde{\varphi}_e).\mathbf{n}\Big)\psi_e + \Big((\mathcal{K}_i\nabla\tilde{\varphi}_e).\mathbf{n}\Big)\Psi\Big\}d\gamma dt + \int_0^T\int_\Omega \tilde{\varphi}_e\mathcal{B}_1 h_1 d\mathbf{x}dt,\\
&\int_0^T\int_\Omega \Big(-\frac{\partial\tilde{u}}{\partial t} + \frac{\partial \mathcal{G}}{\partial u}(\phi,u)\tilde{u}\Big)w d\mathbf{x}dt - \int_0^T\int_\Omega \Big(\sum_{l=1}^{n_2} d_l(\mathbf{x},t)\tilde{u}(\mathbf{x},t)w(\mathbf{x},p_l(t))\Big)d\mathbf{x}dt\\
&\qquad + \int_0^T\int_\Omega \frac{\partial \mathcal{G}}{\partial\phi}(\phi,u)\tilde{u}\Psi d\mathbf{x}dt - \int_0^T\int_\Omega \Big(\sum_{k=1}^{n_1} c_k(\mathbf{x},t)\tilde{u}(\mathbf{x},t)\Psi(\mathbf{x},r_k(t))\Big)d\mathbf{x}dt = 0.
\end{aligned} \tag{3.88}$$

By summing the three relations of the system (3.88), one obtains

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(-\mathfrak{c}_m\frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial \mathcal{I}}{\partial\phi}(\phi,u)\tilde{\phi} + \frac{\partial \mathcal{G}}{\partial\phi}(\phi,u)\tilde{u} - div(\mathcal{K}_i\nabla(\tilde{\phi}+\tilde{\varphi}_e))\Big)\Psi d\mathbf{x}dt\\
&\qquad - \int_0^T\int_\Omega \Big(\sum_{k=1}^{n_1}(a_k(\mathbf{x},t)\tilde{\phi}(\mathbf{x},t) + c_k(\mathbf{x},t)\tilde{u}(\mathbf{x},t))\Psi(\mathbf{x},r_k(t))\Big)d\mathbf{x}dt\\
&- \int_0^T\int_\Omega \Big(div((\mathcal{K}_i+\mathcal{K}_e)\nabla\tilde{\varphi}_e) + div(\mathcal{K}_i\nabla\tilde{\phi})\Big)\psi_e d\mathbf{x}dt\\
&+ \int_0^T\int_\Omega \Big(-\frac{\partial\tilde{u}}{\partial t} + \frac{\partial \mathcal{G}}{\partial u}(\phi,u)\tilde{u} + \frac{\partial \mathcal{I}}{\partial u}(\phi,u)\tilde{\phi}\Big)w d\mathbf{x}dt + \int_{\Omega_{obs}} m_2(\phi(T)-\psi_{obs})\Psi(T)d\mathbf{x}\\
&- \int_0^T\int_\Omega \Big(\sum_{l=1}^{n_2}(b_l(\mathbf{x},t)\tilde{\phi}(\mathbf{x},t) + d_l(\mathbf{x},t)\tilde{u}(\mathbf{x},t))w(\mathbf{x},p_l(t))\Big)d\mathbf{x}dt\\
&= \int_0^T\int_\Omega \tilde{\varphi}_e\mathcal{B}_1 h_1 d\mathbf{x}dt + \int_0^T\int_\Omega \tilde{\phi}\mathcal{B}_2 h_2 d\mathbf{x}dt - \int_0^T\int_\Gamma ((\mathcal{K}_i\nabla\tilde{\phi}).\mathbf{n})(\Psi+\psi_e)d\gamma dt\\
&- \int_0^T\int_\Gamma \Big\{\Big(((\mathcal{K}_i+\mathcal{K}_e)\nabla\tilde{\varphi}_e).\mathbf{n}\Big)\psi_e + \Big((\mathcal{K}_i\nabla\tilde{\varphi}_e).\mathbf{n}\Big)\Psi\Big\}d\gamma dt.
\end{aligned} \tag{3.89}$$

Now we calculate the terms corresponding to delays operators. For this let $s = r_k(t)$ (respectively $s = p_l(t)$), then $t = r_k^{-1}(s) = e_k(s)$ (respectively $t = p_l^{-1}(s) = q_l(s)$) and $dt = e_k'(s)ds$ (respectively $dt = q_l'(s)ds$). So (for $\tilde{a}_k = a_k$ or c_k, and $\tilde{b}_l = b_l$ or d_l)

$$\begin{aligned}\int_0^T \tilde{a}_k(\mathbf{x},t)\Psi(\mathbf{x},r_k(t))\tilde{\phi}(\mathbf{x},t)dt &= \int_{r_k(0)}^{r_k(T)} \tilde{a}_k(\mathbf{x},e_k(s))\Psi(\mathbf{x},s)\tilde{\phi}(\mathbf{x},e_k(s))e_k'(s)ds,\\ \int_0^T \tilde{b}_l(\mathbf{x},t)w(\mathbf{x},p_l(t))\tilde{\phi}(\mathbf{x},t)dt &= \int_{p_l(0)}^{p_l(T)} \tilde{b}_l(\mathbf{x},q_l(s))w(\mathbf{x},s)\tilde{\phi}(\mathbf{x},q_l(s))q_l'(s)ds.\end{aligned}$$

Since Ψ and w are zero on Q_0, we can conclude that

$$\begin{aligned}\int_0^T \sum_{k=1}^{n_1} \tilde{a}_k(\mathbf{x},t)\theta(\mathbf{x},r_k(t))\tilde{\phi}(\mathbf{x},t)dt &= \sum_{i=n_1}^{1} \int_{r_{i+1}(T)}^{r_i(T)} \sum_{k=1}^{i} \tilde{a}_k(\mathbf{x},e_k(s))e_k'(s)\tilde{\phi}(\mathbf{x},e_k(s))\Psi(\mathbf{x},s)ds,\\ \int_0^T \sum_{l=1}^{n_2} \tilde{b}_l(\mathbf{x},t)w(\mathbf{x},p_l(t))\tilde{\phi}(\mathbf{x},t)dt &= \sum_{j=n_2}^{1} \int_{p_{j+1}(T)}^{p_j(T)} \sum_{l=1}^{i} \tilde{b}_l(\mathbf{x},q_l(s))q_l'(s)\tilde{\phi}(\mathbf{x},q_l(s))w(\mathbf{x},s)ds.\end{aligned} \tag{3.90}$$

According to (3.90), system (3.89) becomes

$$\begin{aligned}&\int_0^T\int_\Omega \left(-\mathfrak{c}_m \frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial \mathcal{I}}{\partial \phi}(\phi,u)\tilde{\phi} + \frac{\partial \mathcal{G}}{\partial \phi}(\phi,u)\tilde{u} - div(\mathcal{K}_i\nabla(\tilde{\phi}+\tilde{\varphi}_e))\right)\Psi d\mathbf{x}dt\\ &-\sum_{i=n_1}^{1}\int_\Omega\int_{r_{i+1}(T)}^{r_i(T)}\sum_{k=1}^{i}\Big(a_k(\mathbf{x},e_k(t))\tilde{\phi}(\mathbf{x},e_k(t)) + c_k(\mathbf{x},e_k(t))\tilde{u}(\mathbf{x},e_k(t))\Big)e_k'(t)\Psi(\mathbf{x},t)d\mathbf{x}dt\end{aligned}$$

$$\begin{aligned}&-\int_0^T\int_\Omega\Big(div((\mathcal{K}_i+\mathcal{K}_e)\nabla\tilde{\varphi}_e) + div(\mathcal{K}_i\nabla\tilde{\phi})\Big)\psi_e d\mathbf{x}dt\\ &+\int_0^T\int_\Omega\left(-\frac{\partial\tilde{u}}{\partial t} + \frac{\partial\mathcal{G}}{\partial u}(\phi,u)\tilde{u} + \frac{\partial\mathcal{I}}{\partial u}(\phi,u)\tilde{\phi}\right)wd\mathbf{x}dt\\ &-\sum_{j=n_2}^{1}\int_\Omega\int_{p_{j+1}(T)}^{p_j(T)}\sum_{l=1}^{j}\Big(b_l(\mathbf{x},q_l(t))\tilde{\phi}(\mathbf{x},q_l(t)) + d_l(\mathbf{x},q_l(t))\tilde{u}(\mathbf{x},q_l(t))\Big)q_l'(t)w(\mathbf{x},t)d\mathbf{x}dt\\ &\qquad= \int_0^T\int_{\Omega_c}\tilde{\varphi}_e\mathcal{B}_1h_1d\mathbf{x}dt + \int_0^T\int_{\Omega_d}\tilde{\phi}\mathcal{B}_2h_2d\mathbf{x}dt - \int_{\Omega_{obs}} m_2(\phi(T)-\psi_{obs})\Psi(T)d\mathbf{x}\\ &\qquad-\int_0^T\int_\Gamma((\mathcal{K}_i\nabla\tilde{\phi}).\mathbf{n})(\Psi+\psi_e)d\gamma dt\\ &\qquad-\int_0^T\int_\Gamma\Big\{\Big(((\mathcal{K}_i+\mathcal{K}_e)\nabla\tilde{\varphi}_e).\mathbf{n}\Big)\psi_e + \Big((\mathcal{K}_i\nabla\tilde{\varphi}_e).\mathbf{n}\Big)\Psi\Big\}d\gamma dt.\end{aligned}$$

In order to simplify the previous system, we suppose that $(\tilde{\phi},\tilde{\varphi}_e,\tilde{u})$ satisfies the following "adjoint" system

$$
\begin{aligned}
&-\mathfrak{c}_m \frac{\partial \tilde{\phi}}{\partial t} + \frac{\partial \mathcal{I}}{\partial \phi}(.;\phi,u)\tilde{\phi} + \frac{\partial \mathcal{G}}{\partial \phi}(.;\phi,u)\tilde{u} \\
&\qquad\qquad -div(\mathcal{K}_i \nabla(\tilde{\phi}+\tilde{\varphi}_e)) = m_1(\phi-\phi_{obs})\Upsilon(t)\chi_{\Omega_{obs}}, \text{ in } \Omega\times(r_1(T),T) \\
&-\mathfrak{c}_m \frac{\partial \tilde{\phi}}{\partial t} + \frac{\partial \mathcal{I}}{\partial \phi}(.;\phi,U)\tilde{\phi} + \frac{\partial \mathcal{G}}{\partial \phi}(.;\phi,u)\tilde{u}(\mathbf{x},t) - div(\mathcal{K}_i \nabla(\tilde{\phi}+\tilde{\varphi}_e)) \\
&\qquad\qquad + \sum_{i=1}^{k}\Big(a_i(.,e_i(t))\tilde{\phi}(.,e_i(t)) + c_i(.,e_i(t))\tilde{u}(.,e_i(t))\Big)e_i'(.,t) \\
&\qquad\qquad = m_1(\phi-\phi^{obs})\Upsilon(t)\chi_{\Omega_{obs}}, \text{ in } \Omega\times(r_{k+1}(T),r_k(T)),\, k=n_1,\dots,1 \\
&-div(\mathcal{K}_i\nabla\tilde{\phi}(\mathbf{x},t)) - div((\mathcal{K}_i+\mathcal{K}_e)\nabla\tilde{\varphi}_e(\mathbf{x},t)) = 0, \text{ in } Q \\
&-\frac{\partial \tilde{u}}{\partial t}(\mathbf{x},t) + \frac{\partial \mathcal{G}}{\partial u}(.;\phi,u)\tilde{u}(\mathbf{x},t) + \frac{\partial \mathcal{I}}{\partial u}(.;\phi,u)\tilde{\phi}(\mathbf{x},t) = 0, \text{ in } \Omega\times(p_1(T),T) \\
&-\frac{\partial \tilde{u}}{\partial t} + \frac{\partial \mathcal{G}}{\partial u}(.;\phi,u)\tilde{u} + \frac{\partial \mathcal{I}}{\partial u}(.;\phi,u)\tilde{\phi} \\
&\qquad\qquad + \sum_{i=1}^{l}\Big(b_i(.,q_i(t))\tilde{\phi}(.,q_i(t)) + d_i(.,q_i(t))\tilde{u}(.,q_i(t))\Big)q_i' \\
&\qquad\qquad = 0, \text{ in } \Omega\times(p_{l+1}(T),p_l(T)),\, l=n_2,\dots,1, \\
&(\mathcal{K}_i\nabla(\tilde{\phi}+\tilde{\varphi}_e)).\mathbf{n} = 0, \ (\mathcal{K}_e\nabla\tilde{\varphi}_e).\mathbf{n} = 0, \text{ on } \Sigma, \\
&\tilde{\phi}(.,T) = \frac{m_2}{\mathfrak{c}_m}(\phi(.,T)-\psi_{obs})\chi_{\Omega_{obs}}, \ \tilde{u}(.,T) = 0, \text{ in } \Omega.
\end{aligned}
\tag{3.91}
$$

Since $\int_0^T = \sum_{i=n_1}^{1}\int_{r_{i+1}(T)}^{r_i(T)} + \int_{r_1(T)}^{T} = \sum_{j=n_2}^{1}\int_{p_{j+1}(T)}^{p_j(T)} + \int_{p_1(T)}^{T}$ we can then deduce from the two previous systems that

$$
\begin{aligned}
\int_0^T\int_{\Omega_{obs}} m_1(\phi-\phi_{obs})\Upsilon(t)\Psi d\mathbf{x}dt + \int_{\Omega_{obs}} m_2(\phi(.,T)-\phi_{obs})\Psi(.,T)d\mathbf{x} \\
= \int_0^T\int_{\Omega_c} h_1\mathcal{B}_1^*\tilde{\varphi}_e d\mathbf{x}dt + \int_0^T\int_{\Omega_d} h_2\mathcal{B}_2^*\tilde{\phi}d\mathbf{x}dt.
\end{aligned}
\tag{3.92}
$$

According to (3.92) and (3.84), the expression (3.82) of $\mathcal{J}'$ takes the form

$$
\mathcal{J}'(\xi,\pi)\cdot(h_1,h_2) = (h_1,\chi_{\Omega_c}\aleph_1^*\tilde{\varphi}_e + \alpha\xi)_{U_c} + \int_0^T\int_{\Omega_d} h_2(\mathcal{B}_2^*\tilde{\phi} - \beta\pi)d\mathbf{x}dt. \tag{3.93}
$$

Since (ξ^*,π^*) is an optimal solution we have $(\forall(\xi,\pi)\in\mathcal{U}_{ad}\times\mathcal{V}_{ad})$

$$
\begin{aligned}
&\frac{\partial \mathcal{J}}{\partial \xi}(\xi^*,\pi^*).(\xi-\xi^*) = (\chi_{\Omega_c}\aleph_1^*\tilde{\varphi}_e^* + \alpha\xi^*, \xi-\xi^*)_{U_c} \geq 0, \\
&\frac{\partial \mathcal{J}}{\partial \phi}(\xi^*,\pi^*).(\pi-\pi^*) = \int_0^T\int_{\Omega_d}(-\beta\pi^* + \mathcal{B}_2^*\tilde{\phi}^*)(\pi-\pi^*)d\mathbf{x}dt \leq 0,
\end{aligned}
\tag{3.94}
$$

where $(\tilde{\phi}^*,\tilde{\varphi}_e^*,\tilde{u}^*)$ is the solution of (3.91) corresponding to $\mathcal{F}(\xi^*,\pi^*)$. This completes the proof. □

Remark 3.3. *By using a standard control argument (see e.g. [10], page 207) concerning the sign of the variations* (h_1, h_2) *(depending on the size of* (ξ, π)*), we obtain that*

$$\xi^* = \max\Big(- \tau_1, \min\big(\frac{-\chi_{\Omega_c}\aleph_1^*\tilde{\varphi}_e^*}{\alpha}, \tau_1\big)\Big), \pi^* = \max\Big(- \tau_2, \min\big(\frac{\chi_{\Omega_d}\mathcal{B}_2^*\tilde{\phi}^*}{\beta}, \tau_2\big)\Big). \tag{3.95}$$

□

Let us now give the well-posedness of adjoint system (3.86).

Proposition 3.1. *Assume that assumptions of Theorem 3.2 hold and that* (ϕ, φ_e, u) *is in* $\mathbb{D}$*. Adjoint problem (3.86) admits one unique solution* $(\tilde{\phi}, \tilde{\varphi}_e, \tilde{u}) \in \mathcal{W}(Q)$ *with* $(\frac{\partial \tilde{\phi}}{\partial t}, \frac{\partial \tilde{u}}{\partial t}) \in L^{4/3}(0, T; \mathbb{V}') \times L^2(Q)$ *and* $(\tilde{\phi}, \tilde{u}) \in (C^0([0, T]; L^2(\Omega)))^2$.

Proof. To prove the existence of a unique solution $(\tilde{\phi}, \tilde{\varphi}_e, \tilde{u})$ of linear problem (3.86), which is backward in time, we transform problem (3.86) into an initial-boundary value problem by reversing the sense of time i.e., $t := T - t$. By using the results of Lemma 3.1 on each time interval, we obtain the existence and uniqueness of the solution. □

Remark 3.4. *In this section, our main results investigate Fréchet differentiability properties of solution operator and minimax control problems related to the nonlinear delayed dynamic system (3.1) with an abstract class of ionic models, including some classical models as Rogers-McCulloch, Fitz-Hugh-Nagumo and Aliev-Panfilov. We can consider other ionic model type including Mitchell-Schaeffer model (see [52]). This two-variable model can be defined with operators* $\mathcal{I}$ *and* $\mathcal{G}$ *as (for example)*

$$\begin{aligned}
\mathcal{I}(\phi, u) &= -\frac{u}{\tau_{in}} \frac{(\phi - \phi_{min})^2(\phi_{max} - \phi)}{(\phi_{max} - \phi_{min})} - \frac{1}{\tau_{out}} \frac{\phi - \phi_{min}}{(\phi_{max} - \phi_{min})}, \\
\mathcal{G}(\phi, u) &= \begin{cases} \dfrac{u}{\tau_{open}} - \dfrac{1}{\tau_{open}(\phi_{max} - \phi_{min})^2} & \text{if } \phi < \phi_{gate}, \\ \dfrac{u}{\tau_{close}} & \text{if } \phi \geq \phi_{gate}. \end{cases}
\end{aligned} \tag{3.96}$$

These operators depend on the change-over voltage ϕ_{gate}*, the resting potential* ϕ_{min}*, the maximum potential* ϕ_{max}*, and on times constants* τ_{in}*,* τ_{out}*,* τ_{open} *and* τ_{close}*. The two times* τ_{open} *and* τ_{close}*, respectively controlling the durations of the action potential and of the recovery phase, and the two times* τ_{in} *and* τ_{out}*, respectively controlling the length of depolarization and repolarization phases. This model is well-known to be valid under the assumption* $\tau_{in} \ll \tau_{out} \ll \min(\tau_{open}, \tau_{close})$.

In order to guarantee the well-posedness of system (3.1) *with Mitchell-Schaeffer ionical model, we can use the following regularized version of ionic operator* $\mathcal{G}$

$$\begin{aligned}
\mathcal{G}_\zeta(\phi, u) = &\left(\frac{1}{\tau_{close}} - (\frac{1}{\tau_{close}} - \frac{1}{\tau_{open}})h_\zeta(\phi)\right)\left(u - h_\zeta(\phi)\right) \\
&+ \frac{1}{\tau_{open}}\left(1 - \frac{1}{(\phi_{max} - \phi_{min})^2}\right)h_\zeta(\phi),
\end{aligned} \tag{3.97}$$

where the differentiable function $0 \leq h_\zeta \leq 1$ *is given by*

$$h_\zeta(\phi) = \frac{1}{2}\left(1 - \tanh\left(\frac{\phi - \phi_{gate}}{\zeta}\right)\right),$$

with ζ a positive parameter.

The operator $\mathcal{G}_\zeta(\phi, u)$ can be written as $\mathcal{G}_\zeta(\phi, u) = \mathcal{I}_{2,\zeta}(\phi) + \hbar_\zeta(\phi)u$ where

$$\begin{aligned} \mathcal{I}_{2,\zeta}(\phi) &= -\left(\frac{1}{\tau_{close}} - (\frac{1}{\tau_{close}} - \frac{1}{\tau_{open}})h_\zeta(\phi)\right)h_\zeta(\phi) + \frac{1}{\tau_{open}}\left(1 - \frac{1}{(\phi_{max} - \phi_{min})^2}\right)h_\zeta(\phi), \\ \hbar_\zeta(\phi) &= \frac{1}{\tau_{close}} - \Big(\frac{1}{\tau_{close}} - \frac{1}{\tau_{open}}\Big)h_\zeta(\phi). \end{aligned} \tag{3.98}$$

According to the definition of tanh*, we can deduce that* $\lim_{\zeta \to 0} h_\zeta(\phi) = \begin{cases} 1 \text{ if } \phi < \phi_{gate}, \\ 0 \text{ if } \phi > \phi_{gate} \end{cases}$ *and then* $\lim_{\zeta \to 0} \mathcal{G}_\zeta(\phi, u) = \mathcal{G}(\phi, u)$*. The regularized Mitchell-Schaeffer model has a slightly different structure compared to models in* (2.3) *because in this model, $\hbar_\zeta$ depend on ϕ through the function h_ζ. Since h_ζ is sufficiently regular, the arguments of this paper can be adapted with some necessary modifications to analyse minimax control problems with the regularized Mitchell-Schaeffer ionical model.*

More general, the study developed in this paper remains valid if we consider the operator $\mathcal{G}$ in the form of $\mathcal{G}(\mathbf{x}, t; \phi, u) = \mathcal{I}_2(\mathbf{x}, t; \phi) + \hbar(\mathbf{x}, t; \phi)u$ (i.e. a general form of Hodgkin-Huxley model including Beeler-Reuter and Luo-Rudy ionic models described by continuous or regularized discontinuous functions, see [5, 49, 50]) with $\mathcal{G}$ Carathodory function from $(\Omega \times \mathbb{R}) \times \mathbb{R}^2$ into $\mathbb{R}$ and locally Lipschitz continuous function on (ϕ, u) and, $\mathcal{I}_2$ and $\hbar$ sufficiently regulars. □

We end this section by a description of a gradient algorithm to solve the minimax control problem, by using adjoint model. The method is formulated in terms of continuous variables which are independent of a specific numerical discretization. For more details concerning some optimization strategies in order to solve minimax control problem, by using the adjoint model, in term of the continuous variables and in terms of discrete variables (based on the discretization of continuous direct, adjoint and sensitive models) see Chapter 9 of [10].

3.4. Gradient-iterative algorithm

We present algorithms where the descent direction is calculated by using the adjoint variables, particularly by choosing an admissible step size. For a given observation (ϕ_{obs}, ψ_{obs}), initial states (ϕ_0, u_0) and past states (ϕ_{past}, u_{past}), the resolution of the nonlinear minimax control problem (3.6), with cost functional given by (3.4), by gradient methods requires, at each iteration of the optimization algorithm, the resolution of direct problem (3.1) and its corresponding adjoint problem (3.86).

The gradient algorithm for the resolution of treated saddle point problems is given by:
for $k \geq 1$, (iteration index) we denote by (ξ_k, π_k) the numerical approximation of the control-disturbance at the *kth* iteration of the algorithm.
(**1**) Initialization: $k = 0$ and (ξ_0, π_0) (given initial guess).
(**2**) Resolution of direct problem (3.1) with source term (ξ_k, π_k), gives $(\phi^{(k)}, \varphi_e^{(k)}, u^{(k)}) = \mathcal{F}(\xi_k, \pi_k)$.
(**3**) Resolution of adjoint problem (3.86) (based on $(\xi_k, \pi_k, \mathcal{F}(\xi_k, \pi_k))$, gives $(\tilde{\phi}^{(k)}, \tilde{\varphi}_e^{(k)}, \tilde{u}^{(k)})$.

(**4**) Local expression of the gradient of $\mathcal{J}$ at point (ξ_k, π_k):

$$
(\mathbf{GJ}), \begin{cases} \upsilon_k \overset{def}{=} \dfrac{\partial \mathcal{J}}{\partial \xi}(\xi_k, \pi_k) = \alpha \xi_k + \chi_{\Omega_c} \aleph_1^* \tilde{\varphi}_{e,k}, \\ \varpi_k \overset{def}{=} \dfrac{\partial \mathcal{J}}{\partial \pi}(\xi_k, \pi_k) = -\beta \pi_k + \chi_{\Omega_d} \mathcal{B}_2^* \tilde{\phi}_k, \\ G_k = (\upsilon_k, \varpi_k). \end{cases}
$$

(**5**) Determine (ξ_{k+1}, π_{k+1}):

$$
\begin{cases} \xi_{k+1} := \xi_k - \varsigma_k \upsilon_k, \\ \pi_{k+1} := \pi_k + \delta_k \varpi_k, \end{cases}
$$

where $0 < m \leq \varsigma_k, \delta_k \leq M$ are the sequences of step lengths.
(**6**) IF the gradient is sufficiently small (convergence) THEN end; ELSE set $k := k + 1$ and REPEAT from (**2**) UNTIL convergence.

The approximation of optimal Solution is: $(\xi^*, \pi^*; \phi^*, \varphi_e^*, u^*) := (\xi_k, \pi_k; \phi^{(k)}, \varphi_e^{(k)}, u^{(k)})$. The convergence of the algorithm depends on the second Fréchet derivative of $\mathcal{J}$ (i.e. m, M depend on the second Fréchet derivative of $\mathcal{J}$).

In order to obtain an algorithm which is numerically efficient, the best choice of ς_k, δ_k will be the result of a line minimization and maximization algorithm, respectively. Otherwise, at each iteration step k of previous algorithm, we solve the one-dimensional optimization problem of parameters ς_k and δ_k:

$$
\begin{aligned} \varsigma_k &= \min_{\lambda>0} \mathcal{J}(\xi_k - \lambda \upsilon_k, \pi_k), \\ \delta_k &= \min_{\lambda>0} \mathcal{J}(\xi_k, \pi_k + \lambda \varpi_k). \end{aligned} \tag{3.99}
$$

From the numerical computation viewpoint, it is most efficient to compute (ς_k, δ_k) only approximately, in order to reduce computational cost. To derive an approximation for a pair (ς_k, δ_k) we can use a purely heuristic approach, for example, by taking $\varsigma_k = min(1, \| \upsilon_k \|_\infty^{-1})$ and $\delta_k = min(1, \| \varpi_k \|_\infty^{-1})$ or by using the linearization of $\mathcal{F}(\xi_k - \lambda \upsilon_k, \pi_k)$ at ξ_k and $\mathcal{F}(\xi_k, \pi_k + \lambda \varpi_k)$ at π_k by

$$
\begin{aligned} \mathcal{F}(\xi_k - \lambda \upsilon_k, \pi_k) &\approx \mathcal{F}(\xi_k, \pi_k) - \lambda \frac{\partial \mathcal{F}}{\partial \xi}(\xi_k, \pi_k).\upsilon_k, \\ \mathcal{F}(\xi_k, \pi_k + \lambda \varpi_k) &\approx \mathcal{F}(\xi_k, \pi_k) + \lambda \frac{\partial \mathcal{F}}{\partial \pi}(\xi_k, \pi_k).\varpi_k, \end{aligned}
$$

where

$$
\begin{aligned} (\Psi_c^{(k)}, \psi_{e,c}^{(k)}, w_c^{(k)}) &= \frac{\partial \mathcal{F}}{\partial \xi}(\xi_k, \pi_k).\upsilon_k, \\ (\Psi_d^{(k)}, \psi_{e,d}^{(k)}, w_d^{(k)}) &= \frac{\partial \mathcal{F}}{\partial \pi}(\xi_k, \pi_k).\varpi_k, \end{aligned}
$$

are solutions of the sensitivity problem (3.7).

According to the previous approximation, we can approximate the problem (3.99) by

$$
\varsigma_k = \min_{\lambda>0} H(\lambda) \text{ and } \delta_k = \max_{\lambda>0} R(\lambda), \tag{3.100}
$$

where $H(\lambda) = \mathcal{J}(\xi_k - \lambda \upsilon_k, \pi_k)$ and $R(\lambda) = \mathcal{J}(\xi_k, \pi_k + \lambda \varpi_k)$. Since H and R are polynomial functions of degree 2 (since the functional $\mathcal{J}$ is quadratic), then problem (3.100) can be solved exactly. Consequently, we obtain explicitly the value of the parameters ς_k and δ_k.

Remark 3.5. *1. After derived the gradient of functional $\mathcal{J}$, by using the adjoint model corresponding to sensitivity state (which corresponds to the direct problem), we can use any other classical optimization strategies (as conjugate gradient method, Lagrange-Newton method) to solve control problem considered in this paper.*
2. In the numerical treatment of minimax control problem, the direct system, adjoint system, sensitivity system and objective functional must be discretized (reduction of infinite-dimensional dynamics to finite-dimensional problems). The discretized formulation for direct, sensitivity and adjoint systems can be performed by combining Galerkin and finite element methods to the variational formulations associated to these coupled problems, for space discretization and semi-implicit backward differentiation schemes with an explicit treatment of ionic current, for time discretization, or by using lattice Boltzmann methods. In objective functional, the integrals with respect to time can be approximated by composition trapezoidal rules (see e.g. [7]).
3. Despite its apparent complexity, the proposed gradient algorithm is quite easy to implement. The main difficulty, in practical applications, is due to enormous storage requirements of state solution (and control-disturbance variables) for evaluating the adjoint equation over the whole time interval $[0, T]$*, for large time horizons T or fine space-time meshes (because the computation of the discrete gradient by discrete adjoint methods requires one forward solve of the discrete state system and one backward solve of adjoint system in which state trajectory is an input). Fortunately, these storage requirements can be lowered by using e.g. the so-called "checkpointing" techniques (see e.g. [39]).* □

4. Conclusions

Modeling and control of electrical cardiac activity represent nowadays a very valuable tool to maximize the efficiency and safety of treatment for cardiac disease. For predicting and acting on phenomena and processes occurring inside and surrounding cardiac medium, we have discussed stabilization and regulation processes in order to determine, from some observations (desired target), the best optimal prognostic values of sources, in presence of disturbance and fluctuations. Coupling the proposed method with technical improvements in Magnetic Resonance Imaging (MRI) measurements and genetic, and ionic measurements, will be very beneficial and great help for diagnostics and treatments in medical practices.

The well-posedness and regularity of the governing nonlinear systems are discussed. The Fréchet differentiability and some properties of nonlinear operator solution are derived. Afterwards, minimax control problems have been formulated. Under suitable hypotheses, it is shown that one has existence of an optimal solution, and the appropriate necessary optimality conditions for an optimal solution, by introducing adjoint problems, are derived. These conditions (obtained in a Lagrangian form) correspond to identify the gradient of the cost functional that is necessary to develop numerical optimization methods (gradient methods, Newton methods, etc.). Some numerical methods, combining the obtained optimal necessary conditions and gradient-iterative algorithms, are presented in order to solve the minimax control problems.

It is clear that, in accordance with practical applications and available experimental observations, we can consider other observations, controls and/or disturbances (which can appear in boundary conditions, in initial conditions, in parameters of ionic models or in time-delay functions) in order to take into account at best the influence of uncertainty on the main phenomena and their mutual interactions that take place during the bioelectrical cardiac activity, and we obtain similar results by using similar approach as used in this work (for more details see [10]).

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Conflict of interest

The author declares there is no conflicts of interest in this paper.

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