Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(3): 821–830.
DOI:10.3934/math.2019.3.821
Received: 18 April 2019
Accepted: 30 June 2019
Published: 16 July 2019

Research article

Monotonic solutions for a quadratic integral equation of fractional order

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Abstract: In this paper we present a global existence theorem of a positive monotonic integrable solution for the mixed type nonlinear quadratic integral equation of fractional order

$$x(t) = p(t) + h(t, x(t)) \int_0^t k(t, s)(f_1(s, I^{\alpha} f_2(s, x(s))) + g_1(s, I^{\beta} g_2(s, x(s))))ds, \ t \in [0, 1], \alpha, \beta > 0$$

by applying the technique of measures of weak noncompactness. As an application, we consider an initial value problem of arbitrary (fractional) order differential equations.

Keywords: fractional calculus; quadratic integral equation; fixed point theory; measure of noncompactness
Mathematics Subject Classification: 26A33, 34A60, 45G05

1. Introduction

It is well-known that a useful mathematical tool for physical investigation and description of non-local and anomalous diffusion is fractional calculus, which is that a branch of mathematical analysis dealing with pseudo-differential operators interpreted as integrals and derivatives of non-integer order (see [1, 16, 21, 22]). For example, quadratic integral equations are often applicable in the theory of radiative transfer, of the theory of kinetic gases, the theory of neutron transport and in the theory of traffic. The quadratic integral equation can be very often encountered in many applications (see [6, 7, 9]).
Recently, the existence of positive monotonic continuous and integrable solutions of the mixed type

integral inclusion

$$x(t) \in p(t) + \int_0^1 k(t,s)\ F_1(s, I^{\beta} f_2(s, x(s))ds, \qquad t \in [0,1], \quad \beta > 0 \tag{1.1}$$

has been studied in [13,15] by using Schauder's and nonlinear alternative of Leray-Shauder type fixed-point Theorem. Also, the existence of integrable solution for the nonlinear quadratic integral equation

$$x(t) = a(t) + g(t, x(t)) \int_0^t k(t,s)\ f(s, x(s))ds, \qquad t \in [0,1] \tag{1.2}$$

has been proven in [12] by using Lusin and Dragoni theorems and applying Schauder Tychonoff fixed-point Theorem.
Here we are concerned with the mixed type nonlinear integral equation of fractional order

$$x(t) = p(t) + h(t, x(t)) \int_0^t k(t,s)(f_1(s, I^{\alpha} f_2(s, x(s))) + g_1(s, I^{\beta} g_2(s, x(s))))ds,\ t \in [0,1], \alpha, \beta > 0 \tag{1.3}$$

which is more complicated than the equation assumed in [12]. We will use the technique associated with measures of noncompactness to show a global existence theorem for a positive nondecreasing integrable solution of equation (1.3), where the functions f_1, f_2, g_1 and g_2 satisfy Carathéodory condition.
Moreover, the existence of at least one integrable solution of the fractional-order quadratic integral equation

$$x(t) = p(t) + h(t, x(t)) I^{\theta}(f_1(s, I^{\alpha} f_2(s, x(s)) + g_1(s, I^{\beta} g_2(s, x(s))))),\ \ t \in [0,1],\ \alpha,\ \beta > 0 \tag{1.4}$$

will be studied. Let us mention the result obtained for $h(t, x(t)) = 1$ in integral equation (1.4) extend those obtained in the paper [2]. Also, the results concerning the existence of monotonic positive integrable solution of the nonlinear functional equation

$$x(t) = f_1(t, I^{\alpha} f_2(t, x(t)) + g_1(t, I^{\beta} g_2(t, x(t)),$$

will be given as a special case, which generalized the results proved in [4,14].

2. preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper.
Let $L^1 = L^1(I)$ be the class of Lebesgue integrable function on the interval $I = [a,b]$, where $0 \leq a < b < \infty$, with the standard norm

$$\|x\| = \int_a^b |x(t)|dt.$$

Definition 2.1. The Riemann-Liouville fractional integral of the function $f(.) \in L^1(I)$ of order $\alpha \in R^+$ is defined by (cf. [18,19,22])

$$I_a^{\alpha} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, d(s).$$

For the properties of the fractional order integral (see [17, 18]).

Definition 2.2. The Caputo fractional derivative D^{α} of order $\alpha \in (a, b]$ of the absolutely continuous function g is defined as (see [10, 19, 20, 22])

$$D_a^{\alpha}\, g(t) = I_a^{1-\alpha}\, \frac{d}{dt}\, g(t) \ , \quad t \in [a, b].$$

Now, let E denote an arbitrary Banach space with zero element θ and X a nonempty bounded subset of E. Moreover denote by $B_r = B(\theta, r)$ the closed ball in E centered at θ and with radius r.
The measure of weak noncompactness defined by De Blasi [3, 11] is given by

$$\beta(X) = inf(r > 0; \text{there exists a weakly compact subset } Y \text{ of } E \text{ such that } X \subset Y + B_r). \tag{2.1}$$

The function $\beta(X)$ possesses several useful properties which may be found in De Blasi's paper [11]. The convenient formula for the function $\beta(X)$ in L^1 was given by Appell and De Pascale (see [3]) as follows:

$$\beta(X) = \lim_{\epsilon \to 0} \left(\sup_{x \in X} \left(sup \left[\int_D |x(t)| dt : \ D \subset [a, b], \ meas\ D \leq \epsilon \right] \right) \right), \tag{2.2}$$

where the symbol meas D stands for Lebesgue measure of the set D.

Next, we shall also use the notion of the Hausdorff measure of noncompactness χ (see [4]) defined by

$$\chi(X) = inf(r > 0; \text{ there exist a finite subset } Y \text{ of } E \text{ such that } X \subset Y + B_r). \tag{2.3}$$

In the case when the set X is compact in measure, the Hausdorff and De Blasi measures of noncompactness will be identical. Namely, we have the following (see [3, 11]).

Theorem 2.1. Let X be an arbitrary nonempty bounded subset of L^1. If X is compact in measure, then $\beta(X) = \chi(X)$.

Now, we will recall the fixed point theorem due to Banaś [8].

Theorem 2.2. Let Q be a nonempty, bounded, closed, and convex subset of E, and let $T : Q \to Q$ be a continuous transformation which is a contraction with respect to the Hausdorff measure of noncompactness χ; that is, there exists a constant $\alpha \in [0, 1]$ such that $\chi(TX) \leq \alpha\chi(X)$ for any nonempty subset X of Q. Then, T has at least one fixed point in the set Q.

In the sequel, we will need some criteria for compactness in measure; the complete description of compactness in measure was given in Banaś [4], but the following sufficient condition will be more convenient for our purposes (see [4]).

Theorem 2.3. Let X be a bounded subset of L^1. Assume that there is a family of measurable subsets $(\Omega_c)_{0 \leq c \leq b-a}$ of the interval (a, b) such that meas $\Omega_c = c$. If for every $c \in [0, b - a]$, and for every $x \in X$,

$$x(t_1) \leq x(t_2), \ \ (t_1 \in \Omega_c, \ t_2 \notin \Omega_c)$$

then, the set X is compact in measure.

3. Existence Theorem

To facilitate our discussion, let us first state the following assumptions:

(1) the function $p : [0,1] \to R^+$ is integrable and nondecreasing on $[0,1]$.

(2) $h : [0,1] \times R^+ \to R^+$, satisfies Carathéodory condition i.e., h is measurable in t for any $x \in R^+$ and continuous in x for almost all $t \in [0,1]$. There exists a function $m(t) \in L^1$ such that

$$|h(t,x)| \leq m(t).$$

Moreover it is nondecreasing in the two arguments.

(3) $f_i : [0,1] \times R^+ \to R^+$, and $g_i : [0,1] \times R^+ \to R^+, i = 1,2$ satisfy Carathéodory condition i.e., f_i, g_i are measurable in t for any $x \in R^+$ and continuous in x for almost all $t \in [0,1]$.
There exist four functions $t \to a_i(t)$, $t \to b_i(t)$, $t \to c_i(t)$, and $t \to d_i(t)$. such that

$$|f_i(t,x)| \leq a_i(t) + b_i(t)|x|, \quad i = 1,2 \;\; \forall \;\; t \in [0,1] \; and \; x \in R$$

and

$$|g_i(t,x)| \leq c_i(t) + d_i(t)|x|, \quad i = 1,2 \;\; \forall \;\; t \in [0,1] \; and \; x \in R$$

where $a_i(.), \; c_i(.) \in L^1$, and $b_i(.), d_i(.)$ are measurable and bounded.

(4) $k : [0,1] \times R^+ \to R^+$ is measurable with respect to both variables and the integral operator K defined by

$$(Kx)(t) = \int_0^t k(t,s)x(s)ds, \qquad t \in [0,1]$$

map nondecreasing positive function L^1 into itself and such that

$$\int_0^t k(t,s)\, m(t)dt \; < \; M, \qquad s \in [0,1].$$

Moreover, it is nondecreasing in the first argument.

For the existence of at least one nondecreasing L^1-positive solution of a mixed type integral equation (1.3) we have the following theorem.

Theorem 3.1. Let the assumptions (1)–(4) be satisfied and assume that $\frac{Mb_1b_2}{\Gamma(\alpha+1)} + \frac{Md_1d_2}{\Gamma(\beta+1)} < 1$, then equation (1.3) has at least one solution $x \in L^1$ which is nondecreasing on the interval $[0,1]$.

Proof. Firstly, for $t_1, t_2 \in [0,1]$, $t_1 < t_2$ and $x(t_1) \leq x(t_2)$, we have

$$\begin{aligned} x(t_1) &= p(t_1) + h(t_1, x(t_1)) \int_0^t k(t_1,s)\,(f_1(s, I^\alpha f_2(s,x(s))) + g_1(s, I^\beta g_2(s,x(s))))ds \\ &\leq P(t_2) + h(t_2, x(t_2)) \int_0^t k(t_2,s)\,(f_1(s, I^\alpha f_2(s,x(s))) + g_1(s, I^\beta g_2(s,x(s))))ds \\ &= x(t_2). \end{aligned}$$

This implies that, if the solution of the integral equation (1.3) exists, then it is nondecreasing on [0, 1]. Let the operator T be defined by the formula

$$(Tx)(t) \;=\; p(t) + h(t, x(t)) \int_0^t k(t, s)\,(f_1(s, I^{\alpha} f_2(s, x(s))) + g_1(s, I^{\beta} g_2(s, x(s))))\,ds$$

Let $x \in L^1$, then by assumptions (1)–(4) we find that

$$\begin{aligned}
|(Tx)(t)| &= |p(t)| + |h(t, x(t))| \int_0^t k(t, s)\,|f_1(s, I^{\alpha} f_2(s, x(s))) + g_1(s, I^{\beta} g_2(s, x(s)))|\,ds \\
&\leq |p(t)| + m(t) \int_0^t k(t, s)\,(|f_1(s, I^{\alpha} f_2(s, x(s)))| + |g_1(s, I^{\beta} g_2(s, x(s)))|)\,ds \\
\|Tx\| &= \int_0^1 |(Tx)(t)|\,dt \\
&\leq \int_0^1 |p(t)|\,dt + \int_0^1 m(t) \int_0^t k(t, s)|f_1(s, \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f_2(\tau, x(\tau))\,d\tau|\,ds\,dt \\
&+ \int_0^1 m(t) \int_0^t k(t, s)|g_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} g_2(\tau, x(\tau))\,d\tau|\,ds\,dt \\
&\leq \|p\| + \int_0^1 m(t) \int_0^t k(t, s)(a_1(s) + b_1(s)| \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f_2(\tau, x(\tau))d\tau|)\,ds\,dt \\
&+ \int_0^1 m(t) \int_0^t k(t, s)(c_1(s) + d_1(s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2(\tau, x(\tau))\,d\tau|)\,ds\,dt \\
&\leq \|p\| + \int_0^1 \int_s^1 k(t, s)\,m(t)\,dt\,(a_1(s) + b_1(s)| \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f_2(\tau, x(\tau))d\tau|)ds\,dt \\
&+ \int_0^1 \int_s^1 k(t, s)\,m(t)\,dt\,(c_1(s) + d_1(s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2(\tau, x(\tau))d\tau|)ds\,dt \\
&\leq \|p\| + M \int_0^1 |a_1(s)|\,ds + M \int_0^1 |b_1(s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\alpha)} |f_2(\tau, x(\tau))|d\tau\;\;ds \\
&+ M \int_0^1 |c_1(s)|ds + M \int_0^1 |d_1(s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |f_2(\tau, x(\tau))|d\tau\,ds \\
&\leq \|p\| + M\|a_1\| + Mb_1 \int_0^1 \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} [a_2(\tau) + b_2(\tau)|x(\tau)|]d\tau ds \\
&+ M\|c_1\| + Md_1 \int_0^1 \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} [c_2(\tau) + d_2(\tau)|x(\tau)|]d\tau ds \\
&\leq \|p\| + M\|a_1\| + Mb_1 \int_0^1 \int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} a_2(\tau)dsd\tau + Mb_1 \int_0^1 \int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |b_2(\tau)||x(\tau)|dsd\tau \\
&+ M\|c_1\| + Md_1 \int_0^1 \int_\tau^1 \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} c_2(\tau)dsd\tau + Md_1 \int_0^1 \int_\tau^1 \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |d_2(\tau)||x(\tau)|dsd\tau
\end{aligned}$$

$$\begin{aligned}
&\le \|p\| + M\|a_1\| + Mb_1 \int_0^1 a_2(\tau) \int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} ds d\tau + Mb_1 b_2 \int_0^1 |x(\tau)| \frac{(1-\tau)^\alpha}{\Gamma(\alpha+1)} d\tau \\
&+ M\|c_1\| + Md_1 \int_0^1 c_2(\tau) \int_\tau^1 \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} ds d\tau + Md_1 d_2 \int_0^1 |x(\tau)| \frac{(1-\tau)^\beta}{\Gamma(\beta+1)} d\tau \\
&\le \|p\| + M\|a_1\| + Mb_1 \int_0^1 a_2(\tau) \frac{(1-\tau)^\alpha}{\Gamma(\alpha+1)} d\tau + \frac{Mb_1 b_2}{\Gamma(\alpha+1)} \int_0^1 |x(\tau)| d\tau \\
&+ M\|c_1\| + Md_1 \int_0^1 c_2(\tau) \frac{(1-\tau)^\beta}{\Gamma(\beta+1)} d\tau + \frac{Md_1 d_2}{\Gamma(\beta+1)} \int_0^1 |x(\tau)| d\tau \\
&\le \|p\| + M\|a_1\| + \frac{Mb_1}{\Gamma(\alpha+1)} \int_0^1 |a_2(\tau)|\, d\tau + \frac{Mb_1 b_2 \|x\|}{\Gamma(\alpha+1)} \\
&+ M\|c_1\| + \frac{Md_1}{\Gamma(\beta+1)} \int_0^1 |c_2(\tau)|\, d\tau + \frac{Md_1 d_2 \|x\|}{\Gamma(\beta+1)} \\
&\le \|p\| + M\|a_1\| + \frac{Mb_1\|a_2\|}{\Gamma(\alpha+1)} + \frac{Mb_1 b_2 \|x\|}{\Gamma(\alpha+1)} + \frac{Md_1\|c_2\|}{\Gamma(\beta+1)} + \frac{Md_1 d_2 \|x\|}{\Gamma(\beta+1)}, \\
&\le \|p\| + M\|a_1\| + M\|c_1\| + \frac{Mb_1\|a_2\| + Mb_1 b_2 \|x\|}{\Gamma(\alpha+1)} + \frac{Md_1\|c_2\| + Md_1 d_2 \|x\|}{\Gamma(\beta+1)}
\end{aligned}$$

which gives

$$\|Tx\| \le \|p\| + M\|a_1\| + M\|c_1\| + \frac{Mb_1\|a_2\| + Mb_1 b_2 \|x\|}{\Gamma(\alpha+1)} + \frac{Md_1\|c_2\| + Md_1 d_2 \|x\|}{\Gamma(\beta+1)} \tag{3.1}$$

and proves that $Tx \in L^1$. Moreover, the estimate (3.1) shows that the operator T maps the ball B_r into itself, where

$$r = [\|p\| + M\|a_1\| + M\|c_1\| + \frac{Mb_1\|a_2\|}{\Gamma(\alpha+1)} + \frac{Md_1\|c_2\|}{\Gamma(\beta+1)}][1 - (\frac{Mb_1 b_2}{\Gamma(\alpha+1)} + \frac{Md_1 d_2}{\Gamma(\beta+1)})]^{-1}.$$

Let $Q_r \subset B_r$ consisting of all positive and nondecreasing functions on I. Clearly Q_r is nonempty, bounded, closed and convex (see Banas [4], pp. 780). Now Q_r is a bounded subset of L^1 consisting of all positive and nondecreasing functions on $[0, 1]$, then Theorem 2.3 shows that Q_r is compact in measure (see Lemma 2 in [5] pp. 63).

Thus the operator T maps Q_r into itself, by using assumptions (1)–(4), the operator T is continuous on Q_r, and the operator T transforms a positive and nondecreasing function into the function of the same type (see [5, 23]).

In what follows we show that the operator T is a contraction with respect to the measure of weak noncompactness β. To do this let us fix $\epsilon > 0$ and $X \subset Q_r$. Further, take a measurable subset $D \subset [0, 1]$ such that $meas\ D \le \epsilon$, then for any $x \in X$ by our assumptions and using the same reasoning as in [4, 5] we obtain

$$\begin{aligned}
\|Tx\|_{L^1(D)} &= \int_D |(Tx)(t)| dt \\
&\le \int_D |p(t)| dt + \int_D m(t) \int_0^t k(t,s)[a_1(s) + b_1(s)| \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f_2(\tau, x(\tau)) d\tau | ds dt
\end{aligned}$$

$$
\begin{aligned}
& + \int_D m(t) \int_0^t k(t,s)[c_1(s)+d_1(s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} g_2(\tau, x(\tau)) d\tau| ds dt \\
& \leq \|p\|_D + M\|a_1\|_D + Mb_1 \int_D a_2(\tau) \frac{(1-\tau)^\beta}{\Gamma(\beta+1)} d\tau + \frac{Mb_1 b_2}{\Gamma(\beta+1)} \int_D |x(\tau)| d\tau \\
& + M\|c_1\| + Mb_1 \int_D a_2(\tau) \frac{(1-\tau)^\alpha}{\Gamma(\alpha+1)} d\tau + \frac{Mb_1 b_2}{\Gamma(\alpha+1)} \int_D |x(\tau)| d\tau \\
& \leq \|p\|_D + M\|a_1\|_D + \frac{Mb_1}{\Gamma(\alpha+1)} \int_D |a_2(\tau)|\, d\tau + + \frac{Mb_1 b_2}{\Gamma(\alpha+1)} \int_D |x(\tau)| d\tau \\
& + M\|c_1\| + \frac{Md_1}{\Gamma(\beta+1)} \int_D |c_2(\tau)|\, d\tau + \frac{Md_1 d_2}{\Gamma(\beta+1)} \int_D |x(\tau)| d\tau, \\
& \leq \|p\|_D + M\|a_1\|_D + \frac{Mb_1\|a_2\|_D}{\Gamma(\alpha+1)} + \frac{Mb_1 b_2}{\Gamma(\alpha+1)} \|x\|_D \\
& + M\|c_1\|_D + \frac{Md_1\|c_2\|_D}{\Gamma(\beta+1)} + \frac{Md_1 d_2}{\Gamma(\beta+1)} \|x\|_D \\
\|Tx\|_{L^1(D)} & \leq \|p\|_D + M\|a_1\|_D + M\|c_1\|_D + \frac{Mb_1\|a_2\|_D}{\Gamma(\alpha+1)} + \frac{Md_1\|c_2\|_D}{\Gamma(\beta+1)} + (\frac{Mb_1 b_2}{\Gamma(\alpha+1)} + \frac{Md_1 d_2}{\Gamma(\beta+1)}) \|x\|_D.
\end{aligned}
$$

But

$$\lim_{\epsilon \to 0} \{sup\{ \int_D |p(t)| dt : \ D \subset I, \ meas.D \ < \ \epsilon\}\} = 0$$

$$\lim_{\epsilon \to 0} \{sup\{ \int_D |a_i(t)| dt : \ D \subset I, \ meas.D \ < \ \epsilon\}\} = 0 \quad i = 1, 2$$

and

$$\lim_{\epsilon \to 0} \{sup\{ \int_D |c_i(t)| dt : \ D \subset I, \ meas.D \ < \ \epsilon\}\} = 0 \quad i = 1, 2.$$

We obtain

$$\beta(Tx(t)) \leq (\frac{Mb_1 b_2}{\Gamma(\alpha+1)} + \frac{Md_1 d_2}{\Gamma(\beta+1)})\, \beta(x(t))$$

and

$$\beta(TX) \leq (\frac{Mb_1 b_2}{\Gamma(\alpha+1)} + \frac{Md_1 d_2}{\Gamma(\beta+1)})\, \beta(X) \tag{3.2}$$

where β is the De Blasi measure of weak noncompactness. Keeping in mind Theorem 2.1, we can write (3.2) in the form

$$\chi(TX) \leq (\frac{Mb_1 b_2}{\Gamma(\alpha+1)} + \frac{Md_1 d_2}{\Gamma(\beta+1)})\, \chi(X)$$

where χ is the Hausdorff measure of noncompactness. Since $(\frac{Mb_1 b_2}{\Gamma(\alpha+1)} + \frac{Md_1 d_2}{\Gamma(\beta+1)}) < 1$, it follows, from Theorems 2.2, that T is a contraction with respect to the measure of noncompactness χ and has at least one fixed point in Q_r which proves that the nonlinear quadratic functional integral equation (1.3) has at least one positive nondecreasing solution $x \in L^1[0, 1]$. □

4. Fractional order quadratic functional integral equation

As particular cases of Theorem 3.1, we can obtain theorems on the existence of a positive and nondecreasing solutions belonging to the space L^1.

Theorem 4.1. Let the assumptions of Theorem 3.1 be satisfied with $k(t,s) = \frac{(t-s)^{\theta-1}}{\Gamma(\theta)}$, then the fractional-order quadratic integral equation

$$x(t) = p(t) + h(t, x(t)) I^{\theta}(f_1(s, I^{\alpha} f_2(s, x(s)) + g_1(s, I^{\beta} g_2(s, x(s))))), \quad t \in [0,1], \ \alpha, \ \beta > 0 \tag{4.1}$$

has at least one positive nondecreasing solution $x \in L^1$.

Proof. From the properties of fractional order integral operator, we deduce that the operator

$$(Kx)(t) = \int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} x(s) ds, \quad \theta \in [0,1]$$

satisfy the assumption (4) in Theorem 3.1, and the result follows from the results of Theorem 3.1. □

Corollary 4.1.1. Under the assumptions of Theorem 3.1, with $p(t) = 0$, $h(t, x(t)) = 1$ and letting $\theta \to 0$ the nonlinear functional equation

$$x(t) = f_1(t, I^{\alpha} f_2(t, x(t)) + g_1(t, I^{\beta} g_2(t, x(t)) \tag{4.2}$$

has at least one positive nondecreasing solution $x \in L^1$.

Proof. Fractional-order quadratic integral equation (4.1) will be the functional equation (4.2) and the result follows from Theorem 4.1. □

5. Fractional order functional differential equations

Finally, for the existence of a monotonic positive integrable solution of the nonlinear functional differential equation of fractional order

$$D^{\theta} x(t) = f_1(t, I^{\alpha} f_2(t, x(t)) + g_1(t, I^{\beta} g_2(t, x(t)), \quad t \in (0,1] \quad and \quad I^{1-\theta} x(t)|_{t=0} = p \tag{5.1}$$

where D^{θ} is the Riemann-Liouville fractional order derivative, we have the following theorem.

Theorem 5.1. Under the assumptions of Theorem 3.1, with $p(t) = p\frac{t^{\theta-1}}{\Gamma(\theta)}$ and $h(t, x(t)) = 1$, the Cauchy type problem (5.1) has at least one positive nondecreasing integrable solution.

Proof. Integrating Cauchy problem (5.1) we obtain the integral equation

$$x(t) = p\frac{t^{\theta-1}}{\Gamma(\theta)} + \int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} [f_1(s, I^{\alpha} f_2(s, x(s)) + g_1(s, I^{\beta} g_2(s, x(s))] ds \tag{5.2}$$

which, by Theorem 3.1, has the desired solution, operating by D^{θ} on equation (5.2) we obtain the problem (5.1). So the equivalence between problem (5.1) and integral equation (5.2) is proven and then the results follow from Theorem 3.1. □

Acknowledgment

The authors thank the reviewers for their useful remarks on our paper.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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