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Research article

Existence of positive weak solutions for a nonlocal singular elliptic system

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Abstract: Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, and let $s \in (0,1)$ be such that $s < \frac{n}{2}$. We give sufficient conditions for the existence of a weak solution $(u, v) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ of the nonlocal singular system $(-\Delta)^s u = ad_\Omega^{-\gamma_1} v^{-\beta_1}$ in Ω, $(-\Delta)^s v = bd_\Omega^{-\gamma_2} u^{-\beta_2}$ in Ω, $u = v = 0$ in $\mathbb{R}^n \setminus \Omega$, $u > 0$ in Ω, $v > 0$ in Ω, where a and b are nonnegative bounded measurable functions such that $\inf_\Omega a > 0$ and $\inf_\Omega b > 0$. For the found weak solution (u, v), the behavior of u and v near $\partial\Omega$ is also investigated.

Keywords: fractional singular elliptic systems; positive solutions; sub and supersolutions; Schauder fixed point theorem
Mathematics Subject Classification: Primary 35A15; Secondary 35S15, 47G20, 46E35

1. Introduction and statement of the main results

Singular elliptic problems of the form

$$\begin{cases} -\Delta u = g(.,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.1}$$

with g such that $\lim_{s\to 0^+} g(x,s) = \infty$, have been extensively studied in the literature. Starting with the pioneering works [9, 10, 17], a vast amount of works was devoted to these problems, see for instance, [2, 5, 8, 12–14, 16, 19, 20, 25–27, 30], and [37].

In particular, [26] gives an existence result for classical solutions to problem (1.1), in the case when $g(.,u) = ad_\Omega^{-\gamma} u^{-\beta}$, with $0 \le a \in C^\sigma\left(\overline{\Omega}\right)$ for some $\sigma \in (0,1)$, $\beta > 0$ and $\gamma < 2$; and [13] gives an existence result for very weak solutions of the same problem. Notice that, in this case, $g(x,s)$ becomes singular at $s = 0$ and also at $x \in \partial\Omega$. Let us mention also that existence and uniqueness results for singular problems involving the p-laplacian operator on exterior domains were recently obtained in [6].

The existence of positive solutions of singular elliptic systems is addressed (in the local case), in [22], [29], and [1]. In [22] and [29] the results are obtained via the sub-supersolutions method, while in [1] (where appear also multiplicity results), the methods are variational and topological.

A systematic study of local singular elliptic problems, as well as additional references, can be found in [21, 33]. For a thorough introduction to the variational analysis of nonlinear problems described by nonlocal operators, we refer the reader to the reference [28].

Concerning nonlocal elliptic problems, let us mention that in [32], existence and multiplicity results were obtained for some singular elliptic problems driven by fractional powers of the p-Laplacian operator. In [11], global bifurcation problems for the fractional p-Laplacian were studied and, in [3], existence and multiplicity results were obtained for singular bifurcation problems of the form $(-\Delta)^s u = f(x) u^{-\beta} + \lambda u^p$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, $u > 0$ in Ω, in the case where Ω is a bounded and regular enough domain in $\mathbb{R}^n$, $s \in (0,1)$, $n > 2s, \beta > 0$, $p > 1$, $\lambda > 0$, and f is a nonnegative function belonging to a suitable Lebesgue space. There, it was proved the existence of at least two solutions for this problem when λ is positive and small enough. In [23], a more precise existence and multiplicity result was obtained for the same problem in the case when $f \equiv 1$ and the nonlinearity has critical growth at infinity, (i.e., when $p = 2_s^* - 1$, with $2_s^* = \frac{2n}{n-2s}$). In fact, in [23], it was proved that, under these assumptions, there exists $\Lambda > 0$ such that:
i) There exist exactly two positive solutions when $0 < \lambda < \Lambda$,
ii) There exists at least one positive solution when $\lambda = \Lambda$,
iii) No solution exists when $\lambda > \Lambda$.

Also, in [24], it was investigated the existence of positive weak solutions to problems like $(-\Delta)^s u = -au^{-\beta} + \lambda h$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, $u > 0$ in Ω, in the case where $s \in (0,1)$, $n > 2s, \beta \in (0,1)$, $\lambda > 0$, and where a and h are nonnegative bounded functions with $h \not\equiv 0$.

Our aim in this work is to obtain sufficient conditions on β_1, β_2, γ_1 and γ_2 for the existence of positive weak solutions to the following problem

$$\begin{cases} (-\Delta)^s u = a d_\Omega^{-\gamma_1} v^{-\beta_1} \text{ in } \Omega, \\ (-\Delta)^s v = b d_\Omega^{-\gamma_2} u^{-\beta_2} \text{ in } \Omega, \\ u = v = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \\ u, v \in H^s(\mathbb{R}^n) \\ u > 0 \text{ in } \Omega, \ v > 0 \text{ in } \Omega. \end{cases} \tag{1.2}$$

Here, and from now on, Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, $s \in (0,1)$, $d_\Omega := dist(., \partial\Omega)$, $\beta_1 \in (0,1)$, $\beta_2 \in (0,1)$, $\gamma_1 < 2s$, $\gamma_2 < 2s$, a and b belong to $L^\infty(\Omega)$, and satisfy $\inf_\Omega a > 0$ and $\inf_\Omega b > 0$.

Before stating our main results, let us recall the definition of the fractional Sobolev space $H^s(\mathbb{R}^n)$ and some well known facts related to this space. For $s \in (0,1)$ and $n \in \mathbb{N}$, let

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy < \infty \right\},$$

and for $u \in H^s(\mathbb{R}^n)$, let

$$\|u\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} u^2 + \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}},$$

and let

$$X_0^s(\Omega) := \{u \in H^s(\mathbb{R}^n) : u = 0 \ a.e. \text{ in } \mathbb{R}^n \setminus \Omega\},$$

and for $u \in X_0^s(\Omega)$, let

$$\|u\|_{X_0^s(\Omega)} := \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}.$$

With these norms, $H^s(\mathbb{R}^n)$ and $X_0^s(\Omega)$ are Hilbert spaces (see [36], Lemma 7), $C_c^\infty(\Omega)$ is dense in $X_0^s(\Omega)$ (see [18], Theorem 6). Also, $X_0^s(\Omega)$ is a closed subspace of $H^s(\mathbb{R}^n)$, and from the fractional Poincaré inequality (as stated e.g., in [15], Theorem 6.5; see also Remark 2.1 below), if $n > 2s$ then $\|.\|_{X_0^s(\Omega)}$ and $\|.\|_{H^s(\mathbb{R}^n)}$ are equivalent norms on $X_0^s(\Omega)$.

For $f \in L^1_{loc}(\Omega)$ we will write $f \in \left(X_0^s(\Omega)\right)'$ to mean that exists a positive constant c such that $\left|\int_\Omega f\varphi\right| \leq c\|u\|_{X_0^s(\Omega)}$ for any $\varphi \in X_0^s(\Omega)$. For $f \in \left(X_0^s(\Omega)\right)'$ we will write $((-\Delta)^s)^{-1} f$ for the unique weak solution u (given by the Riesz theorem) of the problem

$$\begin{cases} (-\Delta)^s u = f \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.3}$$

The notion of weak solution that we use in this work is the given by the following definition:

Definition 1.1. Let $s \in (0,1)$, let $f : \Omega \to \mathbb{R}$ be a Lebesgue measurable function such that $f\varphi \in L^1(\Omega)$ for any $\varphi \in X_0^s(\Omega)$. We say that $u : \mathbb{R}^n \to \mathbb{R}$ is a weak solution to the problem

$$\begin{cases} (-\Delta)^s u = f \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \end{cases}$$

if $u \in X_0^s(\Omega)$, $u = 0$ in $\mathbb{R}^n \setminus \Omega$ and, for any $\varphi \in X_0^s(\Omega)$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy = \int_\Omega f\varphi.$$

For $u \in X_0^s(\Omega)$ and $f \in L^1_{loc}(\Omega)$, we will write $(-\Delta)^s u \leq f$ in Ω (respectively $(-\Delta)^s u \geq f$ in Ω) to mean that, for any nonnegative $\varphi \in H_0^s(\Omega)$, it hold that $f\varphi \in L^1(\Omega)$ and

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy \leq \int_\Omega f\varphi \text{ (resp. } \geq \int_\Omega f\varphi).$$

For $u, v \in X_0^s(\Omega)$, we will write $(-\Delta)^s u \leq (-\Delta)^s v$ in Ω (respectively $(-\Delta)^s u \geq (-\Delta)^s v$ in Ω), to mean that $(-\Delta)^s (u - v) \leq 0$ in Ω (resp. $(-\Delta)^s (u - v) \geq 0$ in Ω).

If f and g are measurable real valued functions defined on Ω, we will write $f \approx g$ to mean that there exists a positive constant c, such that $c^1 f \leq g \leq cf$ $a.e.$ in Ω. We will write $f \lessapprox g$ (respectively $f \gtrapprox g$ in Ω) to mean that, for some positive constant c, $f \leq cg$ $a.e.$ in Ω (resp. $f \geq cg$ $a.e.$ in Ω).
Also, we set $\omega_0 := 2diam(\Omega)$. With these notations, our main results read as follow:

Theorem 1.2. *Let $\beta_1 \in (0,1)$, $\beta_2 \in (0,1)$, let $\gamma_1 < 2s$, $\gamma_2 < 2s$, and let a and b be functions in $L^\infty(\Omega)$ such that $a \approx 1$, $b \approx 1$. Assume that one of the following three conditions i) - iii) holds:*
i) $\gamma_1 + s\beta_1 < s$ and $\gamma_2 + s\beta_2 < s$,

ii) $\gamma_1 + s\beta_1 < s$ *and* $\gamma_2 + s\beta_2 = s$,
iii) $\gamma_1 + s\beta_1 = s$ *and* $\gamma_2 + s\beta_2 < s$.
Then problem has a weak solution $(u, v) \in X_0^s(\Omega) \times X_0^s(\Omega)$ *such that* $u \approx \vartheta_1$ *and* $v \approx \vartheta_2$ *in* Ω, *where*

$$\vartheta_1 := d_\Omega^s \text{ and } \vartheta_2 := d_\Omega^s \text{ if i) holds,}$$

$$\vartheta_1 := d_\Omega^s \text{ and } \vartheta_2 := d_\Omega^s \ln\left(\frac{\omega_0}{d_\Omega}\right) \text{ if ii) holds}$$

$$\vartheta_1 := d_\Omega^s \text{ and } \vartheta_2 := d_\Omega^s \text{ if iii) holds.}$$

Theorem 1.3. *Let* $\beta_1 \in (0,1)$, $\beta_2 \in (0,1)$, *let* $\gamma_1 < 2s$, $\gamma_2 < 2s$, *and let a and b be functions in* $L^\infty(\Omega)$ *such that* $a \approx 1$, $b \approx 1$. *Assume that* $\gamma_1 + s\beta_1 = s$ *and* $\gamma_2 + s\beta_2 = s$. *Then problem (1.2) has a weak solution* $(u, v) \in X_0^s(\Omega) \times X_0^s(\Omega)$ *such that* $d_\Omega^s \lessapprox u \lessapprox d_\Omega^s \ln\left(\frac{\omega_0}{d_\Omega}\right)$ *and* $d_\Omega^s \lessapprox v \lessapprox d_\Omega^s \ln\left(\frac{\omega_0}{d_\Omega}\right)$ *in* Ω.

Theorem 1.4. *Let* $\beta_1 \in (0,1)$, $\beta_2 \in (0,1)$, *let* $\gamma_1 < 2s$, $\gamma_2 < 2s$, *and let a and b be functions in* $L^\infty(\Omega)$ *such that* $a \approx 1$, $b \approx 1$. *Assume that one of the following two conditions holds:*
i) $\gamma_1 + s\beta_1 < s$ *and* $s < \gamma_2 + s\beta_2 < \min\left\{2s, \frac{1}{2} + s\right\}$,
ii) $s < \gamma_1 + s\beta_1 < \min\left\{2s, \frac{1}{2} + s\right\}$ *and* $\gamma_2 + \beta_2 s < s$.
Then problem (1.2) has a weak solution $(u, v) \in X_0^s(\Omega) \times X_0^s(\Omega)$ *such that* $u \approx \vartheta_1$ *and* $v \approx \vartheta_2$ *in* Ω, *where*

$$\vartheta_1 := d_\Omega^s \text{ and } \vartheta_2 := d_\Omega^{2s-\gamma_2-s\beta_2} \text{ if i) holds,}$$

$$\vartheta_1 := d_\Omega^{2s-\gamma_1-s\beta_1} \text{ and } \vartheta_2 := d_\Omega^s \text{ if ii) holds.}$$

The article is organized as follows: In Section 2, we quote some known facts and state some preliminary results. Lemma 2.2 quotes a result from [7], which gives accurate two side estimates for the values of the Green operator on negative powers of the distance function d_Ω (where the Green operator is the associated to the fractional laplacian with homogeneous Dirichlet condition on $\mathbb{R}^n \setminus \Omega$). Using this result and some of its consequences, Lemmas 2.4 and 2.5 states that, if the assumptions of Theorem 1.2 (respectively of Theorem 1.4) are assumed, and if ϑ_1 and ϑ_2 are as given in the statement of the respective Theorem, then $d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}$ and $d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}$ belong to $\left(X_0^s(\Omega)\right)'$, $((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\right) \approx \vartheta_1$, and $((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}\right) \approx \vartheta_2$ in Ω. Similarly, using again Lemma 2.2, Lemma 2.6 states that if $\gamma + \beta s = s$ and $\vartheta := d_\Omega^s \ln\left(\frac{\omega_0}{d_\Omega}\right)$, then $d_\Omega^{-\gamma}\vartheta^{-\beta}$ belongs to $\left(X_0^s(\Omega)\right)'$ and $d_\Omega^s \lessapprox ((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma}\vartheta^{-\beta}\right) \lessapprox \vartheta$.

In Section 3, Lemmas 3.1 and 3.2 adapt, to our setting, the ideas of the sub-supersolution method developed, for (local) elliptic systems, in ([29], Theorem 3.2).
In Lemma 3.1 we consider, for $\varepsilon > 0$ and under the hypothesis of either Theorem 1.2 or Theorem 1.4, the set $C_\varepsilon := \left\{(\zeta_1, \zeta_2) \in L^2(\Omega) \times L^2(\Omega) : \varepsilon\vartheta_i \leq \zeta_i \leq \varepsilon^{-1}\vartheta_i \text{ for } i = 1, 2\right\}$, and the operator $T : C_\varepsilon \to L^2(\Omega) \times L^2(\Omega)$ defined by

$$T(\zeta_1, \zeta_2) := \left(((-\Delta)^s)^{-1}\left(ad_\Omega^{-\gamma_1}\zeta_2^{-\beta_1}\right), ((-\Delta)^s)^{-1}\left(bd_\Omega^{-\gamma_2}\zeta_1^{-\beta_2}\right)\right);$$

and we show that T is a continuous and compact map and that, for ε small enough, $T(C_\varepsilon) \subset C_\varepsilon$. Lemma 3.2 says that the same conclusions hold if the hypothesis of Theorem 1.3 are assumed and C_ε is defined by

$$C_\varepsilon := \left\{(\zeta_1, \zeta_2) \in L^2(\Omega) \times L^2(\Omega) : \varepsilon\vartheta_i \leq \zeta_i \leq \varepsilon^{-1}\vartheta_i \text{ for } i = 1, 2\right\}.$$

Finally, Theorems 1.2, 1.3, and 1.4 are proved using the Schauder fixed point theorem combined with Lemmas 3.1 and 3.2.

2. Preliminaries

Remark 2.1. (i) (see e.g., [34], Proposition 4.1 and Corollary 4.2) The following comparison principle holds: If $u, v \in X_0^s(\Omega)$ and $(-\Delta)^s u \geq (-\Delta)^s v$ in Ω, then $u \geq v$ in Ω. In particular, if $v \in X_0^s(\Omega)$, $(-\Delta)^s v \geq 0$ in Ω, and $v \geq 0$ in $\mathbb{R}^n \setminus \Omega$, then $v \geq 0$ in Ω.
(ii) (see e.g., [34], Lemma 7.3) Let $f : \Omega \to \mathbb{R}$ be a nonnegative and nonidentically zero measurable function such that $f \in \left(X_0^s(\Omega)\right)'$, and let u be the weak solution of problem (1.3). Then u satisfies, for some positive constant c,

$$u \geq c d_\Omega^s \text{ in } \Omega. \tag{2.1}$$

(iii) (see e.g., [35], Proposition 1.1) If $f \in L^\infty(\Omega)$ then the weak solution u of problem (1.3) belongs to $C^s(\mathbb{R}^n)$. In particular, there exists a positive constant c such that

$$|u| \leq c d_\Omega^s \text{ in } \Omega. \tag{2.2}$$

For additional regularity resuls see, for instance, [4] and [28].
(iv) (Poincaré inequality, see [15], Theorem 6.5) Let $s \in (0,1)$, let $n > 2s$, and let $2_s^* := \frac{2n}{n-2s}$. Then there exists a positive constant $C = C(n,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\|f\|_{L^{2_s^*}(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x-y|^{n+sp}} dxdy.$$

(v) From the Hölder's inequality and the Poincaré inequality it follows that $v \in \left(X_0^s(\Omega)\right)'$ for any $v \in L^{(2_s^*)'}(\Omega)$.
(vi) (Hardy inequality, see [32], Theorem 2.1) There exists a positive constant c such that, for any $\varphi \in X_0^s(\Omega)$,

$$\left\|d_\Omega^{-s}\varphi\right\|_2 \leq c' \|\varphi\|_{X_0^s(\Omega)}. \tag{2.3}$$

(vii) Let $G : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$ be the Green function for $(-\Delta)^s$ in Ω, with homogeneous Dirichlet boundary condition on $\mathbb{R}^n \setminus \Omega$. Then, for any $f \in C\left(\overline{\Omega}\right)$, the weak solution u of problem (1.3) is given by $u(x) = \int_\Omega G(x,y) f(y)\, dy$ for $x \in \Omega$ and by $u(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$.

Let us recall the following result of [7]:

Lemma 2.2. *(See [7], Lemma 2) Let G be the Green function for* $(-\Delta)^s$ *in* Ω, *with homogeneous Dirichlet boundary condition on* $\mathbb{R}^n \setminus \Omega$. *Then*

$$\int_\Omega G(.,y)\, d_\Omega^{-\rho}(y)\, dy \approx d_\Omega^s \text{ if } \rho < s,$$

$$\int_\Omega G(.,y)\, d_\Omega^{-\rho}(y)\, dy \approx d_\Omega^s \ln\left(\frac{\omega_0}{d_\Omega}\right) \text{ if } \rho = s,$$

$$\int_\Omega G(.,y)\, d_\Omega^{-\rho}(y)\, dy \approx d_\Omega^{2s-\rho} \text{ if } s < \rho < s+1.$$

As a consequence of Lemma 2.2, we have the following

Lemma 2.3. *Let* $\rho \in \left[0, s + \frac{1}{2}\right)$. *Then* $d_\Omega^{-\rho} \in \left(X_0^s(\Omega)\right)'$ *and*

$$((-\Delta)^s)^{-1}\left(d_\Omega^{-\rho}\right) \approx d_\Omega^s \text{ if } \rho < s, \tag{2.4}$$
$$((-\Delta)^s)^{-1}\left(d_\Omega^{-\rho}\right) \approx d_\Omega^s \ln\left(\frac{\omega_0}{d_\Omega}\right) \text{ if } \rho = s,$$
$$((-\Delta)^s)^{-1}\left(d_\Omega^{-\rho}\right) \approx d_\Omega^{2s-\rho} \text{ if } s < \rho < s + \frac{1}{2}$$

Proof. Let $\varphi \in X_0^s(\Omega)$. Since $d_\Omega^{s-\rho} \in L^2(\Omega)$, the Holder and the Hardy inequalities give $\int_\Omega \left|d_\Omega^{-\rho}\varphi\right| \leq \int_\Omega d_\Omega^{s-\rho}\left|\frac{\varphi}{d_\Omega^s}\right| \leq c\left\|d_\Omega^{s-\rho}\right\|_2 \|\varphi\|_{X_0^s(\Omega)} \leq c' \|\varphi\|_{X_0^s(\Omega)}$ with c and c' positive constants independent of φ. Thus $d_\Omega^{-\rho} \in \left(X_0^s(\Omega)\right)'$.

Let G be the Green function for $(-\Delta)^s$ in Ω, with homogeneous Dirichlet boundary condition on $\mathbb{R}^n \setminus \Omega$. To prove (2.4) it is enough (thanks to Lemma 2.2) to show that $((-\Delta)^s)^{-1}\left(d_\Omega^{-\rho}\right) = \int_\Omega G(.,y)\, d_\Omega^{-\rho}(y)\, dy$. Let $\left\{\varepsilon_j\right\}_{j\in N} \subset (0,1)$ be a decreasing sequence such that $\lim_{j\to\infty} \varepsilon_j = 0$, and for $j \in \mathbb{N}$, let $u_{\varepsilon_j} \in X_0^s(\Omega)$ be the weak solution of the problem

$$(-\Delta)^s u_{\varepsilon_j} = \left(d_\Omega + \varepsilon_j\right)^{-\rho} \text{ in } \Omega, \tag{2.5}$$
$$u_{\varepsilon_j} = 0 \text{ on } \mathbb{R}^n \setminus \Omega.$$

Thus $u_{\varepsilon_j} = \int_\Omega G(.,y)\left(d_\Omega(y) + \varepsilon_j\right)^{-\rho} dy$ in Ω and, by Lemma 2.2, there exists a positive constant c, independent of j, such that $u_{\varepsilon_j} \leq c d_\Omega^s$ if $\rho < s$, $u_{\varepsilon_j} \leq c d_\Omega^s \ln\left(\frac{\omega_0}{d_\Omega}\right)$ if $\rho = s$, and $u_{\varepsilon_j} \leq c d_\Omega^{2s-\rho}$ if $s < \rho < \frac{1}{2} + s$. In particular, there exists a positive constant c' such that $\int_\Omega u_{\varepsilon_j} d_\Omega^{-\rho} \leq c'$ for all $j \in \mathbb{N}$. Let $u(x) := \lim_{j\to\infty} u_{\varepsilon_j}(x)$. By the monotone convergence theorem, $u(x) = \int_\Omega G(x,y)\, d_\Omega^{-\beta}(y)\, dy$. Taking u_{ε_j} as a test function in (2.5) we get

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(u_{\varepsilon_j}(x) - u_{\varepsilon_j}(y)\right)^2}{|x-y|^{n+2s}} dxdy = \int_\Omega u_{\varepsilon_j}(y)\left(d_\Omega(y) + \varepsilon_j\right)^{-\rho} dy$$
$$\leq \int_\Omega u_{\varepsilon_j} d_\Omega^{-\rho} \leq c',$$

with c' independent of j. For $j \in \mathbb{N}$, let U_{ε_j} and U be the functions, defined on $\mathbb{R}^n \times \mathbb{R}^n$, by

$$U_{\varepsilon_j}(x,y) := u_{\varepsilon_j}(x) - u_{\varepsilon_j}(y),\ U(x,y) := u(x) - u(y).$$

Then $\left\{U_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ is bounded in $\mathcal{H} = L^2\left(\mathbb{R}^n \times \mathbb{R}^n, \frac{1}{|x-y|^{n+2s}} dxdy\right)$. Thus, after pass to a subsequence if necessary, we can assume that $\left\{U_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ is weakly convergent in $\mathcal{H}$ to some $V \in \mathcal{H}$. Since $\left\{U_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ converges pointwise to U on $\mathbb{R}^n \times \mathbb{R}^n$, we conclude that $U \in \mathcal{H}$ and that $\left\{U_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ converges weakly to U in $\mathcal{H}$. Thus $u \in X_0^s(\Omega)$ and, for any $\varphi \in X_0^s(\Omega)$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy$$

$$= \lim_{j\to\infty}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(u_{\varepsilon_j}(x)-u_{\varepsilon_j}(y)\right)(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy$$
$$= \lim_{j\to\infty}\int_{\Omega}\left(d_\Omega+\varepsilon_j\right)^{-\beta}\varphi=\int_\Omega d_\Omega^{-\beta}\varphi,$$

Therefore $u=((-\Delta)^s)^{-1}\left(d_\Omega^{-\rho}\right)$ and so $((-\Delta)^s)^{-1}\left(d_\Omega^{-\rho}\right)=\int_\Omega G(x,y)\,d_\Omega^{-\beta}(y)\,dy$. □

Lemma 2.4. *Assume the hypothesis of Theorem 1.2 and let ϑ_1 and ϑ_2 be as given there. Then, in each one of the cases i) and ii) of Theorem 1.2, $d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\in\left(X_0^s(\Omega)\right)'$, $d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}\in\left(X_0^s(\Omega)\right)'$, $((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\right)\approx\vartheta_1$, and $((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}\right)\approx\vartheta_2$ in Ω.*

Proof. When the condition *i)* of Theorem 1.2 holds we have $\vartheta_1=\vartheta_2=d_\Omega^s$, and the lemma follows directly from Lemma 2.3. If the condition *ii)* holds, then $\gamma_1+s\beta_1<s$, $\gamma_2+s\beta_2=s$, $\vartheta_1=d_\Omega^s$ and $\vartheta_2=d_\Omega^s\ln\left(\frac{\omega_0}{d_\Omega}\right)$. Since $\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-\beta_1}\in L^\infty(\Omega)$ we have $d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}=d_\Omega^{-\gamma_1-s\beta_1}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-\beta_1}\lessapprox d_\Omega^{-\gamma_1-s\beta_1}$ and so, by Lemma 2.3, $d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\in\left(X_0^s(\Omega)\right)'$ and $((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\right)\lessapprox d_\Omega^s=\vartheta_1$ in Ω. Also, for $\delta>0$ we have $\inf_\Omega d_\Omega^{-\delta}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)\theta^{-\beta_1}>0$, and so

$$d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}=d_\Omega^{-\gamma_1-s\beta_1}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-\beta_1}=d_\Omega^{-(\gamma_1+s\beta_1-\delta)}d_\Omega^{-\delta}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-\beta_1}$$
$$\gtrapprox d_\Omega^{-(\gamma_1+s\beta_1-\delta)}\text{ in }\Omega.$$

Then, by the comparison principle of Remark 2.1 i), and by Lemma 2.3,

$$((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\right)\gtrapprox((-\Delta)^s)^{-1}\left(d_\Omega^{-(\gamma_1+s\beta_1-\delta)}\right)\approx d_\Omega^s=\vartheta_1$$

On the other hand, $d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}=d_\Omega^{-\gamma_2-s\beta_2}=d_\Omega^{-s}$, and so, again by Lemma 2.3, $d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}\in\left(X_0^s(\Omega)\right)'$ and $((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}\right)\approx d_\Omega^s\ln\left(\frac{\omega_0}{d_\Omega}\right)=\vartheta_2$ in Ω.
By replacing β_1, γ_1, ϑ_1 and ϑ_2 by β_2, γ_2, ϑ_2 and ϑ_1 respectively, the same argument proves the lemma in the case *iii).* □

Lemma 2.5. *Assume the hypothesis of Theorem 1.4 and let ϑ_1 and ϑ_2 be as given there. Then the conclusions of Lemma 2.4 remain true for ϑ_1 and ϑ_2.*

Proof. Consider the case when the condition *i)* of Theorem 1.4 holds, i.e., the case when $\gamma_1+s\beta_1<s$, $s<\gamma_2+s\beta_2<\min\left\{2s,\frac{1}{2}+s\right\}$, $\vartheta_1=d_\Omega^s$ and $\vartheta_2=d_\Omega^{2s-\gamma_2-s\beta_2}$. Then $d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}=d_\Omega^{-\gamma_1-\beta_1(2s-\gamma_2-s\beta_2)}$. Since $0<\gamma_1+\beta_1(2s-\gamma_2-s\beta_2)<\gamma_1+s\beta_1<s$, Lemma 2.3 gives that $d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\in\left(X_0^s(\Omega)\right)'$ and that $((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\right)\approx d_\Omega^s=\vartheta_1$ in Ω. On the other hand, $d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}=d_\Omega^{-\gamma_2-s\beta_2}$ and $s<\gamma_2+s\beta_2<\min\left\{2s,\frac{1}{2}+s\right\}$, and so, by Lemma 2.3,

$$d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}\in(X_0^s(\Omega))'\text{ and }((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}\right)\approx d_\Omega^{2s-\gamma_2-s\beta_2}=\vartheta_2\text{ in }\Omega.$$

The proof when the condition *ii)* holds is similar. □

Lemma 2.6. *Let* $\vartheta := d_\Omega^s \ln\left(\frac{\omega_0}{d_\Omega}\right)$. *If* $\gamma + s\beta = s$ *and* $\beta > 0$, *then* $d_\Omega^{-\gamma}\vartheta^{-\beta} \in \left(X_0^s(\Omega)\right)'$ *and* $d_\Omega^s \lessapprox ((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma}\vartheta^{-\beta}\right) \lessapprox \vartheta$ *in* Ω.

Proof. Since $\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-\beta} \in L^\infty(\Omega)$, we have $d_\Omega^{-\gamma}\vartheta^{-\beta} = d_\Omega^{-s}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-\beta} \lessapprox d_\Omega^{-s}$. Then, by Lemma 2.3 and the comparison principle, $d_\Omega^{-\gamma}\vartheta^{-\beta} \in \left(X_0^s(\Omega)\right)'$ and $((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma}\vartheta^{-\beta}\right) \lessapprox ((-\Delta)^s)^{-1}\left(d_\Omega^{-s}\right) \approx d_\Omega^s \ln\left(\frac{\omega_0}{d_\Omega}\right) = \vartheta$ in Ω. On the other hand, since $\inf_\Omega d_\Omega^{-\delta}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-\beta_1} > 0$ for any $\delta > 0$, we have

$$d_\Omega^{-\gamma}\vartheta^{-\beta} = d_\Omega^{-(\gamma+s\beta-\delta)} d_\Omega^{-\delta}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-\beta_1} \gtrapprox d_\Omega^{-(\gamma+s\beta-\delta)} \text{ in } \Omega,$$

and then so, by Lemma 2.3 and the comparison principle, $((-\Delta)^s)^{-1}\left(d_\Omega^{-\gamma}\vartheta^{-\beta}\right) \gtrapprox ((-\Delta)^s)^{-1}\left(d_\Omega^{-(\gamma+s\beta-\delta)}\right) \approx d_\Omega$. □

3. Proof of the main results

Lemma 3.1. *Assume the hypothesis of Theorem 1.2 (respectively of Theorem 1.4), and let* ϑ_1 *and* ϑ_2 *be as defined there. For* $\varepsilon > 0$, *let*

$$C_\varepsilon := \left\{(\zeta_1, \zeta_2) \in L^2(\Omega) \times L^2(\Omega) : \varepsilon\vartheta_i \leq \zeta_i \leq \frac{1}{\varepsilon}\vartheta_i \text{ for } i = 1, 2\right\},$$

and let $T : C_\varepsilon \to L^2(\Omega) \times L^2(\Omega)$ *be defined by*

$$T(\zeta_1, \zeta_2) = \left(((-\Delta)^s)^{-1}\left(a d_\Omega^{-\gamma_1}\zeta_2^{-\beta_1}\right), ((-\Delta)^s)^{-1}\left(b d_\Omega^{-\gamma_2}\zeta_1^{-\beta_2}\right)\right). \tag{3.1}$$

Then:
1) C_ε *is a closed convex set in* $L^2(\Omega) \times L^2(\Omega)$.
2) $T(C_\varepsilon) \subset C_\varepsilon$ *for any* ε *positive and small enough.*
3) $T : C_\varepsilon \to L^2(\Omega) \times L^2(\Omega)$ *is continuous*
4) $T : C_\varepsilon \to L^2(\Omega) \times L^2(\Omega)$ *is a compact map.*

Proof. Clearly C_ε is a closed convex set in $L^2(\Omega) \times L^2(\Omega)$. To see 2), note that, for any $(\zeta_1, \zeta_2) \in C_\varepsilon$, $a d_\Omega^{-\gamma_1}\zeta_2^{-\beta_1} \approx d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}$ and $b d_\Omega^{-\gamma_2}\zeta_1^{-\beta_2} \approx d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2}$ and then, when the hypothesis of Theorem 1.2 hold (respectively of Theorem 1.4 hold), Lemma 2.5 (resp. Lemma 2.4) gives that T is well defined on C_ε and that $T(C_\varepsilon) \subset X_0^s(\Omega) \times X_0^s(\Omega) \subset L^2(\Omega) \times L^2(\Omega)$.
To see *2)* observe that, for any $(\zeta_1, \zeta_2) \in C_\varepsilon$,

$$\varepsilon^{\beta_1} \inf_\Omega(a)\, d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1} \leq a d_\Omega^{-\gamma_1}\zeta_2^{-\beta_1} \leq \varepsilon^{-\beta_1} \sup_\Omega(a)\, d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1} \text{ in } \Omega,$$
$$\varepsilon^{\beta_2} \inf_\Omega(b)\, d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2} \leq a d_\Omega^{-\gamma_2}\zeta_1^{-\beta_2} \leq \varepsilon^{-\beta_2} \sup_\Omega(b)\, d_\Omega^{-\gamma_2}\vartheta_1^{-\beta_2} \text{ in } \Omega$$

and then, by the comparison principle and by Lemmas 2.5 and 2.4, there exist positive constants c_1 and c_2, both independent of ε, ζ_1 and ζ_2, such that

$$c_1\varepsilon^{\beta_1}\vartheta_1 \leq ((-\Delta)^s)^{-1}\left(\varepsilon^{\beta_1} \inf_\Omega(a)\, d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\right) \leq ((-\Delta)^s)^{-1}\left(a d_\Omega^{-\gamma_1}\zeta_2^{-\beta_1}\right)$$

$$\leq ((-\Delta)^s)^{-1}\left(\varepsilon^{-\beta_1}\sup_{\Omega}(a)\, d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\right) \leq c_2\varepsilon^{-\beta_1}\vartheta_1 \text{ in } \Omega$$

and, similarly,

$$c_1\varepsilon^{\beta_2}\vartheta_2 \leq ((-\Delta)^s)^{-1}\left(ad_\Omega^{-\gamma_2}\zeta_1^{-\beta_2}\right) \leq c_2\varepsilon^{-\beta_2}\vartheta_2 \text{ in } \Omega,$$

Since $0 < \beta_1 < 1$ and $0 < \beta_2 < 1$, for ε small enough we have

$$\varepsilon \leq c_1\varepsilon^{\beta_1}, \varepsilon \leq c_1\varepsilon^{\beta_2}, c_2\varepsilon^{-\beta_1} \leq \varepsilon^{-1}, \text{and } c_2\varepsilon^{-\beta_2} \leq \varepsilon^{-1}. \tag{3.2}$$

Thus, for such a ε, $T(C_\varepsilon) \subset C_\varepsilon$.

To prove that $T : C_\varepsilon \to L^2(\Omega) \times L^2(\Omega)$ is continuous, consider an arbitrary $(\zeta_1, \zeta_2) \in C_\varepsilon$, and a sequence $\left\{\left(\zeta_{1,j}, \zeta_{2,j}\right)\right\}_{j\in\mathbb{N}} \subset C_\varepsilon$ that converges to (ζ_1, ζ_2) in $L^2(\Omega) \times L^2(\Omega)$. After pass to a subsequence we can assume that $\left\{\left(\zeta_{1,j}, \zeta_{2,j}\right)\right\}_{j\in\mathbb{N}}$ converges to (ζ_1, ζ_2) *a.e.* in Ω. Since $0 \leq ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1} \leq \sup_\Omega(a)\,\varepsilon^{-\beta_1}d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}$ and since, by Lemmas 2.5 and 2.4, $d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1} \in \left(X_0^1(\Omega)\right)'$, it follows that $\left\{ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1}\right\}_{j\in\mathbb{N}}$ is bounded in $\left(X_0^1(\Omega)\right)'$. Similarly, $\left\{ad_\Omega^{-\gamma_2}\zeta_{1,j}^{-\beta_2}\right\}_{j\in\mathbb{N}}$ is bounded in $\left(X_0^1(\Omega)\right)'$. Let $\left(\xi_{1,j}, \xi_{2,j}\right) := T\left(\zeta_{1,j}, \zeta_{2,j}\right)$. Then $\left\{\left(\xi_{1,j}, \xi_{2,j}\right)\right\}_{j\in\mathbb{N}}$ is bounded in $X_0^1(\Omega) \times X_0^1(\Omega)$. After pass to a further subsequence if necessary, we can assume that there exists $(\xi_1, \xi_2) \in X_0^1(\Omega) \times X_0^1(\Omega)$ such that $\left\{\left(\xi_{1,j}, \xi_{2,j}\right)\right\}_{j\in\mathbb{N}}$ converges to (ξ_1, ξ_2) in $L^2(\Omega) \times L^2(\Omega)$, $\left\{\left(\xi_{1,j}, \xi_{2,j}\right)\right\}_{j\in\mathbb{N}}$ converges (ξ_1, ξ_2) *a.e.* in Ω, and $\left\{\xi_{1,j}, \xi_{2,j}\right\}_{j\in\mathbb{N}}$ converges weakly to (ξ_1, ξ_2) in $X_0^1(\Omega) \times X_0^1(\Omega)$. Let $\varphi \in X_0^1(\Omega)$. We have, for each j,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\xi_{1,j}(x) - \xi_{1,j}(y)\right)(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}dxdy = \int_\Omega ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1}\varphi, \tag{3.3}$$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\xi_{2,j}(x) - \xi_{2,j}(y)\right)(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}dxdy = \int_\Omega bd_\Omega^{-\gamma_2}\zeta_{1,j}^{-\beta_2}\varphi. \tag{3.4}$$

Now, $\left\{\left(\zeta_{1,j}, \zeta_{2,j}\right)\right\}_{j\in\mathbb{N}} \subset C_\varepsilon$ and so $\left|ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1}\varphi\right| \leq \varepsilon^{-\beta_1}\|a\|_\infty\left|d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}\varphi\right| \in L^1(\Omega)$. Therefore, by the Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty}\int_\Omega ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1}\varphi = \int_\Omega ad_\Omega^{-\gamma_1}\zeta_2^{-\beta_1}\varphi. \tag{3.5}$$

Similarly,

$$\lim_{j\to\infty}\int_\Omega bd_\Omega^{-\gamma_2}\zeta_{1,j}^{-\beta_2}\varphi = \int_\Omega bd_\Omega^{-\gamma_2}\zeta_1^{-\beta_2}\varphi. \tag{3.6}$$

Then

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(\xi_1(x) - \xi_1(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}dxdy = \int_\Omega ad_\Omega^{-\gamma_1}\zeta_2^{-\beta_1}\varphi, \tag{3.7}$$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(\xi_2(x) - \xi_2(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}dxdy = \int_\Omega bd_\Omega^{-\gamma_1}\zeta_1^{-\beta_1}\varphi. \tag{3.8}$$

and so $(\xi_1,\xi_2) = T(\zeta_1,\zeta_2)$. Then $\left\{T\left(\zeta_{1,j},\zeta_{2,j}\right)\right\}_{j\in\mathbb{N}}$ converges to $T(\zeta_1,\zeta_2)$ in $L^2(\Omega)\times L^2(\Omega)$. Thus, for any sequence $\left\{\left(\zeta_{1,j},\zeta_{2,j}\right)\right\}_{j\in\mathbb{N}} \subset C_\varepsilon$ that converges to (ζ_1,ζ_2) in $L^2(\Omega)\times L^2(\Omega)$, we have found a subsequence $\left\{\left(\zeta_{1,j_k},\zeta_{2,j_k}\right)\right\}_{k\in\mathbb{N}}$ such that $\left\{T\left(\zeta_{1,j_k},\zeta_{2,j_k}\right)\right\}_{k\in\mathbb{N}}$ converges to $T(\zeta_1,\zeta_2)$ in $L^2(\Omega)\times L^2(\Omega)$. Therefore T is continuous.

To see that $T : C_\varepsilon \to L^2(\Omega)\times L^2(\Omega)$ is a compact map, consider a bounded sequence $\left\{\left(\zeta_{1,j},\zeta_{2,j}\right)\right\}_{j\in\mathbb{N}} \subset C_\varepsilon$. Then $0 \le ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1} \le \sup_\Omega(a)\,\varepsilon^{-\beta_1}d_\Omega^{-\gamma_1}\vartheta_2^{-\beta_1}$ and so, as above, $\left\{ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1}\right\}_{j\in\mathbb{N}}$ is bounded in $\left(X_0^1(\Omega)\right)'$. Then $\left\{((-\Delta)^s)^{-1}\left(ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1}\right)\right\}_{j\in\mathbb{N}}$ is bounded in $X_0^1(\Omega)$. Thus there exists a subsequence $\left\{\left(\zeta_{1,j_k},\zeta_{2,j_k}\right)\right\}_{k\in\mathbb{N}}$ such that $\left(((-\Delta)^s)^{-1}\left(ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1}\right)\right\}_{j\in\mathbb{N}}$ converges in $L^2(\Omega)$. Since $0 \le ad_\Omega^{-\gamma_2}\zeta_{1,j_k}^{-\beta_2} \le \sup_\Omega(b)\,\varepsilon^{-\beta_2}d_\Omega^{-\gamma_1}\vartheta_1^{-\beta_2}$ we can repeat the above argument to obtain (after pass to a further subsequence if necessary) that $\left\{((-\Delta)^s)^{-1}\left(ad_\Omega^{-\gamma_2}\zeta_{1,j_k}^{-\beta_2}\right)\right\}_{k\in\mathbb{N}}$ converges in $L^2(\Omega)$. Therefore $\left\{T\left(\zeta_{1,j_k},\zeta_{2,j_k}\right)\right\}_{j\in\mathbb{N}}$ converges in $L^2(\Omega)\times L^2(\Omega)$. □

Lemma 3.2. *Assume the hypothesis of Theorem 1.3, and let ϑ be as given in Lemma 2.6. For $\varepsilon > 0$, let*

$$C_\varepsilon := \left\{(\zeta_1,\zeta_2)\in L^2(\Omega)\times L^2(\Omega) : \varepsilon d_\Omega \le \zeta_i \le \frac{1}{\varepsilon}\vartheta \text{ for } i=1,2\right\},$$

and let $T : C_\varepsilon \to L^2(\Omega)\times L^2(\Omega)$ be defined by (3.1). Then, for ε positive and small enough, the conclusions 1)-4) of Lemma 3.1 hold for C_ε and T.

Proof. The proof of the lemma is similar to the proof of Lemma 3.1. Clearly 1) holds. To prove 2), consider an arbitrary $(\zeta_1,\zeta_2)\in C_\varepsilon$. Since $0 \le ad_\Omega^{-\gamma_1}\zeta_2^{-\beta_1} \le \varepsilon^{-\beta_1}\sup_\Omega(a)\,d_\Omega^{-s}$ and $0 \le bd_\Omega^{-\gamma_2}\zeta_1^{-\beta_2} \le \varepsilon^{-\beta_1}\sup_\Omega(b)\,d_\Omega^{-s}$ *a.e.* in Ω, we have that $ad_\Omega^{-\gamma_1}\zeta_2^{-\beta_1}$ and $bd_\Omega^{-\gamma_2}\zeta_1^{-\beta_2}$ belong to $\left(X_0^s(\Omega)\right)'$. Then $T(\zeta_1,\zeta_2)$ is well defined and belongs to $L^2(\Omega)\times L^2(\Omega)$. Also, $\varepsilon^{\beta_1}\inf_\Omega(a)\,d_\Omega^{-\gamma_1}\vartheta^{-\beta_1} \le ad_\Omega^{-\gamma_1}\zeta_2^{-\beta_1} \le \varepsilon^{-\beta_1}\sup_\Omega(a)\,d_\Omega^{-s}$ in Ω, and $\varepsilon^{\beta_2}\inf_\Omega(b)\,d_\Omega^{-\gamma_2}\vartheta^{-\beta_2} \le ad_\Omega^{-\gamma_2}\zeta_1^{-\beta_2} \le \varepsilon^{-\beta_2}\sup_\Omega(b)\,d_\Omega^{-s}$ in Ω. Then, by the comparison principle and Lemma 2.6, there exist positive constants c_1 and c_2, both independent of ε, ζ_1, and ζ_2, such that

$$c_1\varepsilon^{\beta_1}d_\Omega^s \le ((-\Delta)^s)^{-1}\left(ad_\Omega^{-\gamma_1}\zeta_2^{-\beta_1}\right) \le c_2\varepsilon^{-\beta_1}\vartheta \text{ in } \Omega, \text{ and}$$
$$c_1\varepsilon^{\beta_2}d_\Omega^s \le ((-\Delta)^s)^{-1}\left(ad_\Omega^{-\gamma_2}\zeta_1^{-\beta_1}\right) \le c_2\varepsilon^{-\beta_2}\vartheta \text{ in } \Omega,$$

and so, as in Lemma 3.1, (3.2) holds for ε small enough. Then, for such a ε, $T(C_\varepsilon)\subset C_\varepsilon$.

To prove 3), consider an arbitrary $(\zeta_1,\zeta_2)\in C_\varepsilon$, and a sequence $\left\{\left(\zeta_{1,j},\zeta_{2,j}\right)\right\}_{j\in\mathbb{N}} \subset C_\varepsilon$ that converges to (ζ_1,ζ_2) in $L^2(\Omega)\times L^2(\Omega)$. After pass to a subsequence we can assume that $\left\{\left(\zeta_{1,j},\zeta_{2,j}\right)\right\}_{j\in\mathbb{N}}$ converges to (ζ_1,ζ_2) *a.e.* in Ω. Since $0 \le ad_\Omega^{--\gamma_1}\zeta_{2,j}^{-\beta_1} \le \sup_\Omega(a)\,\varepsilon^{-\beta_1}d_\Omega^{-s}$, and $0 \le bd_\Omega^{--\gamma_2}\zeta_{1,j}^{-\beta_2} \le \sup_\Omega(b)\,\varepsilon^{-\beta_1}d_\Omega^{-s}$, and taking into account that $d_\Omega^{-s}\in\left(X_0^1(\Omega)\right)'$, it follows that $\left\{ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1}\right\}_{j\in\mathbb{N}}$ and $\left\{bd_\Omega^{-\gamma_2}\zeta_{1,j}^{-\beta_2}\right\}_{j\in\mathbb{N}}$ are bounded in $\left(X_0^1(\Omega)\right)'$. Let $\left(\xi_{1,j},\xi_{2,j}\right) := T\left(\zeta_{1,j},\zeta_{2,j}\right)$. Then $\left\{\left(\xi_{1,j},\xi_{2,j}\right)\right\}_{j\in\mathbb{N}}$ is bounded in $X_0^1(\Omega)\times X_0^1(\Omega)$. Therefore, after pass to a further subsequence if necessary, we can assume that, for some $(\xi_1,\xi_2)\in X_0^1(\Omega)\times X_0^1(\Omega)$, $\left\{\left(\xi_{1,j},\xi_{2,j}\right)\right\}_{j\in\mathbb{N}}$ converges to (ξ_1,ξ_2) in $L^2(\Omega)\times L^2(\Omega)$ and *a.e.* in Ω; and that $\left\{\xi_{1,j},\xi_{2,j}\right\}_{j\in\mathbb{N}}$ converges weakly to (ξ_1,ξ_2) in $X_0^1(\Omega)\times X_0^1(\Omega)$. Let $\varphi\in X_0^1(\Omega)$. Since $\left\{\left(\zeta_{1,j},\zeta_{2,j}\right)\right\}_{j\in\mathbb{N}} \subset C_\varepsilon$ and $\gamma_1 + s\beta_1 = s$, we have $\left|ad_\Omega^{-\gamma_1}\zeta_{2,j}^{-\beta_1}\varphi\right| \le \varepsilon^{-\beta_1}\|a\|_\infty\left|d_\Omega^{-s}\varphi\right|$ and by the Hardy inequality, $\left|d_\Omega^{-s}\varphi\right|\in L^1(\Omega)$. Then, from

(3.3) and (3.4), the Lebesgue dominated convergence theorem gives (3.5). (3.6) is obtained similarly. Then (3.7) and (3.8) hold. Thus $(\xi_1, \xi_2) = T(\zeta_1, \zeta_2)$ and so $\left\{T\left(\zeta_{1,j}, \zeta_{2,j}\right)\right\}_{j\in\mathbb{N}}$ converges to $T(\zeta_1, \zeta_2)$ in $L^2(\Omega) \times L^2(\Omega)$. Then, as in the proof of Lemma 3.1, the conclusion that T is continuous is reached.
To see 4), consider a bounded sequence $\left\{\left(\zeta_{1,j}, \zeta_{2,j}\right)\right\}_{j\in\mathbb{N}} \subset C_\varepsilon$. We have $0 \le a d_\Omega^{-\gamma_1} \zeta_{2,j}^{-\beta_1} \le \sup_\Omega (a)\, \varepsilon^{-\beta_1} d_\Omega^{-s}$ and $0 \le b d_\Omega^{-\gamma_2} \zeta_{1,j_k}^{-\beta_2} \le \sup_\Omega (b)\, \varepsilon^{-\beta_2} d_\Omega^{-s}$ in Ω, and so $\left\{a d_\Omega^{-\gamma_1} \zeta_{2,j}^{-\beta_1}\right\}_{j\in\mathbb{N}}$ and $\left\{b d_\Omega^{-\gamma_2} \zeta_{1,j}^{-\beta_2}\right\}_{j\in\mathbb{N}}$ are bounded in $\left(X_0^1(\Omega)\right)'$. Now 4) follows as in the proof of Lemma 3.1 □

Proof of Theorems 1.2, 1.3, and 1.4. Theorems 1.2, 1.3 and 1.4 follow from the Schauder fixed point theorem (as stated e.g., in [31], Theorem 3.2.20), combined with Lemma 3.1 in the case of Theorems 1.2 and 1.4; and with Lemma 3.2 in the case of Theorem 1.3. □

Conflict of interest

The author declare no conflicts of interest in this paper.

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