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Research article

The new class $L_{z,p,E}$ of $s-$ type operators

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Abstract: The purpose of this study is to introduce the class of s-type $Z\left(u, v; l_p(E)\right)$ operators, which we denote by $L_{z,p,E}(X, Y)$, we prove that this class is an operator ideal and quasi-Banach operator ideal by a quasi-norm defined on this class. Then we define classes using other examples of s-number sequences. We conclude by investigating which of these classes are injective, surjective or symmetric.

Keywords: block sequence space; operator ideal; s-numbers; quasi-norm
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1. Introduction

In this study, the set of all natural numbers is represented by $\mathbb{N}$ and the set of all nonnegative real numbers is represented by $\mathbb{R}^+$.

If the dimension of the range space of a bounded linear operator is finite, it is called a finite rank operator [1].

Throughout this study, X and Y denote real or complex Banach spaces. The space of all bounded linear operators from X to Y is denoted by $\mathcal{B}(X, Y)$ and the space of all bounded linear operators from an arbitrary Banach space to another arbitrary Banach space is denoted by $\mathcal{B}$.

The theory of operator ideals is a very important field in functional analysis. The theory of normed operator ideals first appeared in 1950's in [2]. In functional analysis, many operator ideals are constructed via different scalar sequence spaces. An s- number sequence is one of the most important examples of this. The definition of s- numbers goes back to E. Schmidt [3], who used this concept in the theory of non-selfadjoint integral equations. In Banach spaces there are many different possibilities of defining some equivalents of s- numbers, namely Kolmogorov numbers, Gelfand numbers, approximation numbers, and several others. In the following years, Pietsch give the notion of s- number sequence to combine all s- numbers in one definition [4–6].

A map

$$S : K \to (s_r(K))$$

which assigns a non-negative scalar sequence to each operator is called an *s-number sequence* if for all Banach spaces X, Y, X_0 and Y_0 the following conditions are satisfied:

(i) $\|K\| = s_1(K) \geq s_2(K) \geq \ldots \geq 0$, for every $K \in \mathcal{B}(X, Y)$,
(ii) $s_{p+r-1}(L + K) \leq s_p(L) + s_r(K)$ for every $L, K \in \mathcal{B}(X, Y)$ and $p, r \in \mathbb{N}$,
(iii) $s_r(MLK) \leq \|M\| s_r(L) \|K\|$ for all $M \in \mathcal{B}(Y, Y_0)$, $L \in \mathcal{B}(X, Y)$ and $K \in \mathcal{B}(X_0, X)$, where X_0, Y_0 are arbitrary Banach spaces,
(iv) If $rank(K) \leq r$, then $s_r(K) = 0$,
(v) $s_{n-1}(I_n) = 1$, where I_n is the identity map of n-dimensional Hilbert space l_2^n to itself [7].

$s_r(K)$ denotes the $r - th$ $s-$number of the operator K.

Approximation numbers are frequently used examples of s-number sequence which is defined by Pietsch. $a_r(K)$, the r-th approximation number of a bounded linear operator is defined as

$$a_r(K) = \inf\{\, \|K - A\| : A \in \mathcal{B}(X, Y),\, rank(A) < r \,\},$$

where $K \in \mathcal{B}(X, Y)$ and $r \in \mathbb{N}$ [4]. Let $K \in \mathcal{B}(X, Y)$ and $r \in \mathbb{N}$. The other examples of s-number sequences are given in the following, namely *Gel′fand* number $(c_r(K))$, *Kolmogorov* number $(d_r(K))$, *Weyl* number $(x_r(K))$, *Chang* number $(y_r(K))$, *Hilbert* number $(h_r(K))$, etc. For the definitions of these sequences we refer to [1].

In the sequel there are some properties of $s-$number sequences.

When any isometric embedding $\mathcal{J} \in \mathcal{B}(Y, Y_0)$ is given and an s-number sequence $s = (s_r)$ satisfies $s_r(K) = s_r(\mathcal{J}K)$ for all $K \in \mathcal{B}(X, Y)$ the s-number sequence is called *injective* [8, p.90].

Proposition 1. *[8, p.90–94] The number sequences $(c_r(K))$ and $(x_r(K))$ are injective.*

When any quotient map $\mathcal{S} \in \mathcal{B}(X_0, X)$ is given and an s-number sequence $s = (s_r)$ satisfies $s_r(K) = s_r(K\mathcal{S})$ for all $K \in \mathcal{B}(X, Y)$ the s-number sequence is called *surjective* [8, p.95].

Proposition 2. *[8, p.95] The number sequences $(d_r(K))$ and $(y_r(K))$ are surjective.*

Proposition 3. *[8, p.115] Let $K \in \mathcal{B}(X, Y)$. Then the following inequalities hold:*

i) $h_r(K) \leq x_r(K) \leq c_r(K) \leq a_r(K)$ *and*
ii) $h_r(K) \leq y_r(K) \leq d_r(K) \leq a_r(K)$.

Lemma 1. *[5] Let $S, K \in \mathcal{B}(X, Y)$, then $|s_r(K) - s_r(S)| \leq \|K - S\|$ for $r = 1, 2, \ldots$.*

Let ω be the space of all real valued sequences. Any vector subspace of ω is called a sequence space.

In [9] the space $Z\left(u, v; l_p\right)$ is defined by Malkowsky and Savaş as follows:

$$Z\left(u, v; l_p\right) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left| u_n \sum_{k=1}^{n} v_k x_k \right|^p < \infty \right\}$$

where $1 < p < \infty$ and $u = (u_n)$ and $v = (v_n)$ are positive real numbers.

The Cesaro sequence space ces_p is defined as ([10, 11, 19])

$$ces_p = \left\{ x = (x_k) \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}, \quad 1 < p < \infty.$$

If an operator $K \in \mathcal{B}(X,Y)$ satisfies $\sum_{n=1}^{\infty} (a_n(K))^p < \infty$ for $0 < p < \infty$, K is defined as an l_p type operator in [4] by Pietsch. Afterwards *ces-p* type operators which is a new class obtained via Cesaro sequence space is introduced by Constantin [12]. Later on Tita in [14] proved that the class of l_p type operators and *ces-p* type operators coincide.

In [15], $\varsigma_p^{(s)}$, the class of $s-$type $Z\left(u, v; l_p\right)$ operators is given. For more information about sequence spaces and operator ideals we refer to [1,13,16,18,20].

Let X', the dual of X, be the set of continuous linear functionals on X. The map $x^* \otimes y : X \to Y$ is defined by

$$(x^* \otimes y)(x) = x^*(x)\, y$$

where $x \in X$, $x^* \in X'$ and $y \in Y$.

A subcollection $\mathfrak{I}$ of $\mathcal{B}$ is said to be an *operator ideal* if for each component $\mathfrak{I}(X,Y) = \mathfrak{I} \cap \mathcal{B}(X,Y)$ the following conditions hold:

(i) if $x^* \in X'$, $y \in Y$, then $x^* \otimes y \in \mathfrak{I}(X,Y)$,
(ii) if $L, K \in \mathfrak{I}(X,Y)$, then $L + K \in \mathfrak{I}(X,Y)$,
(iii) if $L \in \mathfrak{I}(X,Y)$, $K \in \mathcal{B}(X_0, X)$ and $M \in \mathcal{B}(Y, Y_0)$, then $MLK \in \mathfrak{I}(X_0, Y_0)$ [6].

Let $\mathfrak{I}$ be an operator ideal and $\rho : \mathfrak{I} \to \mathbb{R}^+$ be a function on $\mathfrak{I}$. Then, if the following conditions hold:

(i) if $x^* \in X'$, $y \in Y$, then $\rho(x^* \otimes y) = \|x^*\| \|y\|$;
(ii) if $\exists C \geq 1$ such that $\rho(L + K) \leq C[\rho(L) + \rho(K)]$;
(iii) if $L \in \mathfrak{I}(X,Y)$, $K \in \mathcal{B}(X_0, X)$ and $M \in \mathcal{B}(Y, Y_0)$, then $\rho(MLK) \leq \|M\| \rho(L) \|K\|$,

ρ is said to be a *quasi-norm* on the operator ideal $\mathfrak{I}$ [6].

For special case $C = 1$, ρ is a norm on the operator ideal $\mathfrak{I}$.

If ρ is a quasi-norm on an operator ideal $\mathfrak{I}$, it is denoted by $[\mathfrak{I}, \rho]$. Also if every component $\mathfrak{I}(X,Y)$ is complete with respect to the quasi-norm ρ, $[\mathfrak{I}, \rho]$ is called a *quasi-Banach operator ideal*.

Let $[\mathfrak{I}, \rho]$ be a quasi-normed operator ideal and $\mathcal{J} \in \mathcal{B}(Y, Y_0)$ be a isometric embedding. If for every operator $K \in \mathcal{B}(X,Y)$ and $\mathcal{J}K \in \mathfrak{I}(X, Y_0)$ we have $K \in \mathfrak{I}(X,Y)$ and $\rho(\mathcal{J}K) = \rho(K)$, $[\mathfrak{I}, \rho]$ is called an *injective quasi-normed operator ideal*. Furthermore, let $[\mathfrak{I}, \rho]$ be a quasi-normed operator ideal and $\mathcal{S} \in \mathcal{B}(X_0, X)$ be a quotient map. If for every operator $K \in \mathcal{B}(X,Y)$ and $K\mathcal{S} \in \mathfrak{I}(X_0, Y)$ we have $K \in \mathfrak{I}(X,Y)$ and $\rho(K\mathcal{S}) = \rho(K)$, $[\mathfrak{I}, \rho]$ is called an *surjective quasi-normed operator ideal* [6].

Let K' be the dual of K. An $s-$ number sequence is called *symmetric* (respectively, *completely symmetric*) if for all $K \in \mathcal{B}$, $s_r(K) \geq s_r(K')$ (respectively,$s_r(K) = s_r(K')$) [6].

Lemma 2. *[6] The approximation numbers are symmetric, i.e., $a_r(K') \leq a_r(K)$ for $K \in \mathcal{B}$.*

Lemma 3. *[6] Let $K \in \mathcal{B}$. Then*

$$c_r(K) = d_r(K') \quad \text{and} \quad c_r(K') \leq d_r(K).$$

In addition , if K is a compact operator then $c_r(K') = d_r(K)$.

Lemma 4. *[8] Let $K \in \mathcal{B}$. Then*

$$x_r(K) = y_r(K^{'}) \quad and \quad y_r(K^{'}) = x_r(K)$$

.

The dual of an operator ideal $\mathfrak{I}$ is denoted by $\mathfrak{I}^{'}$ and it is defined as [6]

$$\mathfrak{I}^{'}(X, Y) = \left\{K \in \mathcal{B}(X, Y) : K' \in \mathfrak{I}\left(Y^{'}, X^{'}\right)\right\}$$

.

An operator ideal $\mathfrak{I}$ is called symmetric if $\mathfrak{I} \subset \mathfrak{I}^{'}$ and is called completely symmetric if $\mathfrak{I} = \mathfrak{I}^{'}$ [6].
Let $E = (E_n)$ be a partition of finite subsets of the positive integers which satisfies

$$\max E_n < \min E_{n+1}$$

for $n \in \mathbb{N}^+$. In [21] Foroutannia defined the sequence space $l_p(E)$ by

$$l_p(E) = \left\{x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left|\sum_{j \in E_n} x_j\right|^p < \infty\right\}, \qquad (1 \leq p < \infty)$$

with the seminorm $|||\cdot|||_{p,E}$, which defined as:

$$|||x|||_{p,E} = \left(\sum_{n=1}^{\infty} \left|\sum_{j \in E_n} x_j\right|^p\right)^{\frac{1}{p}}.$$

For example, if $E_n = \{3n-2, 3n-1, 3n\}$ for all n, then $x = (x_n) \in l_p(E)$ if and only if $\sum\limits_{n=1}^{\infty} |x_{3n-2} + x_{3n-1} + x_{3n}|^p < \infty$. It is obvious that $|||\cdot|||_{p,E}$ is not a norm, since we have $|||x|||_{p,E} = 0$ while $x = (-1, 1, 0, 0, \ldots)$ and $E_n = \{3n-2, 3n-1, 3n\}$ for all n. For the particular case $E_n = \{n\}$ for $n \in \mathbb{N}^+$ we get $l_p(E) = l_p$ and $|||x|||_{p,E} = \|x\|_p$.

For more information about block sequence spaces, we refer the reader to [17, 22–25].

2. Results

Let $u = (u_n)$ and $v = (v_n)$ be positive real number sequences. In this section, by replacing l_p with $l_p(E)$ we get the sequence space $Z\left(u, v; l_p(E)\right)$ defined as follows:

$$Z\left(u, v; l_p(E)\right) = \left\{x \in \omega : \sum_{n=1}^{\infty} \left|u_n \sum_{k=1}^{n} \sum_{j \in E_k} v_j x_j\right|^p < \infty\right\}.$$

An operator $K \in \mathcal{B}(X, Y)$ is in the class of s-type $Z\left(u, v; l_p(E)\right)$ if

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j \in E_k} v_j s_j(K)\right)^p < \infty, \quad (1 < p < \infty).$$

The class of all s-type $Z\left(u, v; l_p(E)\right)$ operators is denoted by $L_{z,p,E}(X, Y)$.

In particular case if $E_n = \{n\}$ for $n = 1, 2, \ldots$, then the class $L_{z,p,E}(X, Y)$ reduces to the class $\varsigma_p^{(s)}$.

Conditions used in Theorem 1 hold throughout the remainder of the paper.

Theorem 1. *Fix* $1 < p < \infty$. *If* $\sum\limits_{n=1}^{\infty} (u_n)^p < \infty$ *and* $\mathcal{M} > 0$ *is such that* $v_{2k-1} + v_{2k} \leq \mathcal{M} v_k, \mathcal{M} > 0$ *for all* $k \in \mathbb{N}$, *then* $L_{z,p,E}$ *is an operator ideal.*

Proof. Let $x^* \in X'$ and $y \in Y$. Since the rank of the operator $x^* \otimes y$ is one, $s_n(x^* \otimes y) = 0$ for $n \geq 2$. By using this fact

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j \in E_k} v_j s_j (x^* \otimes y) \right)^p &= \sum_{n=1}^{\infty} (u_n)^p (v_1 s_1 (x^* \otimes y))^p \\
&= \sum_{n=1}^{\infty} (u_n)^p (v_1)^p \|x^* \otimes y\|^p \\
&= \sum_{n=1}^{\infty} (u_n)^p (v_1)^p \|x^*\|^p \|y\|^p \\
&< \infty.
\end{aligned}$$

Therefore $x^* \otimes y \in L_{z,p,E}(X, Y)$.

Let $L, K \in L_{z,p,E}(X, Y)$. Then

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j \in E_k} v_j s_j (L) \right)^p < \infty, \quad \sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j \in E_k} v_j s_j (K) \right)^p < \infty.$$

To show that $L + K \in L_{z,p,E}(X, Y)$, let us begin with

$$\begin{aligned}
\sum_{k=1}^{n} \sum_{j \in E_k} v_j s_j (L + K) &\leq \sum_{k=1}^{n} \left(\sum_{j \in E_k} v_{2j-1} s_{2j-1} (L + K) + \sum_{j \in E_k} v_{2j} s_{2j} (L + K) \right) \\
&\leq \sum_{k=1}^{n} \sum_{j \in E_k} \left(v_{2j-1} + v_{2j} \right) s_{2j-1} (L + K) \\
&\leq \mathcal{M} \sum_{k=1}^{n} \sum_{j \in E_k} v_j \left(s_j (L) + s_j (K) \right) \\
&\leq \mathcal{M} \sum_{k=1}^{n} \left(\sum_{j \in E_k} v_j s_j (L) + \sum_{j \in E_k} v_j s_j (K) \right).
\end{aligned}$$

By using Minkowski inequality we get;

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \left(\sum_{j \in E_k} v_j s_j (L + K) \right) \right)^p \right)^{\frac{1}{p}} \\
\leq\ & \mathcal{M} \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \left(\sum_{j \in E_k} v_j s_j (L) + \sum_{j \in E_k} v_j s_j (K) \right) \right)^p \right)^{\frac{1}{p}} \\
\leq\ & \mathcal{M} \left[\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j \in E_k} v_j s_j (L) \right)^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j \in E_k} v_j s_j (K) \right)^p \right)^{\frac{1}{p}} \right] < \infty.
\end{aligned}$$

Hence $L+K\in L_{z,p,E}(X,Y)$.

Let $M\in\mathcal{B}(Y,Y_0)$, $L\in L_{z,p,E}(X,Y)$ and $K\in\mathcal{B}(X_0,X)$. Then,

$$\begin{aligned}\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_js_j(MLK)\right)^p &\leq \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}\|M\|\,\|K\|\,v_js_j(L)\right)^p\\ &\leq \|M\|^p\,\|K\|^p\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_js_j(L)\right)^p\right)<\infty.\end{aligned}$$

So $MLK\in L_{z,p,E}(X_0,Y_0)$.

Therefore $L_{z,p,E}(X,Y)$ is an operator ideal. □

Theorem 2. $\|K\|_{z,p,E}=\dfrac{\left(\sum\limits_{n=1}^{\infty}\left(u_n\sum\limits_{k=1}^{n}\sum\limits_{j\in E_k}v_js_j(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}}v_1}$ *is a quasi-norm on the operator ideal* $L_{z,p,E}$.

Proof. Let $x^*\in X'$ and $y\in Y$. Since the rank of the operator $x^*\otimes y$ is one , $s_n(x^*\otimes y)=0$ for $n\geq 2$. Then

$$\begin{aligned}\frac{\left(\sum\limits_{n=1}^{\infty}\left(u_n\sum\limits_{k=1}^{n}\sum\limits_{j\in E_k}v_js_j(x^*\otimes y)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}}v_1} &= \frac{\left(\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)v_1^p\,\|x^*\otimes y\|^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}}v_1}\\ &= \|x^*\otimes y\|=\|x^*\|\,\|y\|\ .\end{aligned}$$

Therefore $\|x^*\otimes y\|_{z,p,E}=\|x^*\|\,\|y\|$.

Let $L,K\in L_{z,p,E}(X,Y)$. Then

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j\in E_k}v_js_j(L+K) &\leq \sum_{k=1}^{n}\sum_{j\in E_k}v_{2j-1}s_{2j-1}(L+K)+\sum_{j\in E_n}v_{2j}s_{2j}(L+K)\\ &\leq \sum_{k=1}^{n}\sum_{j\in E_k}\left(v_{2j-1}+v_{2j}\right)s_{2j-1}(L+K)\\ &\leq \mathcal{M}\sum_{k=1}^{n}\sum_{j\in E_k}v_j\left(s_j(L)+s_j(K)\right).\end{aligned}$$

By using Minkowski inequality we get;

$$\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_js_j(L+K)\right)^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty}\left(Mu_n\sum_{k=1}^{n}\sum_{j\in E_k}v_j\left(s_j(L)+s_j(K)\right)\right)^p\right)^{\frac{1}{p}}$$

$$\leq \mathcal{M}\left[\begin{array}{c}\left(\sum\limits_{n=1}^{\infty}\left(u_n\sum\limits_{k=1}^{n}\sum\limits_{j\in E_k}v_js_j(L)\right)^p\right)^{\frac{1}{p}}\\ +\left(\sum\limits_{n=1}^{\infty}\left(u_n\sum\limits_{k=1}^{n}\sum\limits_{j\in E_k}v_js_j(K)\right)^p\right)^{\frac{1}{p}}\end{array}\right].$$

Hence

$$\|L+K\|_{z,p,E} \leq \mathcal{M}\left(\|S\|_{z,p,E}+\|K\|_{z,p,E}\right).$$

Let $M \in \mathcal{B}(Y, Y_0)$, $L \in L_{z,p,E}(X, Y)$ and $K \in \mathcal{B}(X_0, X)$

$$\begin{aligned}\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_js_j(MLK)\right)^p\right)^{\frac{1}{p}} &\leq \left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}\|M\|\,\|K\|\,v_js_j(L)\right)^p\right)^{\frac{1}{p}}\\ &\leq \|M\|\,\|K\|\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_js_j(L)\right)^p\right)^{\frac{1}{p}}\\ &< \infty\end{aligned}$$

$$\|MLK\|_{z,p,E} \leq \|M\|\,\|K\|\,\|L\|_{z,p,E}.$$

Therefore $\|K\|_{z,p,E}$ is a quasi-norm on $L_{z,p,E}$. ◻

Theorem 3. *Let* $1 < p < \infty$. $\left[L_{z,p,E}(X, Y), \|K\|_{z,p,E}\right]$ *is a quasi-Banach operator ideal.*

Proof. Let X, Y be any two Banach spaces and $1 \leq p < \infty$. The following inequality holds

$$\|K\|_{z,p,E} = \frac{\left(\sum\limits_{n=1}^{\infty}\left(u_n\sum\limits_{k=1}^{n}\sum\limits_{j\in E_k}v_js_j(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}}v_1} \geq \|K\|$$

for $K \in L_{z,p,E}(X, Y)$.

Let (K_m) be Cauchy in $L_{z,p,E}(X, Y)$. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|K_m - K_l\|_{z,p,E} < \varepsilon \tag{2.1}$$

for all $m, l \geq n_0$. It follows that

$$\|K_m - K_l\| \leq \|K_m - K_l\|_{z,p,E} < \varepsilon.$$

Then (K_m) is a Cauchy sequence in $\mathcal{B}(X, Y)$. $\mathcal{B}(X, Y)$ is a Banach space since Y is a Banach space. Therefore $\|K_m - K\| \to 0$ as $m \to \infty$ for some $K \in \mathcal{B}(X, Y)$. Now we show that $\|K_m - K\|_{z,p,E} \to 0$ as $m \to \infty$ for $K \in L_{z,p,E}(X, Y)$.

The operators $K_l - K_m$, $K - K_m$ are in the class $\mathcal{B}(X, Y)$ for $K_m, K_l, K \in \mathcal{B}(X, Y)$.

$$|s_n(K_l - K_m) - s_n(K - K_m)| \leq \|K_l - K_m - (K - K_m)\| = \|K_l - K\|.$$

Since $K_l \to K$ as $l \to \infty$ we obtain

$$s_n(K_l - K_m) \to s_n(K - K_m) \tag{2.2}$$

It follows from (2.1) that the statement

$$\|K_m - K_l\|_{z,p,E} = \frac{\left(\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K_m - K_l)\right)^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}} v_1} < \varepsilon$$

holds for all $m, l \geq n_{0.}$ We obtain from (2.2) that

$$\frac{\left(\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K_m - K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}} v_1} \leq \varepsilon.$$

Hence we have

$$\|K_m - K\|_{z,p,E} < \varepsilon \qquad \text{for all } m \geq n_{0.}$$

Finally we show that $K \in L_{z,p,E}(X,Y)$.

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K)\right)^p &\leq \sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_{2j-1}s_{2j-1}(K) + u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_{2j}s_{2j}(K)\right)^p \\
&\leq \sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k}\left(v_{2j-1} + v_{2j}\right) s_{2j-1}(K - K_m + K_m)\right)^p \\
&\leq \mathcal{M}\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j\left(s_j(K - K_m) + s_j(K_m)\right)\right)^p
\end{aligned}$$

By using Minkowski inequality; since $K_m \in L_{z,p,E}(X,Y)$ for all m and $\|K_m - K\|_{z,p,E} \to 0$ as $m \to \infty$, we have

$$\begin{aligned}
&\mathcal{M}\left(\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j\left(s_j(K - K_m) + s_j(K_m)\right)\right)^p\right)^{\frac{1}{p}} \\
\leq\ &\mathcal{M}\left[\begin{array}{c}\left(\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K - K_m)\right)^p\right)^{\frac{1}{p}} \\ +\left(\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K_m)\right)^p\right)^{\frac{1}{p}}\end{array}\right] < \infty
\end{aligned}$$

which means $K \in L_{z,p,E}(X,Y)$. □

Definition 1. *Let $\mu = (\mu_i(K))$ be one of the sequences $s = (s_n(K))$, $c = (c_n(K))$, $d = (d_n(K))$, $x = (x_n(K))$, $y = (y_n(K))$ and $h = (h_n(K))$. Then the space $L^{(\mu)}_{z,p,E}$ generated via $\mu = (\mu_i(K))$ is defined as*

$$L^{(\mu)}_{z,p,E}(X,Y) = \left\{K \in \mathcal{B}(X,Y) : \sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j\mu_j(K)\right)^p < \infty, (1 < p < \infty)\right\}.$$

And the corresponding norm $\|K\|^{(\mu)}_{z,p,E}$ for each class is defined as

$$\|K\|^{(\mu)}_{z,p,E} = \frac{\left(\sum\limits_{n=1}^{\infty}\left(u_n \sum\limits_{k=1}^{n}\sum\limits_{j\in E_k} v_j\mu_j(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}} v_1}.$$

Proposition 4. *The inclusion $L^{(a)}_{z,p,E} \subseteq L^{(a)}_{z,q,E}$ holds for $1 < p \le q < \infty$.*

Proof. Since $l_p \subseteq l_q$ for $1 < p \le q < \infty$ we have $L^{(a)}_{z,p,E} \subseteq L^{(a)}_{z,q,E}$. □

Theorem 4. *Let $1 < p < \infty$. The quasi-Banach operator ideal $\left[L^{(s)}_{z,p,E}, \|K\|^{(s)}_{z,p,E}\right]$ is injective, if the sequence $s_n(K)$ is injective.*

Proof. Let $1 < p < \infty$ and $K \in \mathcal{B}(X,Y)$ and $\mathcal{J} \in \mathcal{B}(Y,Y_0)$ be any isometric embedding. Suppose that $\mathcal{J}K \in L^{(s)}_{z,p,E}(X,Y_0)$. Then

$$\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(\mathcal{J}K)\right)^p < \infty$$

Since $s = (s_n)$ is injective, we have

$$s_n(K) = s_n(\mathcal{J}K) \text{ for all } K \in \mathcal{B}(X,Y), n = 1,2,\ldots. \tag{2.3}$$

Hence we get

$$\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K)\right)^p = \sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(\mathcal{J}K)\right)^p < \infty$$

Thus $K \in L^{(s)}_{z,p,E}(X,Y)$ and we have from (2.3)

$$\begin{aligned}\|\mathcal{J}K\|^{(s)}_{z,p,E} &= \frac{\left(\sum\limits_{n=1}^{\infty}\left(u_n \sum\limits_{k=1}^{n}\sum\limits_{j\in E_k} v_j s_j(\mathcal{J}K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}} v_1} \\ &= \frac{\left(\sum\limits_{n=1}^{\infty}\left(u_n \sum\limits_{k=1}^{n}\sum\limits_{j\in E_k} v_j s_j(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}} v_1} = \|K\|^{(s)}_{z,p,E}\end{aligned}$$

Hence the operator ideal $\left[L^{(s)}_{z,p,E}, \|K\|^{(s)}_{z,p,E}\right]$ is injective. □

Conclusion 1. *[8, p.90–94] Since the number sequences $(c_n(K))$ and $(x_n(K))$ are injective , the quasi-Banach operator ideals $\left[L^{(c)}_{z,p,E}, \|K\|^{(c)}_{z,p,E}\right]$ and $\left[L^{(x)}_{z,p,E}, \|K\|^{(x)}_{z,p,E}\right]$ are injective.*

Theorem 5. *Let $1 < p < \infty$. The quasi-Banach operator ideal $\left[L^{(s)}_{z,p,E}, \|K\|^{(s)}_{z,p,E}\right]$ is surjective, if the sequence $(s_n(K))$ is surjective.*

Proof. Let $1 < p < \infty$ and $K \in \mathcal{B}(X,Y)$ and $\mathcal{S} \in \mathcal{B}(X_0, X)$ be any quotient map. Suppose that $K\mathcal{S} \in L^{(s)}_{z,p,E}(X_0, Y)$. Then

$$\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K\mathcal{S})\right)^p < \infty.$$

Since $s = (s_n)$ is surjective, we have

$$s_n(K) = s_n(K\mathcal{S}) \text{ for all } K \in \mathcal{B}(X,Y), n = 1,2,\ldots. \tag{2.4}$$

Hence we get

$$\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K)\right)^p = \sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K\mathcal{S})\right)^p < \infty.$$

Thus $K \in L^{(s)}_{z,p,E}(X,Y)$ and we have from (2.4)

$$\begin{aligned}
\|K\mathcal{S}\|^{(s)}_{z,p,E} &= \frac{\left(\sum\limits_{n=1}^{\infty}\left(u_n \sum\limits_{k=1}^{n}\sum\limits_{j\in E_k} v_j s_j(K\mathcal{S})\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}} v_1} \\
&= \frac{\left(\sum\limits_{n=1}^{\infty}\left(u_n \sum\limits_{k=1}^{n}\sum\limits_{j\in E_k} v_j s_j(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}} v_1} = \|K\|^{(s)}_{z,p,E}.
\end{aligned}$$

Hence the operator ideal $\left[L^{(s)}_{z,p,E}, \|K\|^{(s)}_{z,p,E}\right]$is surjective. □

Conclusion 2. *[8, p.95] Since the number sequences $(d_n(K))$ and $(y_n(K))$ are surjective , the quasi-Banach operator ideals $\left[L^{(d)}_{z,p,E}, \|K\|^{(d)}_{z,p,E}\right]$ and $\left[L^{(y)}_{z,p,E}, \|K\|^{(y)}_{z,p,E}\right]$ are surjective.*

Theorem 6. *Let $1 < p < \infty$. Then the following inclusion relations holds:*

i $L^{(a)}_{z,p,E} \subseteq L^{(c)}_{z,p,E} \subseteq L^{(x)}_{z,p,E} \subseteq L^{(h)}_{z,p,E}$
ii $L^{(a)}_{z,p,E} \subseteq L^{(d)}_{z,p,E} \subseteq L^{(y)}_{z,p,E} \subseteq L^{(h)}_{z,p,E}$.

Proof. Let $K \in L^{(a)}_{z,p,E}$. Then

$$\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n}\sum_{j\in E_k} v_j s_j(K)\right)^p < \infty$$

where $1 < p < \infty$. And from Proposition 3, we have;

$$\begin{aligned}\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jh_j(K)\right)^p &\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jx_j(K)\right)^p\\ &\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jc_j(K)\right)^p\\ &\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_ja_j(K)\right)^p\\ &< \infty\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jh_j(K)\right)^p &\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jy_j(K)\right)^p\\ &\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jd_j(K)\right)^p\\ &\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_ja_j(K)\right)^p\\ &< \infty.\end{aligned}$$

So it is shown that the inclusion relations are satisfied. □

Theorem 7. *For* $1 < p < \infty$, $L^{(a)}_{z,p,E}$ *is a symmetric operator ideal and* $L^{(h)}_{z,p,E}$ *is a completely symmetric operator ideal.*

Proof. Let $1 < p < \infty$.

Firstly, we show that $L^{(a)}_{z,p,E}$ is symmetric in other words $L^{(a)}_{z,p,E} \subseteq \left(L^{(a)}_{z,p,E}\right)'$ holds. Let $K \in L^{(a)}_{z,p,E}$. Then

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_ja_j(K)\right)^p < \infty.$$

It follows from [6, p.152] $a_n\left(K'\right) \le a_n(K)$ for $K \in \mathcal{B}$. Hence we get

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_ja_j(T')\right)^p \le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_ja_j(K)\right)^p < \infty.$$

Therefore $K \in \left(L^{(a)}_{z,p,E}\right)'$. Thus $L^{(a)}_{z,p,E}$ is symmetric.

Now we prove that the equation $L^{(h)}_{z,p,E} = \left(L^{(h)}_{z,p,E}\right)'$ holds. It follows from [8, p.97] that $h_n\left(K'\right) = h_n(K)$ for $K \in \mathcal{B}$. Then we can write

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jh_j(K')\right)^p = \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jh_j(K)\right)^p.$$

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jh_j(K')\right)^p=\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}\sum_{j\in E_k}v_jh_j(K)\right)^p.$$

Hence $L^{(h)}_{z,p,E}$ is completely symmetric. □

Theorem 8. *Let* $1<p<\infty$. *The equation* $L^{(c)}_{z,p,E}=\left(L^{(d)}_{z,p,E}\right)^{'}$ *and the inclusion relation* $L^{(d)}_{z,p,E}\subseteq\left(L^{(c)}_{z,p,E}\right)^{'}$ *holds. Also, for a compact operator K,* $K\in L^{(d)}_{z,p,E}$ *if and only if* $K^{'}\in\left(L^{(c)}_{z,p,E}\right)$.

Proof. Let $1<p<\infty$. For $K\in\mathcal{B}$ it is known from [8] that $c_n(K)=d_n\left(K^{'}\right)$ and $c_n\left(K^{'}\right)\leq d_n(K)$. Also, when K is a compact operator, the equality $c_n\left(K^{'}\right)=d_n(K)$ holds. Thus the proof is clear. □

Theorem 9. $L^{(x)}_{z,p,E}=\left(L^{(y)}_{z,p,E}\right)^{'}$ *and* $L^{(y)}_{z,p,E}=\left(L^{(x)}_{z,p,E}\right)^{'}$ *hold for* $1<p<\infty$.

Proof. Let $1<p<\infty$. For $K\in\mathcal{B}$ we have from [8] that $x_n(K)=y_n\left(K^{'}\right)$ and $y_n(K)=x_n\left(K^{'}\right)$. Thus the proof is clear. □

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Conflict of interest

The authors declare no conflict of interest.

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