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Research article

On (complete) normality of m-pF subalgebras in BCK/BCI-algebras

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Abstract: In this paper, we introduce the concepts of normal m-polar fuzzy subalgebras, maximal m-polar fuzzy subalgebras and completely normal m-polar fuzzy subalgebras in BCK/BCI-algebras. We discuss some properties of normal (resp., maximal, completely normal) m-polar fuzzy subalgebras. We prove that any non-constant normal m-polar fuzzy subalgebra which is a maximal element of $(\mathcal{NO}(X), \subseteq)$ takes only the values $\widehat{0} = (0, 0, ..., 0)$ and $\widehat{1} = (1, 1, ..., 1)$, and every maximal m-polar fuzzy subalgebra is completely normal. Moreover, we state an m-polar fuzzy characteristic subalgebra in BCK/BCI-algebras.

Keywords: BCK/BCI-algebras; m-polar fuzzy sets; m-polar fuzzy subalgebras; normal m-polar fuzzy subalgebras; completely normal m-polar fuzzy subalgebras; maximal m-polar fuzzy subalgebras
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1. Introduction

Imai and Iséki [14] in 1966 introduced a significant algebraic structure called a BCK-algebra. In the same year, Iséki [15] introduced the notion of a BCI-algebra as a generalization of a BCK-algebra. Today, BCK/BCI-algebras have been extensively studied by several researchers and they have been applied to several fields of mathematics, such as fuzzy set theory, group theory, ring theory, functional analysis, and so on.

The theory of fuzzy sets (FSs), initiated by Zadeh [28] in 1965, has obtained more attention by authors in a wide range of scientific domains, including decision theory, robotics, management sciences and numerous other disciplines. In 1986, Atanassov [11] introduced the notion of intuitionistic fuzzy sets (IFSs) in which there are two functions, membership function and non-membership function. In 1994, Zhang [29] introduced the new notion of bipolar fuzzy sets (BFSs) in which there are two functions, positive membership function and negative membership function. Applications of BFSs and IFSs appear in different areas, including decision-making, optimization problems, and medical diagnosis. In algebraic structures, Xi [27] implemented the idea of FSs into BCK/BCI-algebras and gave the notions of fuzzy subalgebras and ideals, while Lee [16] generalized the Xi's idea and gave the notions of bipolar fuzzy subalgebras and ideals in BCK/BCI-algebras. After that, many researchers used the ideas of fuzzy sets and hybrid models of fuzzy sets and gave several results in various algebraic structures, for instance BCK/BCI-algebras [6–8, 22, 23, 25, 26], B-algebras [18, 24], G-algebras [21] and BG-algebras [19, 20]. In several real-life situations, information sometimes comes from m factors ($m \geq 2$), that is, multi-attribute data arise which cannot be handled using the existing ideals (e.g., fuzzy ideals, bipolar fuzzy ideals, etc.). For the time being, experts trust that the real world is proceeding to multipolarity. Multi-polar vagueness in information performs a crucial role in different domains of the sciences, such as technology and neurobiology.

In view of this motivation, the notion of m-polar fuzzy (m-pF) sets was initiated by Chen et al. [12] in 2014 which is a generalization of the BFSs. In an m-pF set, the degree of membership of an object ranges over $[0, 1]^m$, which depicts m distinct characteristics of the object. Akram et al. [3], for the first time, introduced the new concept of m-pF Lie subalgebras of a Lie algebra, which is a generalization of BF Lie subalgebras. Al-Masarwah and Ahmad [9] defined the idea of m-pF subalgebras and ideals in BCK/BCI-algebras and described several properties of m-pF BCK/BCI-algebras. After that, many authors applied the idea of m-pF sets to other mathematical theories such as groups [13], Lie algebras [2], BCK/BCI-algebras [10], matroid theory [17] and graph Theory [1, 4, 5].

In this paper, we establish the normalization of m-pF subalgebras in BCK/BCI-algebras. We introduce the concepts of normal m-pF subalgebras, maximal m-pF subalgebras and completely normal m-pF subalgebras in BCK/BCI-algebras. We discuss some properties of normal (resp., maximal, completely normal) m-pF subalgebras. We prove that any non-constant normal m-pF subalgebra which is a maximal element of $(\mathcal{NO}(X), \subseteq)$ takes only the values $\widehat{0} = (0, 0, ..., 0)$ and $\widehat{1} = (1, 1, ..., 1)$, and every maximal m-pF subalgebra is completely normal. Moreover, we state an m-pF characteristic subalgebra in BCK/BCI-algebras.

2. Preliminaries

We first recall some elementary aspects which are used to present the paper. In this paper, X always denotes a BCK/BCI-algebra without any specifications.

By a BCI-algebra we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,

(a2) $(x * (x * y)) * y = 0$,

(a3) $x * x = 0$,

(a4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom (a5) $0 * x = 0$ for all $x \in X$, then X is called a BCK-algebra. A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$. Any BCK/BCI-algebra X satisfies the following axioms:

(1) $(x * y) * z = (x * z) * y$,

(2) $x * y \leq x$,

(3) $(x * y) * z \leq (x * z) * (y * z)$,

(4) $x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x$.

for all $x, y, z \in X$. A non-empty subset I of X is called a subalgebra of X if $x * y \in I$ for any $x, y \in I$.

Definition 2.1. [12] A function $\widehat{\mathcal{H}}$ is defined from $X(\neq \phi)$ to a m-tuple of real number in $[0, 1]$ is said to be an m-pF set, that is, a mapping $\widehat{\mathcal{H}} : X \rightarrow [0, 1]^m$. The membership degree of any element $x \in X$ is denoted by

$$\widehat{\mathcal{H}}(x) = (p_1 \circ \widehat{\mathcal{H}}(x), p_2 \circ \widehat{\mathcal{H}}(x), ..., p_m \circ \widehat{\mathcal{H}}(x))$$

where $p_j \circ \widehat{\mathcal{H}} : [0, 1]^m \rightarrow [0, 1]$ is defined the j-th projection mapping. The smallest and largest values in $[0, 1]^m$ are $\widehat{0} = (0, 0, ..., 0)$ and $\widehat{1} = (1, 1, ..., 1)$, respectively.

By $K_{\widehat{\mathcal{H}}}$ we denote the set $\{x \in X \mid \widehat{\mathcal{H}}(x) = \widehat{\mathcal{H}}(0)\}$. For any m-pF sets $\widehat{\mathcal{H}}$ and $\widehat{C}$ in a set X, we define

$$\widehat{\mathcal{H}} \subseteq \widehat{C} \Leftrightarrow \widehat{\mathcal{H}}(x) \leq \widehat{C}(x), \forall x \in X.$$

Definition 2.2. [9] Let $\widehat{\mathcal{H}}$ be an m-pF set of X. Then, $\widehat{\mathcal{H}}_{\widehat{t}} = \{x \in X \mid \widehat{\mathcal{H}}(x) \geq \widehat{t}\}$ is said to be the level cut subset of $\widehat{\mathcal{H}}$ for all $\widehat{t} \in (0, 1]^m$.

If M is a nonempty subsets of X, then the m-pF characteristic function $\widehat{C}_M$ denoted and defined by

$$\widehat{C}_M(x) = \begin{cases} \widehat{1} = (1, 1, ..., 1), & \text{if } x \in M \\ \widehat{0} = (0, 0, ..., 0), & \text{otherwise.} \end{cases}$$

Clearly, the m-pF characteristic function of any subset of X is an m-pF subset of X.

3. Normality of m-polar fuzzy subalgebras

In the current section, we present the concepts of normal m-pF subalgebras, maximal m-pF subalgebras and completely normal m-pF subalgebras in X and investigate several fundamental properties.

Definition 3.1. [9] An m-pF set $\widehat{\mathcal{H}}$ in X is called an m-pF subalgebra of X if

$$\widehat{\mathcal{H}}(x * y) \geq \inf\{\widehat{\mathcal{H}}(x), \widehat{\mathcal{H}}(y)\}, \forall x, y \in X.$$

Example 3.1. [9] Consider a BCK-algebra $X = \{0, a, b, c\}$ with the Cayley table which is given in Table 1.

Table 1. Cayley table for the operation $*$.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $\widehat{\mathcal{H}} : X \to [0,1]^m$ be an m-pF set in X defined by:

$$\widehat{\mathcal{H}}(x) = \begin{cases} (0.8, 0.8, ..., 0.8), & \text{if } x = 0, a, c \\ (0.5, 0.5, ..., 0.5), & \text{if } x = b. \end{cases}$$

By routine computations, we can verify that $\widehat{\mathcal{H}}$ is an m-pF subalgebra of X.

Lemma 3.2 ([9]). *If $\widehat{\mathcal{H}}$ is an m-pF subalgebra of X, then $\widehat{\mathcal{H}}(0) \geq \widehat{\mathcal{H}}(x)$, $\forall x \in X$.*

Theorem 3.3. *Let $\phi \neq M \subseteq X$ and let $\widehat{\mathcal{H}}_M : X \to [0,1]^m$ be an m-pF set in X defined by*

$$\widehat{\mathcal{H}}_M(x) = \begin{cases} \hat{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_m), & \text{if } x \in M \\ \hat{\beta} = (\beta_1, \beta_2, ..., \beta_m), & \text{otherwise,} \end{cases}$$

for all $x \in X$ and $\hat{\alpha}, \hat{\beta} \in [0,1]^m$ with $\hat{\alpha} > \hat{\beta}$. Then, $\widehat{\mathcal{H}}_M$ is an m-pF subalgebra of X if and only if M is a subalgebra of X. Moreover, in this case $K_{\widehat{\mathcal{H}}_M} = \{x \in X \mid \widehat{\mathcal{H}}_M(x) = \widehat{\mathcal{H}}_M(0)\} = M$.

Proof. Let $\widehat{\mathcal{H}}$ be an m-pF subalgebra of X. Let $x, y \in X$ be such that $x, y \in M$. Then, we have

$$\begin{aligned} \widehat{\mathcal{H}}_M(x * y) &\geq \inf\{\widehat{\mathcal{H}}_M(x), \widehat{\mathcal{H}}_M(y)\} \\ &= \{\hat{\alpha}, \hat{\alpha}\} \\ &= \hat{\alpha}, \end{aligned}$$

and so $x * y \in M$. Hence, M is a subalgebra of X.

Conversely, suppose that M is a subalgebra of X and let $x, y \in X$. Then, we have the following cases:
Case(1). If $x, y \in M$, then $x * y \in M$. Therefore

$$\widehat{\mathcal{H}}_M(x * y) = \hat{\alpha} = \inf\{\widehat{\mathcal{H}}_M(x)\widehat{\mathcal{H}}_M(y)\}.$$

Case(2). If $x \notin M$ or $y \notin M$, then

$$\widehat{\mathcal{H}}_M(x * y) \geq \hat{\beta} = \inf\{\widehat{\mathcal{H}}_M(x), \widehat{\mathcal{H}}_M(y)\}.$$

This shows that $\widehat{\mathcal{H}}_M$ is an m-pF subalgebra of X.

Moreover, we have $K_{\widehat{\mathcal{H}}_M} = \{x \in X \mid \widehat{\mathcal{H}}_M(x) = \widehat{\mathcal{H}}_M(0)\} = \{x \in X \mid \widehat{\mathcal{H}}_M(x) = \hat{\alpha}\} = M$. □

Now, we introduce and characterize normal m-pF subalgebras of a BCK/BCI-algebra X.

Definition 3.4. An m-pF subalgebra $\widehat{\mathcal{H}}$ of X is said to be normal if there exists $x \in X$ such that $\widehat{\mathcal{H}}(x) = \widehat{1} = (1, 1, ..., 1)$.

Example 3.2. Let X be a BCK-algebra in Example 3.1. Then, an m-pF subalgebra $\widehat{\mathcal{H}}$ in X defined by

$$\widehat{\mathcal{H}}(x) = \begin{cases} (1, 1, ..., 1), & \text{if } x = 0, a, c \\ (0.7, 0.7, ..., 0.7), & \text{if } x = b, \end{cases}$$

is a normal m-pF subalgebra of X.

We know that if $\widehat{\mathcal{H}}$ is a normal m-pF subalgebra of X, then clearly $\widehat{\mathcal{H}}(0) = \widehat{1} = (1, 1, ..., 1)$, and hence $\widehat{\mathcal{H}}$ is normal if and only if $\widehat{\mathcal{H}}(0) = \widehat{1} = (1, 1, ..., 1)$.

Theorem 3.5. *Given an m-pF subalgebra $\widehat{\mathcal{H}}$ of X and let $\widehat{\mathcal{H}}^+$ be an m-pF set in X defined by*

$$\widehat{\mathcal{H}}^+(x) = \widehat{\mathcal{H}}(x) + \widehat{\mathcal{H}}^c(0)), \forall x \in X.$$

Then, $\widehat{\mathcal{H}}^+$ is a normal m-pF subalgebra of X which contains $\widehat{\mathcal{H}}$.

Proof. Let $x, y \in X$. Then, we have

$$\begin{aligned} \widehat{\mathcal{H}}^+(x * y) &= \widehat{\mathcal{H}}(x * y) + \widehat{\mathcal{H}}^c(0)) \\ &\geq \inf\{\widehat{\mathcal{H}}(x), \widehat{\mathcal{H}}(y)\} + \widehat{\mathcal{H}}^c(x) \\ &= \inf\{\widehat{\mathcal{H}}(x) + \widehat{\mathcal{H}}^c(0), \widehat{\mathcal{H}}(y) + \widehat{\mathcal{H}}^c(0)\} \\ &= \inf\{\widehat{\mathcal{H}}^+(x), \widehat{\mathcal{H}}^+(y)\}. \end{aligned}$$

Moreover, $\widehat{\mathcal{H}}^+(0) = \widehat{\mathcal{H}}(0) + \widehat{\mathcal{H}}^c(0) = \widehat{1}$. Therefore, $\widehat{\mathcal{H}}^+$ is a normal m-pF subalgebra of X. Clearly, $\widehat{\mathcal{H}} \subseteq \widehat{\mathcal{H}}^+$. Thus, $\widehat{\mathcal{H}}^+$ is a normal m-pF subalgebra of X which contains $\widehat{\mathcal{H}}$. □

Corollary 3.6. *Let $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{H}}^+$ be as in Theorem 3.5. If there is $x \in X$ such that $\widehat{\mathcal{H}}^+(x) = \widehat{0}$, then $\widehat{\mathcal{H}}(x) = \widehat{0}$.*

Proof. Since $\widehat{\mathcal{H}} \subseteq \widehat{\mathcal{H}}^+$, it is straightforward. □

Using Theorem 3.3, we know that for any subalgebra M of X. The m-pF characteristic function $\widehat{C}_M$ of M is a normal m-pF subalgebra of X. It is clear that $\widehat{\mathcal{H}}$ is normal if and only if $\widehat{\mathcal{H}}^+ = \widehat{\mathcal{H}}$.

Proposition 3.7. *If $\widehat{\mathcal{H}}$ is an m-pF subalgebra of X, then $(\widehat{\mathcal{H}}^+)^+ = \widehat{\mathcal{H}}^+$. Moreover, if $\widehat{\mathcal{H}}$ is normal, then $(\widehat{\mathcal{H}}^+)^+ = \widehat{\mathcal{H}}$.*

Proof. Straightforward. □

Theorem 3.8. *If $\widehat{\mathcal{H}}$ and $\widehat{C}$ are m-pF subalgebras of X, such that $\widehat{\mathcal{H}} \subseteq \widehat{C}$ and $\widehat{\mathcal{H}}(0) = \widehat{C}(0)$, then $K_{\widehat{\mathcal{H}}} \subseteq K_{\widehat{C}}$.*

Proof. Let $x \in K_{\widehat{\mathcal{H}}}$. Then,

$$\widehat{C}(x) \geq \widehat{\mathcal{H}}(x) = \widehat{\mathcal{H}}(0) = \widehat{C}(0)$$

and so $\widehat{C}(x) = \widehat{C}(0)$, i.e., $x \in K_{\widehat{C}}$. Hence, $K_{\widehat{\mathcal{H}}} \subseteq K_{\widehat{C}}$. □

Corollary 3.9. *If $\widehat{\mathcal{H}}$ and $\widehat{C}$ are normal m-pF subalgebras of X such that $\widehat{\mathcal{H}} \subseteq \widehat{C}$, then $K_{\widehat{\mathcal{H}}} \subseteq K_{\widehat{C}}$.*

Theorem 3.10. *Let $\widehat{\mathcal{H}}$ be an m-pF subalgebra of X. If there exists an m-pF subalgebra $\widehat{C}$ of X such that $\widehat{C}^+ \subseteq \widehat{\mathcal{H}}$, then $\widehat{\mathcal{H}}$ is normal.*

Proof. Suppose that there exists an m-pF subalgebra $\widehat{C}$ of X such that $\widehat{C}^+ \subseteq \widehat{\mathcal{H}}$. Then, $\widehat{1} = \widehat{C}^+(0) \leq \widehat{\mathcal{H}}(0)$, and so $\widehat{\mathcal{H}}(0) = \widehat{1}$. This completes the proof. □

Theorem 3.11. *Let $\psi : [0, 1]^m \to [0, 1]^m$ be an increasing function and $\widehat{\mathcal{H}}$ be an m-pF set of X. Then, an m-pF set $\widehat{\mathcal{H}}_\psi : X \to [0, 1]^m$ defined by*

$$\widehat{\mathcal{H}}_\psi(x) = \psi(\widehat{\mathcal{H}}(x)), \forall x \in X$$

is an m-pF subalgebra of X if and only if $\widehat{\mathcal{H}}$ is an m-pF subalgebra of X. In particular, if $\psi(\widehat{\mathcal{H}}(0)) = \widehat{1}$, then $\widehat{\mathcal{H}}_\psi$ is normal, and if $\psi(\widehat{t}) = \widehat{t}$ for all $\widehat{t} \in [0, 1]^m$, then $\widehat{\mathcal{H}}$ is contained in $\widehat{\mathcal{H}}_\psi$.

Proof. Let $\widehat{\mathcal{H}}_\psi$ be an m-pF subalgebra of X. Then, for all $x, y \in X$, we have

$$\begin{aligned} \psi(\widehat{\mathcal{H}}(x*y)) &= \widehat{\mathcal{H}}_\psi(x*y) \\ &\geq \inf\{\widehat{\mathcal{H}}_\psi(x), \widehat{\mathcal{H}}_\psi(y)\} \\ &= \inf\{\psi(\widehat{\mathcal{H}}(x)), \psi(\widehat{\mathcal{H}}(y))\} \\ &= \psi(\inf\{\widehat{\mathcal{H}}(x), \widehat{\mathcal{H}}(y)\}). \end{aligned}$$

Since ψ is an increasing, it follows that

$$\widehat{\mathcal{H}}(x*y) \geq \inf\{\widehat{\mathcal{H}}(x), \widehat{\mathcal{H}}(y)\}.$$

Hence, $\widehat{\mathcal{H}}$ is an m-pF subalgebra of X.

Conversely, if $\widehat{\mathcal{H}}$ is an m-pF subalgebra of X, then for all $x, y \in X$, we have

$$\begin{aligned} \widehat{\mathcal{H}}_\psi(x*y) &= \psi(\widehat{\mathcal{H}}(x*y)) \\ &\geq \psi(\inf\{\widehat{\mathcal{H}}(x), \widehat{\mathcal{H}}(y)\}) \\ &= \inf\{\psi(\widehat{\mathcal{H}}(x)), \psi(\widehat{\mathcal{H}}(y))\} \\ &= \inf\{\widehat{\mathcal{H}}_\psi(x), \widehat{\mathcal{H}}_\psi(y)\}. \end{aligned}$$

Hence, $\widehat{\mathcal{H}}_\psi$ is an m-pF subalgebra of X.

Now, if $\psi(\widehat{\mathcal{H}}(0)) = \widehat{1} = (1, 1, ..., 1)$, then clearly $\widehat{\mathcal{H}}_\psi$ is normal. Assume that $\psi(\widehat{t}) = \widehat{t}$ for all $\widehat{t} \in [0, 1]^m$. Then,

$$\widehat{\mathcal{H}}_\psi(x) = \psi(\widehat{\mathcal{H}}(x)) \geq \widehat{\mathcal{H}}(x)$$

for all $x \in X$, which proves that $\widehat{\mathcal{H}}$ is contained in $\widehat{\mathcal{H}}_\psi$. □

Denote by $\mathcal{NO}(X)$ the set of all normal m-pF subalgebras of X. Note that $\mathcal{NO}(X)$ is a poset under the set inclusion.

Theorem 3.12. *Let $\widehat{\mathcal{H}} \in \mathcal{NO}(X)$ be a non-constant such that it is a maximal element of $(\mathcal{NO}(X), \subseteq)$. Then, $\widehat{\mathcal{H}}$ takes only the values $\widehat{0} = (0, 0, ..., 0)$ and $\widehat{1} = (1, 1, ..., 1)$.*

Proof. Let $\widehat{\mathcal{H}}$ be a non-constant maximal element of $(\mathcal{NO}(X), \subseteq)$. Since $\widehat{\mathcal{H}}$ is normal, so $\widehat{\mathcal{H}}(0) = \widehat{1}$. Let $x \in X$ be such that $\widehat{\mathcal{H}}(x) \neq \widehat{1}$. We claim that $\widehat{\mathcal{H}}(x) = \widehat{0}$. If not, then there exists $b \in X$ such that $\widehat{0} < \widehat{\mathcal{H}}(b) < \widehat{1}$. Let $\widehat{C} : X \to [0, 1]^m$ be an m-pF set in X defined by

$$\widehat{C}(x) = \frac{1}{2}(\widehat{\mathcal{H}}(x) + \widehat{\mathcal{H}}(b)).$$

for all $x \in X$. Then, clearly $\widehat{C}$ is well defined, and for all $x, y \in X$, we have

$$\begin{aligned} \widehat{C}(x*y) &= \frac{1}{2}(\widehat{\mathcal{H}}(x*y) + \widehat{\mathcal{H}}(b)) \\ &\geq \frac{1}{2}(\inf\{\widehat{\mathcal{H}}(x), \widehat{\mathcal{H}}(y)\} + \widehat{\mathcal{H}}(b)) \\ &= \inf\{\frac{1}{2}(\widehat{\mathcal{H}}(x) + \widehat{\mathcal{H}}(b)), \frac{1}{2}(\widehat{\mathcal{H}}(y) + \widehat{\mathcal{H}}(b))\} \end{aligned}$$

$$= \inf\{\widehat{C}(x), \widehat{C}(y)\}.$$

Hence, $\widehat{C}$ is an m-pF subalgebra of X. It follows from Theorem 3.5 that $\widehat{C}^+ \in \mathcal{NO}(X)$ where $\widehat{C}^+$ is defined by $\widehat{C}^+(x) = \widehat{C}(x) + \widehat{C}^c(0)$, $\forall x \in X$. Clearly, $\widehat{C}^+(x) \geq \widehat{\mathcal{H}}(x)$, $\forall x \in X$. Note that

$$\begin{aligned} \widehat{C}^+(b) &= \widehat{C}(b) + \widehat{C}^c(0)) \\ &= \widehat{C}(b) + \widehat{1} - \widehat{C}(0) \\ &= \frac{1}{2}(\widehat{\mathcal{H}}(b) + \widehat{\mathcal{H}}(b)) + \widehat{1} - \frac{1}{2}(\widehat{\mathcal{H}}(0) + \widehat{\mathcal{H}}(b)) \\ &= \frac{1}{2}(\widehat{\mathcal{H}}(b) + \widehat{1}) \\ &> \widehat{\mathcal{H}}(b) \end{aligned}$$

and $\widehat{C}^+(b) < \widehat{1} = \widehat{C}^+(0)$. Hence, $\widehat{C}^+$ is a non-constant and $\widehat{\mathcal{H}}$ is not a maximal element of $\mathcal{NO}(X)$. This is a contradiction. This completes the proof. □

Definition 3.13. Let $\widehat{\mathcal{H}}$ be an m-pF subalgebra of X. Then, $\widehat{\mathcal{H}}$ is said to be maximal if

(i) $\widehat{\mathcal{H}}$ is non-constant.

(ii) $\widehat{\mathcal{H}}^+$ is a maximal element of the poset $(\mathcal{NO}(X), \subseteq)$.

Theorem 3.14. *A maximal m-pF subalgebra $\widehat{\mathcal{H}}$ of X is normal and takes the values* $\widehat{0} = (0, 0, ..., 0)$ *and* $\widehat{1} = (1, 1, ..., 1)$.

Proof. Let $\widehat{\mathcal{H}}$ be a maximal m-pF subalgebra of X. Then, $\widehat{\mathcal{H}}^+$ is a non-constant maximal element of the poset $(\mathcal{NO}(X), \subseteq)$. It follows that from Theorem 3.12 that $\widehat{\mathcal{H}}^+$ takes only the values $\widehat{0}$ and $\widehat{1}$. Note that $\widehat{\mathcal{H}}^+(x) = \widehat{1}$ if and only if $\widehat{\mathcal{H}}(x) = \widehat{\mathcal{H}}(0)$, and $\widehat{\mathcal{H}}^+(x) = \widehat{0}$ if and only if $\widehat{\mathcal{H}}(x) = \widehat{\mathcal{H}}(0) - \widehat{1}$. By Corollary 3.6, we have $\widehat{\mathcal{H}}(x) = \widehat{0}$ i.e., $\widehat{\mathcal{H}}(0) = \widehat{1}$. Hence, $\widehat{\mathcal{H}}$ is normal, and clearly $\widehat{\mathcal{H}}^+ = \widehat{\mathcal{H}}$. This completes the proof. □

Theorem 3.15. *If $\widehat{\mathcal{H}}$ is a maximal m-pF subalgebra of X, then* $\widehat{\mathcal{H}}_{K_{\widehat{\mathcal{H}}}} = \widehat{\mathcal{H}}$.

Proof. Clearly, $\widehat{\mathcal{H}}_{K_{\widehat{\mathcal{H}}}} \subseteq \widehat{\mathcal{H}}$ and $\widehat{\mathcal{H}}_{K_{\widehat{\mathcal{H}}}}$ takes only the values $\widehat{0}$ and $\widehat{1}$. Let $x \in X$. If $\widehat{\mathcal{H}}(x) = 0$, then obviously $\widehat{\mathcal{H}} \subseteq \widehat{\mathcal{H}}_{K_{\widehat{\mathcal{H}}}}$. If $\widehat{\mathcal{H}}(x) = 1$, then $x \in K_{\widehat{\mathcal{H}}}$, and so $\widehat{\mathcal{H}}_{K_{\widehat{\mathcal{H}}}}(x) = \widehat{1}$. This shows that $\widehat{\mathcal{H}} \subseteq \widehat{\mathcal{H}}_{K_{\widehat{\mathcal{H}}}}$. □

Theorem 3.16. *For a maximal m-pF subalgebra $\widehat{\mathcal{H}}$ of X, $K_{\widehat{\mathcal{H}}}$ is a maximal subalgebra of X.*

Proof. Let $K_{\widehat{\mathcal{H}}}$ be a proper subalgebra of X because $\widehat{\mathcal{H}}$ is non-constant. Let M be a subalgebra of X such that $K_{\widehat{\mathcal{H}}} \subseteq M$. Noticing that for every subalgebras M and N of X, $M \subseteq N$ if and only if $\widehat{\mathcal{H}}_M \subseteq \widehat{\mathcal{H}}_N$, then we obtain $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_{K_{\widehat{\mathcal{H}}}} \subseteq \widehat{\mathcal{H}}_M$. Since $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{H}}_M$ are normal and since $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}^+$ is a maximal element of $\mathcal{NO}(X)$, we have that either $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_M$ or $\widehat{\mathcal{H}}_M = \widehat{\mathbf{1}}$, where $\widehat{\mathbf{1}} : X \to [0, 1]^m$ is an m-pF set defined by $\widehat{\mathbf{1}}(x) = (1, 1, ..., 1) = \widehat{1}$ for all $x \in X$. The other case implies that $M = X$. If $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_M$, then $K_{\widehat{\mathcal{H}}} = K_{\widehat{\mathcal{H}}_M} = M$ by Theorem 3.3. This proves that $K_{\widehat{\mathcal{H}}}$ is a maximal subalgebra of X. This completes the proof. □

Definition 3.17. A normal m-pF subalgebra $\widehat{\mathcal{H}}$ of X is said to be completely normal if there exists $x \in X$ such that $\widehat{\mathcal{H}}(x) = \widehat{0}$. Denote by $\mathcal{CN}(X)$ the set of all completely normal m-pF subalgebra of X.

We note that $\mathcal{CN}(X) \subseteq \mathcal{NO}(X)$ and the restriction of partial ordering $\subseteq$ of $\mathcal{NO}(X)$ gives a partial ordering on $\mathcal{CN}(X)$.

Theorem 3.18. *A non-constant maximal element of* $(\mathcal{NO}(X), \subseteq)$ *is also a maximal element of* $(\mathcal{CN}(X), \subseteq)$.

Proof. Let $\widehat{\mathcal{H}}$ be a non-constant maximal element of $(\mathcal{N}(X), \subseteq)$. By Theorem 3.12, $\widehat{\mathcal{H}}$ takes only the values $\widehat{0}$ and $\widehat{1}$. Now, $\widehat{\mathcal{H}}(0) = \widehat{1}$ and $\widehat{\mathcal{H}}(x) = \widehat{0}$ for some $x \in X$. Thus, $\widehat{\mathcal{H}} \in \mathcal{CN}(X)$. Suppose there exists $\widehat{C} \in \mathcal{CN}(X)$ such that $\widehat{\mathcal{H}} \subseteq \widehat{C}$. It follows that $\widehat{\mathcal{H}} \subseteq \widehat{C}$ in $\mathcal{NO}(X)$. Since $\widehat{\mathcal{H}}$ is maximal in $(\mathcal{NO}(X), \subseteq)$ and since $\widehat{C}$ is non-constant, therefore $\widehat{\mathcal{H}} = \widehat{C}$. Hence, $\widehat{\mathcal{H}}$ is maximal element of $(\mathcal{CN}(X), \subseteq)$. This completes the proof. □

Theorem 3.19. *Every maximal m-pF subalgebra of X is completely normal.*

Proof. Let $\widehat{\mathcal{H}}$ be a maximal m-pF subalgebra of X. Then, by Theorem 3.14, $\widehat{\mathcal{H}}$ is normal and $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}^+$ takes only the values $\widehat{0}$ and $\widehat{1}$. Since $\widehat{\mathcal{H}}$ is a non constant, it follows that $\widehat{\mathcal{H}}(0) = \widehat{1}$ and $\widehat{\mathcal{H}}(x) = \widehat{0}$ for some $x \in X$. Hence, $\widehat{\mathcal{H}}$ is completely normal. This completes the proof. □

4. m-Polar fuzzy characteristic subalgebras

Definition 4.1. For an endomorphism Ψ of X and an m-pF set $\widehat{\mathcal{H}}$ in X. We define a new m-pF set $\widehat{\mathcal{H}}[\Psi] : X \to [0, 1]^m$ by $\widehat{\mathcal{H}}[\Psi](x) = \widehat{\mathcal{H}}(\Psi(x))$ for all $x \in X$.

Theorem 4.2. *If* $\widehat{\mathcal{H}}$ *is an m-pF subalgebra of X, then so is* $\widehat{\mathcal{H}}[\Psi]$.

Proof. Let $x, y \in X$. Then,

$$\begin{aligned} \widehat{\mathcal{H}}[\Psi](x * y) &= \widehat{\mathcal{H}}(\Psi(x * y)) \\ &= \widehat{\mathcal{H}}(\Psi(x) * \Psi(y)) \\ &\geq \inf\{\widehat{\mathcal{H}}(\Psi(x)), \widehat{\mathcal{H}}(\Psi)(y)\} \\ &= \inf\{\widehat{\mathcal{H}}[\Psi](x), \widehat{\mathcal{H}}[\Psi](y)\} \end{aligned}$$

Hence, $\widehat{\mathcal{H}}[\Psi]$ is an m-pF subalgebra of X. □

Example 4.1. Consider a BCK-algebra $X = \{0, a, b\}$ with the Cayley table which is given in Table 2.

Table 2. Cayley table for the operation $*$.

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	b	0

Let $\widehat{\mathcal{H}} : X \to [0, 1]^m$ be an m-pF set in X defined by:

$$\widehat{\mathcal{H}}(x) = \begin{cases} \widehat{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_m), & \text{if } x = 0, a \\ \widehat{\delta} = (\delta_1, \delta_2, ..., \delta_m), & \text{if } x = b, \end{cases}$$

where $\widehat{\gamma} > \widehat{\delta}$. By routine computations, we can verify that $\widehat{\mathcal{H}}$ is an m-pF subalgebra of X. There are four endomorphisms of X as follows:

$$\begin{aligned}\Psi_1 &: 0 \to 0, a \to 0, b \to 0,\\ \Psi_2 &: 0 \to 0, a \to 0, b \to a,\\ \Psi_3 &: 0 \to 0, a \to 0, b \to b,\\ \Psi_4 &: 0 \to 0, a \to a, b \to b.\end{aligned}$$

By Theorem 4.2, we have $\widehat{\mathcal{H}}[\Psi_i]$ for $i = 1, 2, 3, 4$ are m-pF subalgebras.

Definition 4.3. A subalgebra K of X is called characteristic if $\Psi(K) = K$ for all $\Psi \in Aut(X)$, where $Aut(X)$ is the set of all automorphisms of X.

Definition 4.4. An m-pF subalgebra $\widehat{\mathcal{H}}$ of X is called an m-pF characteristic if $\widehat{\mathcal{H}}[\Psi](x) = \widehat{\mathcal{H}}(x)$ for all $x \in X$ and $\Psi \in Aut(X)$.

Example 4.2. In Example 4.1, Ψ_4 is an automorphism of X. It is clear that $\Psi_4(\widehat{\mathcal{H}}(x)) = \widehat{\mathcal{H}}(x)$ for all $x \in X$. Therefore, $\widehat{\mathcal{H}}$ is characteristic. Also, $\widehat{\mathcal{H}}[\Psi_4](x) = \widehat{\mathcal{H}}(\Psi_4(x)) = \widehat{\mathcal{H}}(x)$ for all $x \in X$. Hence, $\widehat{\mathcal{H}}$ is an m-pF characteristic.

Lemma 4.5. *Let $\widehat{\mathcal{H}}$ be an m-pF subalgebra of X and let $x \in X$. Then, $\widehat{\mathcal{H}}(x) = \widehat{t}$ if and only if $x \in \widehat{\mathcal{H}}_{\widehat{t}}$ and $x \notin \widehat{\mathcal{H}}_{\widehat{s}}$ for all $\widehat{s} > \widehat{t}$.*

Proof. Let $\widehat{\mathcal{H}}$ be an m-pF subalgebra of X and let $x \in X$. Suppose $\widehat{\mathcal{H}}(x) = \widehat{t}$, so that $x \in \widehat{\mathcal{H}}_{\widehat{t}}$. If possible, let $x \in \widehat{\mathcal{H}}_{\widehat{s}}$ for $\widehat{s} > \widehat{t}$. Then, $\widehat{\mathcal{H}}(x) \geq \widehat{s} > \widehat{t}$. this contradicts the fact that $\widehat{\mathcal{H}}(x) = \widehat{t}$, concluding that $x \notin \widehat{\mathcal{H}}_{\widehat{s}}$ for all $\widehat{s} > \widehat{t}$.

Conversely, let $x \in \widehat{\mathcal{H}}_{\widehat{t}}$ and $x \notin \widehat{\mathcal{H}}_{\widehat{s}}$ for all $\widehat{s} > \widehat{t}$. Now, let $x \in \widehat{\mathcal{H}}_{\widehat{t}} \Rightarrow \widehat{\mathcal{H}}(x) \geq \widehat{t}$, since $x \notin \widehat{\mathcal{H}}_{\widehat{s}}$ for all $\widehat{s} > \widehat{t}$. Therefore, $\widehat{\mathcal{H}}(x) = \widehat{t}$. □

Theorem 4.6. *For an m-pF subalgebra $\widehat{\mathcal{H}}$ of X, the following are equivalent:*

(1) $\widehat{\mathcal{H}}$ is an m-pF characteristic.

(2) Each level cut subset $\widehat{\mathcal{H}}_{\widehat{t}}$ is characteristic subalgebra.

Proof. Suppose $\widehat{\mathcal{H}}$ is an m-pF characteristic and let $\widehat{t} \in Im(\widehat{\mathcal{H}})$, $\Psi \in Aut(X)$ and $x \in \widehat{\mathcal{H}}_{\widehat{t}}$. Then,

$$\widehat{\mathcal{H}}[\Psi](x) = \widehat{\mathcal{H}}(\Psi(x)) = \widehat{\mathcal{H}}(x) \geq \widehat{t},$$

i.e., $\widehat{\mathcal{H}}(\Psi(x)) \geq \widehat{t}$. Thus, $\Psi(x) \in \widehat{\mathcal{H}}_{\widehat{t}}$, i.e., $\Psi(\widehat{\mathcal{H}}_{\widehat{t}}) \subseteq \widehat{\mathcal{H}}_{\widehat{t}}$. Now, let $x \in \widehat{\mathcal{H}}_{\widehat{t}}$ and $y \in X$ be such that $\Psi(y) = x$. Then,

$$\widehat{\mathcal{H}}(y) = \widehat{\mathcal{H}}[\Psi](y) = \widehat{\mathcal{H}}(\Psi(y)) = \widehat{\mathcal{H}}(x).$$

Hence, $y \in \widehat{\mathcal{H}}_{\widehat{t}}$, so that $x = \Psi(y) \in \Psi(\widehat{\mathcal{H}}_{\widehat{t}})$. Consequently, $\widehat{\mathcal{H}}_{\widehat{t}} \subseteq \Psi(\widehat{\mathcal{H}}_{\widehat{t}})$. Therefore, $\widehat{\mathcal{H}}_{\widehat{t}} = \Psi(\widehat{\mathcal{H}}_{\widehat{t}})$ and $\widehat{\mathcal{H}}$ is characteristic.

Conversely, suppose that each level cut subset $\widehat{\mathcal{H}}_{\widehat{t}}$ is characteristic subalgebra and let $x \in X, \Psi \in Aut(X)$ and $\widehat{\mathcal{H}}(x) = \widehat{t}$. Then, by Lemma 4.5, $x \in \widehat{\mathcal{H}}_{\widehat{t}}$ and $x \notin \widehat{\mathcal{H}}_{\widehat{s}}$ for all $\widehat{s} > \widehat{t}$. Thus, $\Psi(x) \in \Psi(\widehat{\mathcal{H}}_{\widehat{t}}) = \widehat{\mathcal{H}}_{\widehat{t}}$, so that $\widehat{\mathcal{H}}(\Psi(x)) \geq \widehat{t}$. Let $\widehat{t_1} = \widehat{\mathcal{H}}[\Psi](x)$ and suppose $\widehat{t_1} > \widehat{t}$. Then, $\Psi(x) \in \widehat{\mathcal{H}}_{\widehat{t_1}} = \Psi(\widehat{\mathcal{H}}_{\widehat{t_1}})$, which implies from the injectivity of Ψ that $x \in \widehat{\mathcal{H}}_{\widehat{t_1}}$, a contradiction. Thus, $\widehat{\mathcal{H}}(\Psi(x)) = \widehat{\mathcal{H}}(x)$. Therefore, $\widehat{\mathcal{H}}[\Psi]$ is an m-pF characteristic. □

5. Conclusion

The idea of m-pF algebraic structures plays a significant rule in several fields of applied mathematics, computer sciences and information systems. In [9], we have already introduced the concepts of m-pF subalgebras and ideals of BCK/BCI-algebras and investigated some of their related properties. In this study, as a continuation of [9], we have introduced the concepts of normal m-pF subalgebras, maximal m-pF subalgebras and completely normal m-pF subalgebras in BCK/BCI-algebras and discussed some of their properties. We have proved that any non-constant normal m-pF subalgebra which is a maximal element of $(\mathcal{NO}(X), \subseteq)$ takes only the values $\widehat{0} = (0, 0, ..., 0)$ and $\widehat{1} = (1, 1, ..., 1)$, and every maximal m-pF subalgebra is completely normal. Moreover, we have stated an m-pF characteristic subalgebra in BCK/BCI-algebras. In the future, the results of this work can be further expanded to several algebraic structures, for instance UP-algebras, BRK-algebras, KU-algebras, etc.

Conflict of Interest

We declare that we have no conflict of interest.

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