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Research article

Approximate solutions to nonlinear fractional order partial differential equations arising in ion-acoustic waves

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Abstract: An approximative procedure named asymptotic homotopy perturbation method (AHPM) is introduced to obtain solutions of the non-linear fractional order models. The two special cases, FZK$(3,3,3)$ and FZK$(2,2,2)$ of fractional Zakharov-Kuznetsov equations are chosen for the illustrative purpose of our method. AHPM is a very recent new procedure as compare with other existing homotopy perturbation procedures. A new auxiliary function has been introduced in AHPM. The AHPM solutions are compared with solutions of fractional complex transform FCT using variational iteration method VIM and exact solutions. Further, the surface graph of AHPM solutions are compared with surface graph of solutions of homotopy perturbation transform method (HPTM) solutions. In comparison, the solutions computed by AHPM are in agreement with exact solutions of the problems. The simulation section reveals that our new developed procedure is effective and explicit.

Keywords: asymptotic homotopy perturbation method; fractional Zakharov-Kuznetsov equations
Mathematics Subject Classification: 35A22, 35A25, 35K57

1. Introduction

In recent decades, importance of fractional order models is well disclosed fact in many fields of engineering and science. Numerous fractional order partial differential equations(FPDEs) have been used by many authors to describe various important biological and physical processes like in the fields of chemistry, biology, mechanics, polymer, economics, biophysics control theory and aerodynamics. In this concern, many researchers have studied various schemes and aspects of PDEs and FPDEs as well, see [1–10]. However, the great attention has been given very recently to obtaining the solution of fractional models of the physical interest. Keeping in views, the computation complexities involved in fractional order models is very crucial and is the difficulty in solving these fractional models. Some times, the exact analytic solution of each and every FPDE can not be obtained using the traditional schemes and methods. However, there exists some schemes and methods, which have been proved to be efficient in obtaining the approximation to solution of the fractional order problems. Among them, we bring the attention of readers to these methods and schemes [11–21] which are used successfully. These methods and schemes have their own demerits and merits. Some of them provide a very good approximation with convenient way. For example, see the methods and schemes in the articles [22–39].

The main aim of this work is to develop a new procedure which is easy with respect to application and more efficient as compare with existing procedures. In this concern, we introduced asymptotic homotopy perturbation method (AHPM) to obtain the solution of nonlinear fractional order models. It is a new version of perturbation techniques. In simulation section, we have testified our proposed procedure by considering the test problems of non linear fractional order Zakharov-Kuznetsov $ZK(m,n,r)$ equations of the form [11, 12]

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} + a_0(u^m(x,y,t))_x + a_1(u^n(x,y,t))_{xxx} + a_2(u^r(x,y,t))_{yyx} = 0,\ \ 0 < \alpha \leq 1. \tag{1.1}$$

Where a_0, a_1, a_2 are arbitrary constants and m, n, r are non zero integers. If $\alpha = 1$, then equation (1.1) becomes classical Zakharov-Kuznetsov $ZK(m,n,r)$ equation given as:

$$\frac{\partial u(x,y,t)}{\partial t} + a_0(u^m(x,y,t))_x + a_1(u^m(x,y,t))_{xxx} + a_2(u^r(x,y,t))_{yyx} = 0. \tag{1.2}$$

The ZK equation has been firstly modelized for depicting weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma [40]. The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [41, 42].

The plan of the rest paper is as follows: Section 2 provides theory of the AHPM; Section 3 provides implementation of AHPM. Finally, a brief conclusion and the further work has been listed.

2. Basic idea of AHPM

Here, we provide that the Caputo type fractional order derivative will be used throughout this paper for the computation of derivative.

Let us consider the nonlinear problem in the form as

$$T\left(u(x,y,t)\right) + g(x,y,t) = 0, \tag{2.1}$$

$$B\left(u\left(x,y,t\right), \frac{\partial u\left(x,y,t\right)}{\partial t}\right) = 0. \tag{2.2}$$

Where $T\left(u\left(x,y,t\right)\right)$ denotes a differential operator which may consists ordinary, partial or space-fractional or time-fractional differential derivative. $T\left(u\left(x,y,t\right)\right)$ can be expressed for fractional model as follows:

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} + N(u(x,y,t)) + g(x,y,\ t) = 0 \tag{2.3}$$

subject to the condition

$$B\left(u(x,y,t), \frac{\partial u(x,y,t)}{\partial t}\right) = 0, \tag{2.4}$$

where the operator $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the Caputo derivative operator, N is non linear operator and B denotes a boundary operator, $u(x,y,t)$ is unknown exact solution of Eq. (2.1), $g(x,y,t)$ denotes known function and x, y and t denote special and temporal variables respectively. Let us construct a homotopy $\Phi(x,y,t;p) : \Omega \times [0,\ 1] \to R$ which satisfies

$$\frac{\partial^\alpha \Phi(x,y,t;p)}{\partial t^\alpha} + g(x,y,t) - p\left[N(\Phi(x,y,t;p)\right] = 0, \tag{2.5}$$

where $p \in [0,\ 1]$ is said to be an embedding parameter. At this phase of our work it is pertinent that our proposed deformation Eq. (2.5) is an alternate form of the deformation equations as

$$(1-p)\left[L(\Phi(x,y,t;p)) - L(u_0(x,y,t)) + g(x,y,t)\right] + p\left[T(\Phi(x,y,t;p) + g(x,y,t)\right] = 0, \tag{2.6}$$

$$(1-p)\left[L(\Phi(x,y,t;p)) - L(u_0(x,y,t))\right] = hp\left[T(\Phi(x,y,t;p) + g(x,y,t)\right] \tag{2.7}$$

and

$$(1-p)\left[L(\Phi(x,y,t;p)) + g(x,y,t)\right] - H(p)\left[T(\Phi(x,y,t;p) + g(x,y,t)\right] = 0. \tag{2.8}$$

in HPM, HAM, OHAM proposed by Liao in [43], He in [44] and Marinca in [45] respectively. Basically, according to homotopy definition, when $p = 0$ and $p = 1$ we have

$$\Phi(x,y,t;p) = u_0(x,y,t), \quad \phi(x,y,t;p) = u(x,y,t).$$

Obviously, when the embedding parameter p varies from 0 to 1, the defined homotopy ensures the convergence of $\phi(x,y,t;p)$ to the exact solution $u(x,y,t)$. Consider $\phi(x,y,t;p)$ in the form

$$\Phi(x,y,t;p) = u_0(x,y,t) + \sum_{i=1}^{\infty} u_i(x,y,t)p^i \tag{2.9}$$

and assuming $N(\Phi(x,y,t;p))$ as follows

$$N(\Phi(x,y,t;p)) = B_1 N_0 + \sum_{i=1}^{\infty}\left(\sum_{m=0}^{i} B_{i+1-m} N_m\right)p^i,\ B_1 + B_2 + B_3 + ... = -1. \tag{2.10}$$

Where

$$B_i = B_i(x, y, t, c_i), for, \ i = 1, 2, 3, \ldots. \tag{2.11}$$

are arbitrary auxiliary functions, will be discussed later. Thus, if $p = 0$ and $p = 1$ in Eq. (2.5), we have

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} + f(x) = 0, \text{ and } \frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} + N(u(x, y, t)) + g(x, y, t) = 0,$$

respectively.

It is obvious that the construction of introduced auxiliary function in Eq. (2.10) is different from the auxiliary functions that are proposed in articles [43–45]. Hence the procedure proposed in our paper is different from the procedures proposed by Liao, He, Marinca in aforesaid papers [43–45] as well as Optimal Homotopy perturbation method (OHPM) in [46].

Furthermore, when we substitute Eq. (2.9) and Eq. (2.10) in Eq. (2.5) and equate like power of p, the obtained series of simpler linear problems are

$$\begin{aligned}
p^0 &: \ \frac{\partial^\alpha u_0(x, y, t)}{\partial t^\alpha} + g = 0, \\
p^1 &: \ \frac{\partial^\alpha u_1(x, y, t)}{\partial t^\alpha} = B_1 N_0, \\
p^2 &: \ \frac{\partial^\alpha u_2(x, y, t)}{\partial t^\alpha} = B_2 N_0 + B_1 N_1, \\
p^3 &: \ \frac{\partial^\alpha u_3(x, y, t)}{\partial t^\alpha} = B_3 N_0 + B_2 N_1 + B_1 N_2, \\
&\vdots \\
p^k &: \ \frac{\partial^\alpha u_k(x, y, t)}{\partial t^\alpha} = \sum_{i=0}^{k-1} B_{k-i} N_i.
\end{aligned}$$

We obtain the series solutions by using the integral operator J^α on both sides of the above each simple fractional differential equation. The convergence of the series solution Eq. (2.9) to the exact solution depends upon the auxiliary parameters (functions) $B_i(x, y, t, c_i)$. The choice of $B_i(x, y, t, c_i)$ is purely on the basis of terms appear in nonlinear part of the Eq. (2.1). The Eq.(2.9) converges to the exact solution of Eq. (2.1) at $p = 1$:

$$\widetilde{u}(x, y, t) = u_0(x, y, t) + \sum_{k=1}^{\infty} u_k(x, y, t; c_i), \ i = 1, 2, 3, \ldots. \tag{2.12}$$

Particularly, we can truncate the Eq. (2.12) into finite m-terms to obtain the solution of nonlinear problem. The auxiliary convergence control constants $c_1, \ c_2, c_3, \ldots$ can be found by solving the system

$$R(\partial x_1, \partial y_1) = R(\partial x_2, \partial y_2) = R(\partial x_3, , \partial y_3) = \cdots = R(\partial x_m, , \partial y_m) = 0, \ \ \partial x_i, \ \partial y_i \in [a, \ b]. \tag{2.13}$$

It can be verified to observe that HPM is only a case of Eq. (2.5) when $p = -p$ and

$$N(\Phi(x, y, t; p)) = N_0 + \sum_{i=1}^{\infty} N_i p^i.$$

The HAM is also a case of Eq. (2.5) when $p = ph$ and

$$N(\Phi(x, y, t; p)) = N_0 + \sum_{i=1}^{\infty} N_i p^i.$$

The OHAM is also another case when

$$B_{k-1} = B_{k-2} + h_k\left(t, c_j\right) + \sum_{i=0}^{k-2} h_{\left(k-(i+1)\right)}\left(t, c_j\right) B_i, \text{ and } h_k\left(t, c_j\right) = c_k$$

in Eq. (2.10), we obtain exactly the series problems which are obtained by OHAM after expanding and equating the like power of p in deformation equation. Furthermore, concerning the Optimal Homotopy Asymptotic Method (OHAM) mentioned in this manuscript and presented in [45], that the version of OHAM proposed in 2008 was improved in time and the most recent improvement, which also contains an auxiliary functions, are presented in the papers [47, 48]. We also have improved the version of OHAM by introducing a very new auxiliary function in Eq. (2.10). Our method proposed in this paper uses a very new and more general form of auxiliary function

$$N(\phi(x, y, t; p)) = B_1 N_0 + \sum_{i=1}^{\infty}\left(\sum_{m=0}^{i} B_{i+1-m} N_m\right) p^i$$

which depends on arbitrary parameters B_1, $B_2, B_3, \ldots$ and is useful for adjusting and controlling the convergence of nonlinear part as well as linear part of the problem with simple way.

3. Applications

In this portion, we apply AHPM to obtain solution of the following problems to show the accuracy and appropriateness of the new procedure for to solve nonlinear problems.

Problem 3.1. *Let us consider FZK*$(2, 2, 2)$ *in the form:*

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} + \left(u^2(x, y, t)\right)_x + \frac{1}{8}\left(u^2(x, y, t)\right)_{xxx} + \frac{1}{8}\left(u^2(x, y, t)\right)_{yyx} = 0, \quad 0 < \alpha \leq 1, \tag{3.1}$$

subject to the condition

$$u(x, y, 0) = \frac{4}{3} k \sinh^2(x + y).$$

When $\alpha = 1$. *Then the exact solution of Eq.* (3.1)*:*

$$u(x, y, t) = \frac{4}{3} k \sinh^2(x + y - kt).$$

As in Eq. (3.1), the non linear part

$$N(u(x, y, t)) = \left(u^2(x, y, t)\right)_x + \frac{1}{8}\left(u^2(x, y, t)\right)_{xxx} + \frac{1}{8}\left(u^2(x, y, t)\right)_{yyx}.$$

Now, follow the procedure of AHPM, we obtain series of the simpler linear problems as:
Zero order problem:

$$\frac{\partial^\alpha u_0}{\partial t^\alpha} = 0,\ u_0 = \frac{4}{3} k \sinh^2(x + y). \tag{3.2}$$

First order problem:

$$\frac{\partial^\alpha u_1}{\partial t^\alpha} = B_1 N_0,\ u_1 = 0. \tag{3.3}$$

Second order problem:

$$\frac{\partial^\alpha u_2}{\partial t^\alpha} = B_2 N_0 + B_1 N_1,\ u_2 = 0. \tag{3.4}$$

Third order problem:

$$\frac{\partial^\alpha u_3}{\partial t^\alpha} = B_3 N_0 + B_2 N_1 + B_1 N_2,\ u_3 = 0. \tag{3.5}$$

Fourth order problem:

$$\frac{\partial^\alpha u_4}{\partial t^\alpha} = B_4 N_0 + B_3 N_1 + B_2 N_2 + B_1 N_3,\ u_4 = 0 \tag{3.6}$$

and so on.

Solving the above equations, the respective solutions from Eqs. (3.2)–(3.6) are given as follow:

$$u_0(x, y, t) = \frac{4}{3} k \sinh^2(z) \tag{3.7}$$

$$u_1(x, y, t) = B_1 N_0 \frac{t^\alpha}{\Gamma(\alpha + 1)} \tag{3.8}$$

$$u_2(x, y, t) = B_2 N_0 \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{64}{27} {B_1}^2 k^3 \left[13 \cosh(2z) - 70 \cosh(4z) + 75 \cosh(6z)\right] \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \tag{3.9}$$

$$\begin{aligned} & u_3(x, y, t) \\ & = B_3 N_0 \tfrac{t^\alpha}{\Gamma(\alpha+1)} + \tfrac{64}{27} B_1 B_2 k^3 \left[13 \cosh(2z) - 70 \cosh(4z) + 75 \cosh(6z)\right] \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & + B_1 \tfrac{64}{81} k^3 \left[\begin{array}{l} B_2 \left(39 \cosh(2z) - 210 \cosh(4z) + 225 \cosh(6z)\right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ + {B_1}^2 k \left(\begin{array}{l} 160 \sinh(2z) + 320 \sinh(4z) - 2400 \sinh(6z) \\ +3400 \sinh(8z) \end{array} \right) \frac{\Gamma(2\alpha+1) t^{2\alpha}}{(\Gamma(\alpha+1))^2 \Gamma(3\alpha+1)} \\ + {B_1}^2 k \left(\begin{array}{l} -768 \sinh(2z) + 9120 \sinh(4z) - 26400 \sinh(6z) \\ +20400 \sinh(8z) \end{array} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{array} \right] \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&u_4(x,y,t) = \\
&B_4 N_0 \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{64}{27} B_1 B_3 k^3 \left[13\cosh(2z) - 70\cosh(4z) + 75\cosh(6z)\right] \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + B_2 \tfrac{64}{81} k^3 \\
&\left[\begin{array}{l}
B_2\left(39\cosh(2z) - 210\cosh(4z) + 225\cosh(6z)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
+B_1{}^2 k \begin{pmatrix} 160\sinh(2z) + 320\sinh(4z) - \\ 2400\sinh(6z) + 3400\sinh(8z) \end{pmatrix} \frac{\Gamma(2\alpha+1)t^{2\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} \\
+B_1{}^2 k \begin{pmatrix} -768\sinh(2z) + 9120\sinh(4z) - \\ 26400\sinh(6z) + 20400\sinh(8z) \end{pmatrix} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}
\end{array}\right] + B_1 \tfrac{64}{243} k^3 \\
&\left[\begin{array}{l}
B_1{}^3 k^2\left(1022400\cosh(6z) - 15200\cosh(4z) - 2502400\cosh(8z) + 1768000\cosh(10z)\right) \\
\frac{2\Gamma(2\alpha)t^{4\alpha}}{\alpha(\Gamma(\alpha))^2\Gamma(4\alpha+1)} + B_1{}^3 k^2 \begin{pmatrix} 85248\cosh(2z) - 1816320\cosh(4z) + 9878400\cosh(6z) \\ -18278400\cosh(8z) + 10608000\cosh(10z) \end{pmatrix} \\
\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + B_3\left(117\cosh(2z) - 630\cosh(4z) + 675\cosh(6z)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + B_1{}^3 k^2 \\
\begin{pmatrix} -56640\cosh(2z) + 119040\cosh(4z) + 496800\cosh(6z) \\ -2121600\cosh(8z) + 2340000\cosh(10z) \end{pmatrix} \frac{\Gamma(3\alpha+1)t^{4\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)} \\
+B_1 B_2 k \begin{pmatrix} 1440\sinh(2z) + 2880\sinh(4z) - 21600\sinh(6z) \\ +30600\sinh(8z) \end{pmatrix} \frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} \\
+B_1 B_2 k \begin{pmatrix} -4608\sinh(2z) + 54720\sinh(4z) - 158400\sinh(6z) \\ +122400\sinh(8z) \end{pmatrix} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}
\end{array}\right]
\end{aligned} \tag{3.11}
$$

so on....

In similar way, we can compute the solution of the next simpler linear problems which are difficult to compute by using OHAM procedure. we choose $B_1 = c_1, B_2 = c_2, B_3 = c_3, B_4 = c_4$ *and consider*

$$\widetilde{u}(x) = u_0(x) + u_1(x, c_1) + u_2(x, c_1,\ c_2) + u_3(x, c_1,\ c_2,\ c_3) + u_4(x, c_1,\ c_2,\ c_3,\ c_4)\ . \tag{3.12}$$

The residual:

$$R\left(\widetilde{u}(x,y,t)\right) = \frac{\partial^{\alpha}\widetilde{u}(x,y,t)}{\partial t^{\alpha}} + \left(\widetilde{u}^2(x,y,t)\right)_x + \frac{1}{8}\left(\widetilde{u}^2(x,y,t)\right)_{xxx} + \frac{1}{8}\left(\widetilde{u}^2(x,y,t)\right)_{yyx}. \tag{3.13}$$

We obtain number of optimal values of auxiliary constants by using the Eq. (2.13) and choose those optimal values whose sum is in $[-1,\ 0)$. *Now, substituting the optimal values of auxiliary constants (from the Table 1) into the Eq. (3.12), we obtain the AHPM solutions for different values of* α *at* $k = 0.001$

If $\alpha = 1$, *then we have*

$$
\begin{aligned}
&\widetilde{u}(x,y,t) \\
&= 6.67e^{-4}\cosh(2z) - 1.5e^{-36}\cosh(2x-2y) + 1.52e^{-10}t^2\cosh(2z) - 8.2e^{-10}t^2\cosh(4z) - 9.82e^{-19}t^4 \\
&\cosh(2z) - 1.96\,e^{-17}t^4\cosh(4z) - 3.44\,e^{-16}t^4\cosh(8z) + 9.74e^{-15}t^3\sinh(2z) - 2.7\,e^{-13}t^3\sinh(4z) - \\
&7.91\,e^{-13}t^3\sinh(8z) + 2.45e^{-16}t^4\cosh(10z) + 8.79e^{-10}t^2\cosh(6z) + 1.56e^{-16}t^4\cosh(6z) + 8.85e^{-13}t^3 \\
&\sinh(6z) + 1.02e^{-7}t\sinh(2z) - 1.27e^{-7}t\sinh(4z) - 6.67e^{-4}.
\end{aligned}
$$

Table 1. The auxiliary control constants for the problem 3.1.

Aux. Const.	$\alpha = 1$	$\alpha = 0.75$	$\alpha = 0.67$
c_1	−0.03206298594	0.02857059949	0.02811316381
c_2	−0.05626044816	−0.09201640919	−0.09255449827
c_3	−0.03255192821	0.23864503710	0.24163106460
c_4	0.09230916402	−0.58155364530	−0.59008434920
$c_1 + c_2 + c_3 + c_4$	−0.0286	−0.4064	−0.4129

Table 2. Solution of the problem 3.1 for various values of α, x, y and t at $k = 0.001$.

x	y	t	VIM [1] ($\alpha = 0.67$)	AHPM ($\alpha = 0.67$)	VIM [1] ($\alpha = 0.75$)	AHPM ($\alpha = 0.75$)
0.1	0.1	0.2	$5.312992862e-5$	0.000053661825095	$5.325267164e-5$	0.000053719590053
		0.3	$5.285029317e-5$	0.000053541299605	$5.297615384e-5$	0.000053602860321
		0.4	$5.260303851e-5$	0.000053433636044	$5.272490734e-5$	0.000053495694982
0.6	0.6	0.2	$2.953543396e-3$	0.0029991909006	$2.964363202e-3$	0.0030049612043
		0.3	$2.928652795e-3$	0.0029871641418	$2.939926307e-3$	0.0029932937681
		0.4	$2.905913439e-3$	0.0029764476067	$2.917239345e-3$	0.002982607386
0.9	0.9	0.2	$1.045289537e-2$	0.011096367296	$1.064555345e-2$	0.011161869914
		0.3	$0.990546789e-2$	0.010960230827	$1.017186398e-2$	0.011029208413
		0.4	$0.927982231e-2$	0.010839735959	$0.960539982e-2$	0.010908463773

If $\alpha = 0.75$, *then we have*

$$\widetilde{u}(x,y,t) = 6.67e^{-4}\cosh(2z) - 1.5e^{-36}\cosh(2z) - 9.63e^{-19}t^{3}\cosh(2z) - 5.12e^{-17}t^{3}\cosh(4z) + 4.09e^{-10}t^{3/2}\cosh(2z) - 7.79e^{-16}t^{3}\cosh(8z) - 2.2e^{-9}t^{3/2}\cosh(4z) + 1.57e^{-6}t^{3/4}\sinh(2z) - 1.97e^{-6}t^{3/4}\sinh(4z) + 2.65e^{-14}t^{9/4}\sinh(2z) - 6.14\,e^{-13}t^{9/4}\sinh(4z) - 1.74\,e^{-12}t^{9/4}\sinh(8z) + 5.34\,e^{-16}t^{3}\cosh(10z) + 3.66\,e^{-16}t^{3}\cosh(6z) + 2.36\,e^{-9}t^{3/2}\cosh(6z) + 1.98e^{-12}t^{9/4}\sinh(6z) - 6.67e^{-4}.$$

If $\alpha = 0.67$, *then we have*

$$\widetilde{u}(x,y,t) = 6.67e^{-4}\cosh(2z) - 1.5e^{-36}\cosh(2x-2y) - 8.33e^{-19}t^{67/25}\cosh(2z) - 7.12e^{-17}t^{67/25}\cosh(4z) - 1.05e^{-15}t^{67/25}\cosh(8z) + 4.57\,e^{-10}t^{67/50}\cosh(2z) - 2.46\,e^{-9}t^{67/50}\cosh(4z) + 1.63\,e^{-6}t^{67/100}\sinh(2z) - 2.03\,e^{-6}t^{67/100}\sinh(4z) + 3.45\,e^{-14}t^{201/100}\sinh(2z) - 7.54\,e^{-13}t^{201/100}\sinh(4z) - 2.1\,e^{-12}t^{201/100}\sinh(8z) + 7.09\,e^{-16}t^{67/25}\cosh(10z) + 4.97\,e^{-16}t^{67/25}\cosh(6z) + 2.64\,e^{-9}t^{67/50}\cosh(6z) + 2.41\,e^{-12}t^{201/100}\sinh(6z) - 6.67e^{-4}.$$

Table 3. Solution and absolute error of the problem 3.1 for various values of x, y and t at $k = 0.001, \alpha = 1$.

x	y	t	VIM [1]	AHPM	Exact	VIM [1] Error	AHPM Error
0.1	0.1	0.2	$5.355612471e-5$	0.00005403406722	$5.393877159e-5$	$3.83e-7$	$9.53e-8$
		0.3	$5.331384269e-5$	0.000054026996643	$5.388407669e-5$	$5.7e-7$	$1.43e-7$
		0.4	$5.307396595e-5$	0.000054019939232	$5.382941057e-5$	$7.55e-7$	$1.91e-7$
0.6	0.6	0.2	$2.991347666e-3$	0.0030365547595	$3.036507411e-3$	$4.52e-5$	$4.73e-8$
		0.3	$2.969760240e-3$	0.0030358658613	$3.035778955e-3$	$6.6e-5$	$8.69e-8$
		0.4	$2.948601126e-3$	0.0030351876618	$3.035050641e-3$	$8.64e-5$	$1.37e-7$
0.9	0.9	0.2	$1.102746671e-2$	0.011526053047	$1.153697757e-2$	$5.1e-4$	$1.09e-5$
		0.3	$1.073227877e-2$	0.011518772848	$1.153454074e-2$	$8.02e-4$	$1.58e-5$
		0.4	$1.035600465e-2$	0.011511900292	$1.153210438e-2$	0.00118	$2.02e-5$

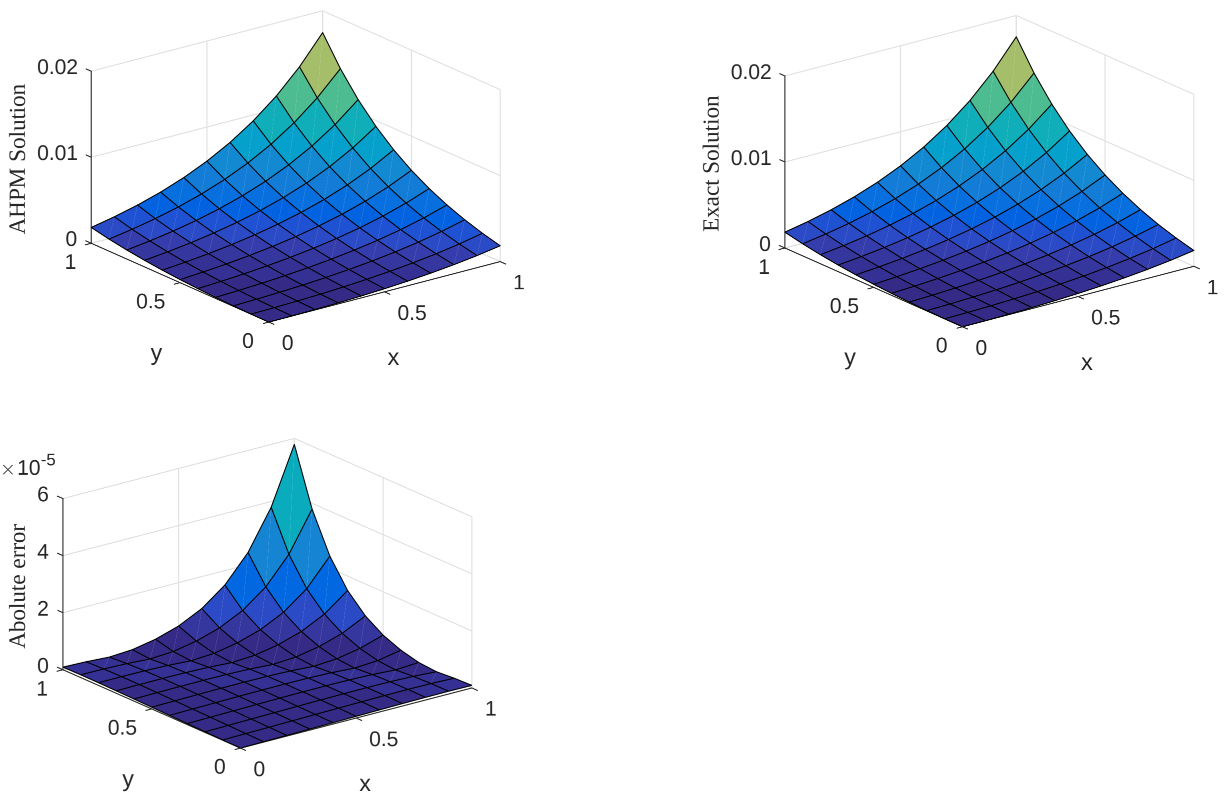

Figure 1. AHPM solution, exact solution and absolute error of AHPM solution of Problem 3.1 at α=1 and t=0.5 when k=0.001.

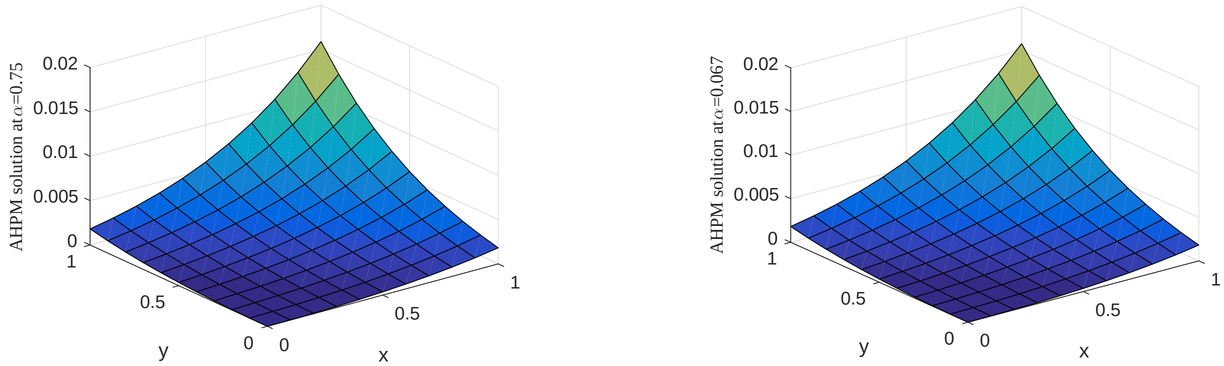

Figure 2. AHPM solution of the Problem 3.1 for various values of x and y at t=0.5 when k=0.001.

Tables 2 and 3 show the AHPM solution,VIM solution, exact solution and absolute error of AHPM solution. It is obvious from Tables 2 and 3 that AHPM solution results are more accurate to the exact solution results as compare with VIM [11] solution results. The AHPM solution, exact solution and absolute error of AHPM solution are plotted for different values of α, x, y and t in Figures 1 and 2. The curves of AHPM and exact solution are exactly matching as compare with homotopy perturbation transform method (HPTM) [12]. It is obvious from the Tables 2 and 3, Figures 1 and 2, that the AHPM solution of the problem 3.1 is in very good agreement with exact solution.

Problem 3.2. *Now, we consider FZK*$(3,3,3)$ *in the form:*

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} + \left(u^3(x,y,t)\right)_x + 2\left(u^3(x,y,t)\right)_{xxx} + 2\left(u^3(x,y,t)\right)_{yyx} = 0, \quad 0 < \alpha \le 1, \tag{3.14}$$

subject to condition

$$u(x,y,0) = \frac{3}{2}k\sinh\left(\frac{1}{6}(x+y)\right).$$

When $\alpha = 1$. *Then the exact solution of equation* (3.14)*:*

$$u(x,y,t) = \frac{3}{2}k\sinh\left(\frac{1}{6}(x+y-kt)\right).$$

As in Eq. (3.14), the non linear part:

$$N(u(x,y,t)) = \left(u^3(x,y,t)\right)_x + 2\left(u^3(x,y,t)\right)_{xxx} + 2\left(u^3(x,y,t)\right)_{yyx}.$$

Now, follow the procedure of AHPM, we obtain series of the simpler linear problems as follow:

Zero order problem:

$$\frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, \ u_0 = \frac{3}{2}k\sinh\left(\frac{1}{6}(x+y)\right). \tag{3.15}$$

First order problem:

$$\frac{\partial^{\alpha} u_1}{\partial t^{\alpha}} = B_1 N_0, \ u_1 = 0. \tag{3.16}$$

Second order problem:

$$\frac{\partial^{\alpha} u_2}{\partial t^{\alpha}} = B_2 N_0 + B_1 N_1, \ u_2 = 0. \tag{3.17}$$

Third order problem:

$$\frac{\partial^{\alpha} u_3}{\partial t^{\alpha}} = B_3 N_0 + B_2 N_1 + B_1 N_2, \ u_3 = 0. \tag{3.18}$$

Fourth order problem:

$$\frac{\partial^{\alpha} u_4}{\partial t^{\alpha}} = B_4 N_0 + B_3 N_1 + B_2 N_2 + B_1 N_3, \quad u_4 = 0. \tag{3.19}$$

The respective solutions of the Eqs. (3.15)–(3.19) are given as follow:

$$\begin{aligned}
u_0(x,y,t) &= \frac{3}{2} k \sinh\left(\tfrac{1}{6}z\right), \\
u_1(x,y,t) &= B_1 N_0 \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
u_2(x,y,t) &= B_2 N_0 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{3}{32} k^5 {B_1}^2 \left[\begin{array}{l} 801 \sinh^3\left(\tfrac{1}{6}z\right) + 765 \sinh^4\left(\tfrac{1}{6}z\right) \\ +127 \sinh\left(\tfrac{1}{6}z\right) \end{array}\right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},
\end{aligned}$$

$$u_3(x,y,t) = B_3 N_0 \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{3}{32} k^5 B_1 B_2 \left[801 \sinh^3\left(\tfrac{1}{6}z\right) + 765 \sinh^4\left(\tfrac{1}{6}z\right) + 127 \sinh\left(\tfrac{1}{6}z\right)\right] \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+B_1 \tfrac{3}{8192} k^5 \left[\begin{array}{l} B_2 \left(1120 \sinh\left(\tfrac{1}{6}z\right) - 9936 \sinh\left(\tfrac{1}{2}z\right) + 12240 \sinh\left(\tfrac{5}{6}z\right)\right) \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ {B_1}^2 k^2 \left(\begin{array}{l} 1350 \cosh\left(\tfrac{1}{2}z\right) + 2770 \cosh\left(\tfrac{1}{6}z\right) - 29070 \cosh\left(\tfrac{5}{6}z\right) \\ +32886 \cosh\left(\tfrac{7}{6}z\right) \end{array}\right) \tfrac{\Gamma(2\alpha+1) t^{3\alpha}}{(\Gamma(\alpha+1))^2 \Gamma(3\alpha+1)} \\ {B_1}^2 k^2 \left(\begin{array}{l} 56079 \cosh\left(\tfrac{1}{2}z\right) - 4155 \cosh\left(\tfrac{1}{6}z\right) - 182835 \cosh\left(\tfrac{5}{6}z\right) \\ +155295 \cosh\left(\tfrac{7}{6}z\right) \end{array}\right) \tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{array}\right],$$

$$
\begin{aligned}
&u_4(x,y,t) = \\
&B_4 N_0 \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{3}{32}k^5 B_1 B_3 \left[801\sinh^3\left(\tfrac{1}{6}z\right) + 765\sinh^4\left(\tfrac{1}{6}z\right) + 127\sinh\left(\tfrac{1}{6}z\right)\right]\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + B_2 \tfrac{3}{8192}k^5 \\
&\left[\begin{array}{l}
B_2\left(1120\sinh\left(\tfrac{1}{6}z\right) - 9936\sinh\left(\tfrac{1}{2}z\right) + 12240\sinh\left(\tfrac{5}{6}z\right)\right)\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
B_1{}^2k^2\left(\begin{array}{l} 1350\cosh\left(\tfrac{1}{2}z\right) + 2770\cosh\left(\tfrac{1}{6}z\right) - 29070\cosh\left(\tfrac{5}{6}z\right) \\ +32886\cosh\left(\tfrac{7}{6}z\right)\end{array}\right)\tfrac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} \\
B_1{}^2k^2\left(\begin{array}{l} 56079\cosh\left(\tfrac{1}{2}z\right) - 4155\cosh\left(\tfrac{1}{6}z\right) - 182835\cosh\left(\tfrac{5}{6}z\right) \\ +155295\cosh\left(\tfrac{7}{6}z\right)\end{array}\right)\tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}
\end{array}\right] + B_1\tfrac{1}{131072}k^5 \\
&\left[\begin{array}{l}
B_1{}^3k^3\left(937040\cosh\left(\tfrac{1}{3}z\right) - 36000\cosh\left(\tfrac{2}{3}z\right) + 16083360\cosh\left(\tfrac{4}{3}z\right)\right)\tfrac{\Gamma(3\alpha+1)t^{4\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)} \\
+B_3\left(-476928\sinh\left(\tfrac{1}{2}z\right) + 53760\sinh\left(\tfrac{1}{6}z\right) + 587520\sinh\left(\tfrac{5}{6}z\right)\right)\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
+B_1{}^3k^4\left(\begin{array}{l} 552744\sinh\left(\tfrac{1}{2}z\right) + 1771470\sinh\left(\tfrac{3}{2}z\right) - 63900\sinh\left(\tfrac{1}{6}z\right) \\ -275400\sinh\left(\tfrac{5}{6}z\right) - 1479870\sinh\left(\tfrac{7}{6}z\right)\end{array}\right)\tfrac{\Gamma(3\alpha+1)t^{4\alpha}}{(\Gamma(\alpha+1))^2\Gamma(4\alpha+1)} \\
+B_1{}^3k^4\left(\begin{array}{l} -2349000\sinh\left(\tfrac{1}{2}z\right) + 39956490\sinh\left(\tfrac{3}{2}z\right) - 21300\sinh\left(\tfrac{1}{6}z\right) \\ +23555880\sinh\left(\tfrac{5}{6}z\right) - 57758778\sinh\left(\tfrac{7}{6}z\right)\end{array}\right)\tfrac{\Gamma(2\alpha+1)t^{4\alpha}}{(\Gamma(\alpha+1))^2\Gamma(4\alpha+1)} \\
+B_1{}^3k^4\left(\begin{array}{l} 2948400\sinh\left(\tfrac{1}{2}z\right) + 33461100\sinh\left(\tfrac{3}{2}z\right) - 540600\sinh\left(\tfrac{1}{6}z\right) \\ +9510480\sinh\left(\tfrac{5}{6}z\right) - 39704364\sinh\left(\tfrac{7}{6}z\right) - 10167120\cosh(z)\end{array}\right)\tfrac{\Gamma(3\alpha+1)t^{4\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)} \\
+B_1{}^3k^4\left(\begin{array}{l} -24230988\sinh\left(\tfrac{1}{2}z\right) + 188683425\sinh\left(\tfrac{3}{2}z\right) + 903510\sinh\left(\tfrac{1}{6}z\right) + \\ 147146220\sinh\left(\tfrac{5}{6}z\right) - 300495825\sinh\left(\tfrac{7}{6}z\right)\end{array}\right)\tfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\
+B_1B_2k^2\left(\begin{array}{l} 5383584\cosh\left(\tfrac{1}{2}z\right) - 398880\cosh\left(\tfrac{1}{6}z\right) - 17552160\cosh\left(\tfrac{5}{6}z\right) \\ +14908320\cosh\left(\tfrac{7}{6}z\right)\end{array}\right)\tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
+B_1B_2k\left(1399680\sinh(z) - 108160\sinh\left(\tfrac{1}{3}z\right) - 576000\sinh\left(\tfrac{2}{3}z\right)\right)\tfrac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} \\
+B_1B_2k^2\left(\begin{array}{l} 129600\cosh\left(\tfrac{1}{2}z\right) + 265920\cosh\left(\tfrac{1}{6}z\right) - 2790720\cosh\left(\tfrac{5}{6}z\right) \\ +3157056\cosh\left(\tfrac{7}{6}z\right)\end{array}\right)\tfrac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}
\end{array}\right]
\end{aligned}
$$

and so on.

In similar way, we can compute the solution of the next simpler linear problems. We choose $B_1 = c_1, B_2 = c_2, B_3 = c_3, B_4 = c_4$ *and consider*

$$\widetilde{u}(x) = u_0(x) + u_1(x, c_1) + u_2(x, c_1,\ c_2) + u_3(x, c_1,\ c_2,\ c_3) + u_3(x, c_1,\ c_2,\ c_3,\ c_4). \tag{3.20}$$

We compute the residual;

$$R\left(\widetilde{u}(x,y,t)\right) = \frac{\partial^{\alpha}\widetilde{u}(x,y,t)}{\partial t^{\alpha}} + \left(\widetilde{u}^3(x,y,t)\right)_x + 2\left(\widetilde{u}^3(x,y,t)\right)_{xxx} + 2\left(\widetilde{u}^3(x,y,t)\right)_{yyx}.$$

We obtain number of optimal values of auxiliary constants by using the Eq. (2.13) and choose those optimal values whose sum is in $[-1,\ 0)$. *Now, substituting the optimal values of auxiliary constants (from Table 4) into the Eq. (3.20), we obtain the AHPM solutions for different values of* α *at* $k = 0.001$.

If $\alpha = 1$, *then we have*

$\widetilde{u}(x, y, t) =$

$1.5\ e^{-3}\sinh(0.17z) - 1.3\ e^{-21}t^3\cosh(0.5z) - 6.2\ e^{-16}t^2\sinh(0.5z) - 6.4\ e^{-28}t^4\sinh(0.5z) - 3.4\ e^{-21}\ t^3\cosh(1.2z) - 1.4e^{-21}t^3\cosh(1.2z) - 2.1e^{-26}t^4\sinh(1.2z) + 1.8e^{-24}t^4\cosh(1.3z) + 1.1e^{-25}t^4\cosh(0.33z) - 2.2e^{-23}t^3\cosh(0.17z) - 4.1e^{-27}t^4\cosh(0.67z) - 8.9e^{-24}t^3c\cosh(0.17z) + 1.4e^{-20}t^3\sinh(0.33z) + 2.3\ e^{-17}t^2\sinh(0.17z) + 7.7e^{-20}t^3\sinh(0.67z) + 4.6\,e^{-17}t^2\sinh(0.17z) - 4.3\ e^{-29}t^4\sinh(0.17z) + 1.4\ e^{-26}t^4\sinh(1.5z) + 5.3\ e^{-21}t^3\ \cosh(0.83z) + 5.1\ e^{-16}\ t^2\ \sinh(0.83z) + 8.4\ e^{-27}\ t^4\ \sinh(0.83z) + 2.5\ e^{-16}\ t^2\sinh(0.83z) - 1.2\ e^{-24}t^4\cosh(z) - 1.9e^{-19}t^3\sinh(z) - 3.8\ e^{-10}t\cosh(0.17z)(9.0\cosh^2(0.17z) - 8) + 3.1\ e^{-17}t^2\sinh(0.17z)(800.0\sinh^2(0.17z) + 766\sinh^4(0.17z) + 133).$

If $\alpha = 0.75$, *then we have*

$\widetilde{u}(x, y, t) =$

$1.5e^{-3}\sinh(0.17z) - 3.3e^{-21}t^{2.25}\cosh(0.5z) - 3.3e^{-27}t^3\sinh(0.5z) - 9.1e^{-16}t^{1.5}\sinh(0.5z) - 8.3e^{-21}t^{2.25}\cosh(1.2z) - 3.6e^{-21}t^{2.25}\cosh(1.2z) - 8e^{-26}t^3\sinh(1.2z) + 5.6e^{-24}t^3\cosh(1.3z) + 3.3e^{-25}t^3\cosh(0.33z) - 8.2e^{-24}t^{2.25}\cosh(0.17z) - 1.3e^{-26}t^3\cosh(0.67z) - 3.5e^{-24}t^{2.25}\cosh(0.17z) + 3.1\ e^{-20}t^{(2.25}\sinh(0.33z) + 3.9\,e^{-17}t^{1.5}\sinh(0.17z) + 1.6\,e^{-19}t^{2.25}\sinh(0.67z) - 7.8\,e^{-29}t^3\sinh(0.17z) + 6.3e^{-17}\ t^{1.5}\ \sinh(0.17z) + 5.5\,e^{-26}t^3\sinh(1.5z) + 13\ \ e^{-21}t^{2.25}\cosh(0.83z) + 3.4e^{-26}t^3\sinh(0.83z) + 11.2\,e^{-16}t^{1.5}\sinh(0.83z) - 3.5\ e^{-24}\ t^3\cosh(z) - 4\ \ e^{-19}\ t^{2.25}\sinh(z) - 4.1\ \ e^{-10}\ t^{0.75}\cosh(0.17z)\ \ (9\cosh^2(0.17z) - 8) + 4.7\,e^{-17}t^{1.5}\sinh(0.17z)\ \ (800\sinh^2(0.17z) + 766\sinh^4(0.17z) + 133).$

If $\alpha = 0.67$, *then we have*

$\widetilde{u}(x, y, t) =$

$1.5\ e^{-3}\ \sinh(0.17z) - 2.6\ e^{-21}t^{2.01}\ \cosh(0.5z) - 9.3\ e^{-29}t^{2.68}\sinh(0.5z) - 5.3\ e^{-15}t^{1.34}\sinh(0.5z) - 3.3e^{-21}t^{2.01}\cosh(1.2z) - 5.8e^{-21}t^{2.01}\ \cosh(1.2z) - 2.1e^{-27}t^{2.68}\ \sinh(1.2z) + 1.4e^{-25}t^{2.68}\cosh(1.3z) + 7.9\ e^{-27}t^{2.68}\cosh(0.33z) + 1.6\ e^{-24}\ t^{2.01}\cosh(0.17z) - 3.0\ e^{-28}\ t^{2.68}\cosh(0.67z) + 2.7\ e^{-24}\ t^{2.01}\cosh(0.17z) + 4.7\ e^{-20}t^{2.01}\sinh(0.33z) + 9.9\ e^{-16}t^{1.34}\ \sinh(0.17z) + 2.5\ e^{-19}t^{2.01}\ \sinh(0.67z) - 1.4\ e^{-30}t^{2.68}\sinh(0.17z) - 3.9\,e^{-16}t^{1.34}\sinh(0.17z) + 1.4\,e^{-27}t^{2.68}\sinh(1.5z) + 10\ e^{-21}t^{2.01}\cosh(0.83z) + 9.0e^{-28}t^{2.68}\sinh(0.83z) - 4.3e^{-15}t^{1.34}\sinh(0.83z) + 1.1e^{-14}t^{1.34}\sinh(0.83z) - 8.6e^{-26}t^{2.68}\cosh(z) - 6.1e^{-19}t^{2.01}\sinh(z) - 4.2e^{-10}t^{0.67}\cosh(0.17z)(9\cosh^2(0.17z) - 8) - 5.8e^{-17}t^{1.34}\sinh(0.17z)(800\sinh^2(0.17z) + 766\sinh^4(0.17z) + 133).$

Tables 5–7 show the AHPM solution,VIM solution, exact solution and absolute error of AHPM solution. The AHPM solution, exact solution and absolute error of AHPM solution are plotted for different values of α, x, y and t in Figures 3 and 4. It is obvious from the Tables 5–7, Figures 3 and 4, that the AHPM solution of the problem 3.2 is in very good agreement with exact solution.

4. Discussion and conclusions

In this article, asymptotic homotopy perturbation method (AHPM) is developed to solve non-linear fractional models. It is a different procedure from the procedures of HAM, HPM and OHAM. The

Table 4. The auxiliary control constants for the problem 3.2.

Aux.Const.	$\alpha = 1$	$\alpha = 0.75$	$\alpha = 0.67$
c_1	−0.5877657093	−0.6018150968	−0.2216842316
c_2	−0.1517357373	−0.1655454563	−1.594288158
c_3	−0.3855312838	−0.3415880346	5.172517408
c_4	0.1250311669	0.10894720410	−4.35654487
$c_1 + c_2 + c_3 + c_4$	−1.0000	−1.0000	−1.0000

Table 5. Solution of the problem 3.2 for varios values of α, x, y and t at $k = 0.001$.

x	y	t	VIM [1] ($\alpha = 0.67$)	AHPM ($\alpha = 0.67$)	VIM [1] ($\alpha = 0.75$)	AHPM ($\alpha = 0.75$)
0.1	0.1	0.2	$5.000911707e-5$	0.000050009117063	$5.000913646e-5$	0.000050009136457
		0.3	$5.000907252e-5$	0.000050009072517	$5.000909264e-5$	0.000050009092629
		0.4	$5.000903274e-5$	0.000050009032711	$5.000905240e-5$	0.00005000905238
0.6	0.6	0.2	$3.020038072e-4$	0.00030200380721	$3.020038341e-4$	0.00030200383392
		0.3	$3.020037458e-4$	0.00030200374584	$3.020037735e-4$	0.00030200377354
		0.4	$3.020036910e-4$	0.000302003691	$3.020037181e-4$	0.00030200371809
0.9	0.9	0.2	$4.567801693e-4$	0.00045678016935	$4.567802061e-4$	0.00045678020615
		0.3	$4.567800847e-4$	0.00045678008481	$4.567801231e-4$	0.00045678012298
		0.4	$4.567800092e-4$	0.00045678000927	$4.567800467e-4$	0.0004567800466

Table 6. Solution and absolute error of the problem 3.2 for various values of x, y and t at $k = 0.001$ and $\alpha = 1$.

x	y	t	VIM [1]	AHPM	Exact
0.1	0.1	0.2	$5.000918398e-5$	0.000050009183981	$4.995923204e-5$
		0.3	$5.000914609e-5$	0.000050009146085	$4.993421817e-5$
		0.4	$5.000910820e-5$	0.000050009108189	$4.990920434e-5$
0.6	0.6	0.2	$3.020038992e-4$	0.0003020038994	$3.019530008e-4$
		0.3	$3.020038472e-4$	0.00030200384719	$3.019274992e-4$
		0.4	$3.020037950e-4$	0.00030200379498	$3.019019978e-4$
0.9	0.9	0.2	$4.567802964e-4$	0.00045678029634	$4.567281735e-4$
		0.3	$4.567802242e-4$	0.00045678022442	$4.567020404e-4$
		0.4	$4.567801525e-4$	0.00045678015251	$4.566759074e-4$

Table 7. AHPM solution and exact solution and absolute error of AHPM solution of the problem 3.2 for various values of α, x, y and t at $k = 0.001$.

x	y	t	AHPM ($\alpha = 0.67$)	AHPM ($\alpha = 0.75$)	AHPM ($\alpha = 1$)	Exact ($\alpha = 1$)	Error
1	1	0.2	0.00050931053201	0.0005093105733	0.0005093106745	0.00050925803257	$5.26e-8$
		0.4	0.00050931035239	0.00050931039427	0.00050931051311	0.00050920522983	$1.05e-7$
		0.6	0.00050931020147	0.00050931023732	0.00050931035172	0.00050915242765	$1.58e-7$
		0.8	0.00050931006661	0.00050931009319	0.00050931019033	0.00050909962604	$2.11e-7$
		1	0.00050930994256	0.00050930995788	0.00050931002895	0.00050904682499	$2.63e-7$
5	5	0.2	0.003829187741	0.0038291908792	0.00382919857	0.0038290737537	$1.25e-7$
		0.4	0.0038291740912	0.0038291772737	0.0038291863046	0.003828936676	$2.5e-7$
		0.6	0.0038291626226	0.0038291653465	0.0038291740396	0.0038287996024	$3.74e-7$
		0.8	0.0038291523746	0.0038291543934	0.003829161775	0.0038286625332	$4.99e-7$
		1	0.0038291429482	0.003829144112	0.0038291495107	0.0038285254682	$6.24e-7$
10	10	0.2	0.020993470055	0.020993944094	0.02099510678	0.020996261505	$1.15e-6$
		0.4	0.020991408888	0.020991888681	0.02099325193	0.020995559858	$2.31e-6$
		0.6	0.020989679031	0.020990088717	0.020991398624	0.020994858233	$3.46e-6$
		0.8	0.020988134787	0.020988437313	0.020989546856	0.020994156633	$4.61e-6$
		1	0.020986715582	0.02098688854	0.020987696623	0.020993455055	$5.76e-6$

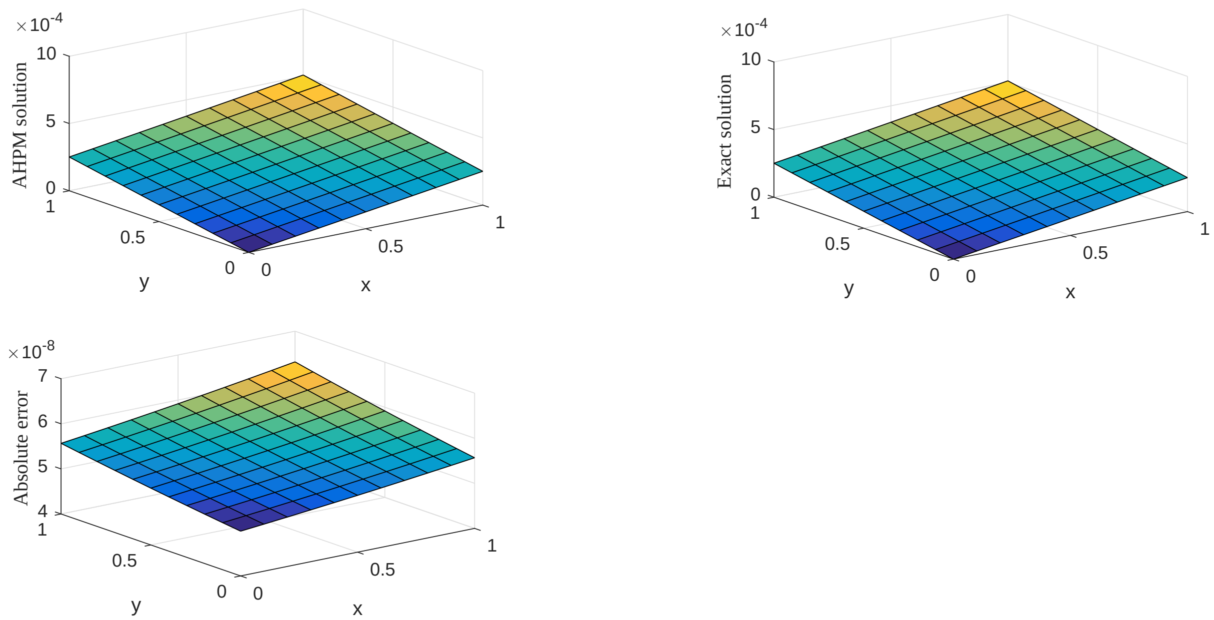

Figure 3. AHPM solution, exact solution and absolute error of AHPM solution of Problem 3.2 at α=1 and t=0.2 when k=0.001.

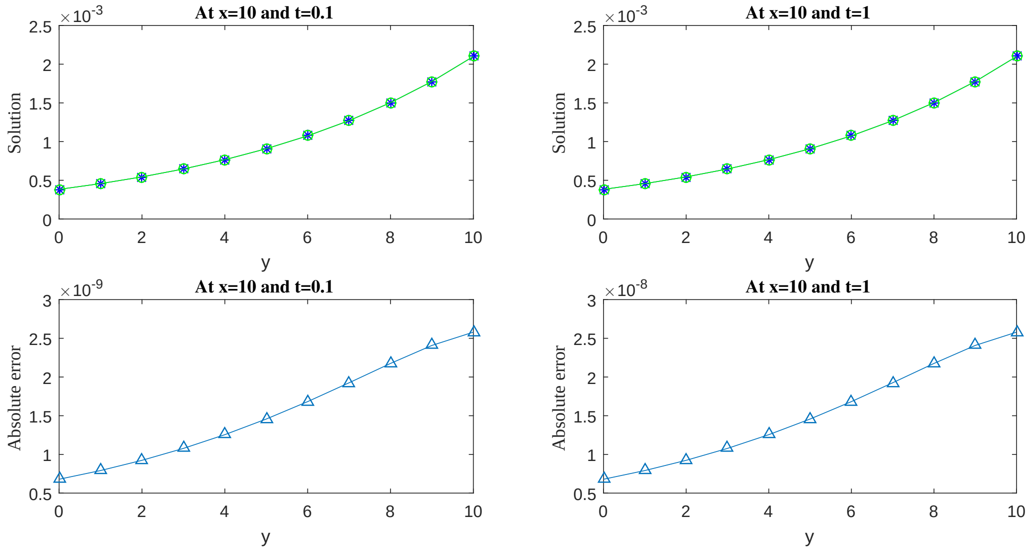

Figure 4. AHPM solution, exact solution and absolute error of AHPM solution of Problem 3.2 at α=1 and k = 0.0001.

two special cases, $ZK(2, 2, 2)$ and $ZK(3, 3, 3)$ of fractional Zakharov-Kuznetsov model are considered to illustrate a very simple procedure of the homotopy methods. The numerical results in simulation section of AHPM solutions are more accurate to the exact solutions as comparing with fractional complex transform (FCT) using variational iteration method (VIM). In the field of fractional calculus, it is necessary to introduce various procedures and schemes to compute the solution of non-linear fractional models. In this concern, we expect that this new proposed procedure is a best effort. The best improvement and the application of this new procedures to the solution of advanced non-linear fractional models with computer software codes will be our further consideration.

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Conflict of interest

The authors declare no conflict of interest.

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