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Research article

Proof of the completeness of the system of eigenfunctions for one boundary-value problem for the fractional differential equation

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Abstract: The present paper is devoted to the spectral analysis of operators induced by differential expressions of fractional order and boundary conditions of Sturm-Liouville type. In particular, this paper establishes the completeness of the system of eigenfunctions and associated functions of one class for non-self-adjoint integral operators associated with boundary-value problems for fractional-order differential equations.

Keywords: Mittag-Leffler function; spectrum; eigenvalue; fractional derivative; completeness
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1. Introduction

The problems of completeness of eigenfunctions and associated functions of the operators, generated by the ordinary differential expressions of fractional order and boundary conditions of Sturm-Liouville type are in the focus of many researchers. This mainly deals with the fact that such problems arise in solving boundary value problems for fractional differential equations for advection-diffusion using the method of separation of variables. In present paper, we resolve this very important problem by well-known Livshits theorem on spectral decomposition of linear nonself-adjoint operators.

2. Main Results

In paper [1] was studied operator in the space $L_2(0,1)$

$$A_\rho u = \int_0^1 G(x,t)u(t)\,dt = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt - \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)\,dt\right],$$

which was first considered in [2,3], where $0<\rho<2$ and

$$G(x,t) = \begin{cases} \dfrac{(1-t)^{\frac{1}{\rho}-1}x^{\frac{1}{\rho}-1} - (x-t)^{\frac{1}{\rho}-1}}{\Gamma(\rho^{-1})}, \; 0 \le t \le x \le 1 \\ \dfrac{(1-t)^{\frac{1}{\rho}-1}x^{\frac{1}{\rho}-1}}{\Gamma(\rho^{-1})}, \; 0 \le x \le t \le 1 \end{cases}$$

is the Green function of the following problem S (with $\lambda = 0$):

$$\frac{1}{\Gamma(n-\rho^{-1})}\frac{d^n}{dx^n}\int_0^x (x-s)^{n-\rho^{-1}-1}u(s)ds + \lambda\, u = 0,$$

$(n-1 \le \rho^{-1} < n,\; n = [\rho^{-1}]+1$, where $[\rho^{-1}]$ is the integer part of $\rho^{-1})$

$$u(0) = 0, u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, u(1) = 0.$$

In this case [1,4], if $\gamma_0 = \gamma_1 = \cdots = \gamma_n = 1$, then problem S takes the form

$$u^{(n)} + \lambda\, u = 0,$$

$$u(0) = 0, u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, u(1) = 0.$$

Its Green function $G(x,t)$ (for $\lambda = 0$) reads

$$G(x,t) = \begin{cases} \dfrac{(1-t)^{n-1}x^{n-1} - (x-t)^{n-1}}{(n-1)!}, 0 \le t \le x \le 1 \\ \dfrac{(1-t)^{n-1}x^{n-1}}{(n-1)!}, \; 0 \le x \le t \le 1 \end{cases}.$$

In particular, in this paper we provide very important proof of the completeness of the system of eigenfunctions and associated functions in L_2 of the operator A_ρ for $0<\rho<2$ (this fact plays main role in solving boundary value problems for advection-diffusion equation of fractional order by the method of separation of variables [5], based on well-known Livshits theorem [6].

Theorem (Livshits):

If $K(x,y)$ $(a \le x, y \le b)$ – is a limited kernel, and "real part" $\frac{1}{2}(K+K^*)$ of it is non-negative kernel, then the inequality is hold

$$\sum_{j=1}^{\infty} Re(\frac{1}{\lambda_j}) \le \int_a^b ReK(t,t)dt,$$

where λ_j – is the characteristic numbers of kernel K. The system of main functions of the kernel K is complete in domain of values of the integral operator Kf if and only if, when there is an equal sign in inequality above.

In his paper [7] M. M. Dzhrbashian wrote, that "the question about the completeness of the systems of eigenfunctions of the operator A_ρ or a finer question about whether these systems compose a basis in L_2, has a certain interest but its solving is apparently associated with significant analytic difficulties". The questions of the completeness of the systems of eigenfunctions and associated functions for similar problems were studied by A. V. Agibalova in [8,9]. Undoubtedly, we shall note the fundamental results of M. M. Malamud and L. L. Oridoroga [10–13], obtained in this direction. In [14, 15] (see also [2]), using the theorem of Matsaev and Palant , it was established that the system of eigenfunctions of the operator A_ρ is complete in L_2.

As noted above, in this paper, a similar result was obtained using the well-known Livshchits theorem [6].

The following proof of the completeness of the system of eigenfunctions is simpler than the previously presented proofs, which makes the results of this paper very significant.

Now we give the main result of paper.

Theorem 1. *The system of eigenfunctions and associated functions of the operator A_ρ, where $0 < \rho < 2$, is complete in L_2.*

Proof. Let us designate the kernel of A_ρ as $K(x, y)$. In [14] the authors have proved that this kernel is non-negative by the following way: Let us rewrite A_ρ as

$$A_\rho u = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^1 (x - xt)^{\frac{1}{\rho}-1} u(t)dt - \int_0^x (x - t)^{\frac{1}{\rho}-1} u(t)\, dt\right].$$

Clearly, for $\rho > 1$, the kernel of A_ρ is non-negative.

By the same way, we may show that the kernel $K^*(x, y)$ for adjoint operator

$$A^*_\rho u = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^1 (t - xt)^{\frac{1}{\rho}-1} u(x)dx - \int_x^1 (t - x)^{\frac{1}{\rho}-1} u(x)\, dx\right]$$

is non-negative too. Thus $\frac{1}{2}(K + K^*)$ is non-negative. Let us show that the following expression holds

$$\sum_{j=1}^{\infty} Re(\frac{1}{\lambda_j}) = \int_0^1 ReK(t, t)dt.$$

From [14], we know that the value λ_j is an eigenvalue of the operator A_ρ if and only if λ_j is a zero of the function $E_\rho(\lambda_j; \frac{1}{\rho})$, where [7]

$$E_\rho(z; \mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + k\rho^{-1})},\ \rho > 0.$$

The asymptotics of zeros for function $E_\rho(\lambda_j; \frac{1}{\rho}) = 0$ is well known. In particular, we have the following well-known Dzhrbaschian-Nersisian lemma [16, p.142].

Lemma (Dzhrbaschian-Nersisian): *1. All zeros of functions $E_\rho(z;\mu)$ (where $\rho > \frac{1}{2}, \rho \neq 1; Im\mu = 0$) with largest absolute values, are prime.*

2. The following asymptotic formulas are valid

$$\gamma_k^\pm = e^{\pm i\frac{\pi}{2\rho}}(2\pi k)^{1/\rho}\left(1 + O(\frac{log_k}{k})\right), \; k \to \infty.$$

So, if $\lambda_j = \alpha_j + i\beta_j$ is an eigenvalue of the operator A_ρ, the adjoint number $\overline{\lambda}_j = \alpha_j - i\beta_j$ will be an eigenvalue of the operator A_ρ. Therefore

$$spA_\rho = \sum_{j=1}^{\infty}\frac{1}{\lambda_j} = \sum_{j=1}^{\infty}Re(\frac{1}{\lambda_j}).$$

To find the trace spA_ρ of the operator A_ρ, let's rewrite A_ρ as $A_\rho u = A_1 u - A_0 u$ where

$$A_0 u = \frac{1}{\Gamma(\rho^{-1})}\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt,$$

$$A_1 u = \frac{1}{\Gamma(\rho^{-1})}\int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)dt.$$

Clearly, for $0 < \rho < 2$, the operators A_0 and A_1 are trace class. Hence

$$spA_\rho = sp(A_1 - A_0) = sp(A_1) - sp(A_0).$$

Moreover, it's clear that $sp(A_0) = 0$. Thus

$$spA_\rho = sp(A_1).$$

Since operator A_1 is one-dimensional, it's easy to find a trace. Consider the equation

$$u(x) - \frac{\lambda}{\Gamma(\rho^{-1})}\int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)dt = 0$$

The Fredhold determinant

$$d(\lambda) = |1 - \lambda K_{11}|,$$

where

$$K_{11} = \frac{1}{\Gamma(\rho^{-1})}\int_0^1 t^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}dt = \frac{\Gamma(2-\nu)}{\Gamma(4-2\nu)}\,(\nu = 2-\rho^{-1}).$$

From above follow that

$$sp(A_1) = \frac{\Gamma(2-\nu)}{\Gamma(4-2\nu)}$$

which proves the Theorem 1.

Remark. Since the operator A_ρ doesn't generate associated functions [17], we proved that the system of functions

$$\chi_n(x) = x^{\frac{1}{\rho}-1} E_\rho(\lambda_n x^{\frac{1}{\rho}}; \frac{1}{\rho})$$

is complete in L_2 (but the system of these functions, unfortunately, is not orthogonal).

Similarly, it is possible to provide a spectral analysis of the operator

$$A_\rho^{[\alpha^{-1},\rho]} u = \frac{1}{\Gamma(\rho^{-1})} \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\alpha-1} u(t)dt - \frac{1}{\Gamma(\rho^{-1})} \int_0^x (x-t)^{\frac{1}{\rho}-1} u(t)dt,$$

considered in ([14], see the references therein).

Theorem 2. *Let* $0 < \rho < 2$*,* $\alpha < \frac{1}{\rho}$*. Then, the system of eigenfunctions and associated functions of the operator* $A_\rho^{[\alpha^{-1},\rho]}$ *is complete in* L_2*.*

Proof. We carry out the proof of Theorem 2 in the same way as the proof of Theorem 1. It can be easily shown that the kernel $M(x, t)$ of the operator $A_\rho^{[\alpha^{-1},\rho]}$ is non-negative. Elementary calculations show that the kernel $M^*(x, t)$ of the operator adjoint to the operator $A_\rho^{[\alpha^{-1},\rho]}$ will be non-negative too. Thus $\frac{1}{2}(M + M^*)$ will be non-negative too. The fact that

$$\sum_{j=1}^{\infty} Re(\frac{1}{\mu_j}) = \int_0^1 ReM(t, t)dt$$

where μ_j are eigenvalues of the operator $A_\rho^{[\alpha^{-1},\rho]}$, shown in the same way as in Theorem 1.

3. Conclusion

In present paper, we provide the proof of the completeness of eigenfunctions and associated functions of the operators, generated by the ordinary differential expressions of fractional order and boundary conditions of Sturm-Liouville type.

Conflict of interest

The authors declare no conflict of interest.

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