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Research article

Depth and Stanley depth of edge ideals associated to some line graphs

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Abstract: In this paper, we compute some upper and lower bounds for depth and Stanley depth of edge ideals associated to line graphs of the ladder and circular ladder graphs. Furthermore, we determine some bounds for the dimension of the quotient rings of the edge ideals associated to these graphs.

Keywords: depth; Stanley depth; dimension; monomial ideal; line graph
Mathematics Subject Classification: 13C15, 13P10, 13F20, 05E99

1. Introduction

Let $S := K[x_1, \dots, x_n]$ be a polynomial ring over a field K, and A be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $a \in A$ be a homogeneous element and $X \subset \{x_1, x_2, \dots, x_n\}$. We denote by $aK[X]$ the K-subspace of A generated by all elements ab where b is a monomial in $K[X]$. The $\mathbb{Z}^n$-graded K-subspace $aK[X]$ of A is called a Stanley space of dimension $|X|$, if $aK[X]$ is a free $K[X]-$module. A Stanley decomposition of A is a presentation of K-vector space A as a finite direct sum of Stanley spaces

$$\mathcal{D} : A = \bigoplus_{i=1}^{r} a_i K[X_i].$$

The number $\text{sdepth}(\mathcal{D}) = \min\{|X_i| : i = 1, \dots, r\}$ is called the Stanley depth of $\mathcal{D}$. Let $\text{sdepth}(A) = \max\{\text{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } A\}$, then $\text{sdepth}(A)$ is called the Stanley depth of A. Stanley conjectured in [1] that $\text{sdepth}(A) \geq \text{depth}(A)$ for any $\mathbb{Z}^n$-graded S-module A. Let $I \subset J \subset S$ are monomial ideals and $A = J/I$, Ichim et al. reduced this conjecture to the case when J and I are the squarefree monomial ideals; see [2]. The above conjecture was disproved by Duval et al. by providing a counterexample; see [3]. Let $\mathfrak{m}$ be the unique graded maximal ideal of S. For an S-module A, the depth of A is an important algebraic invariant which is defined to be the maximal length of a regular sequence on A in $\mathfrak{m}$; see [4] for definition and results regarding depth. Herzog, Vladoiu and Zheng gave an algorithm for computing Stanley depth of modules of the type J/I by using some posets related to J/I; see [5]. However, it is too hard to compute Stanley depth by using their method, see for instance, [6–10]. Recently, Ichim et al. gave another algorithm in [11] for computing Stanley depth of any finitely generated $\mathbb{Z}^n$-graded S-module. But it is still hard to compute the Stanley depth even by using this new algorithm. Therefore, it's worth giving values and bounds for Stanley depth of some classes of modules. For some known results on Stanley depth, we refer the readers to [12–16].

The paper is organized as follows. In Section 2, we give definitions, notation, and discussion of some necessary results. In the third section, we find bounds for depth and Stanley depth of cyclic modules associated to line graphs of the ladder and circular ladder graphs; see Theorem 3.2 and 3.6. We also compute some bounds for Krull dimension of these cyclic modules; see Proposition 3.8 and 3.9.

2. Definitions and notation

Let G be a graph having vertex set $V(G) = \{a_1, a_2, \dots, a_n\}$ and edge set $E(G)$, then the edge ideal $I(G) = (x_i x_j : \{a_i, a_j\} \in E(G))$ associated with G is a squarefree monomial ideal of S. If G is a graph on vertices $\{a_1, a_2, \dots, a_n\}$, then G is said to be a path if $E(G) = \{\{a_i, a_{i+1}\} : i \in [n-1]\}$ and G is called a cycle if $E(G) = \{\{a_i, a_{i+1}\} : i \in [n-1]\} \bigcup \{\{a_1, a_n\}\}$. We use the notations P_n and C_n for path and cycle on n vertices respectively. A vertex cover of a graph G is a subset B of $V(G)$ such that for every edge $e \in E(G)$, $e \cap B \neq \emptyset$ and B is minimal with respect to this property, that is for any proper subset B' of B, then there exists an edge $e \in E(G)$ with $e \cap B' = \emptyset$. A prime ideal Q is a minimal prime of an ideal I if $I \subset Q$ and if Q' is a prime ideal with $I \subset Q' \subset Q$, then $Q' = Q$. It is easy to verify that B is a minimal vertex cover of G if and only if the prime ideal Q generated by the variables corresponding to vertices of B is a minimal prime of $I(G)$. Let $\alpha(G) := \min\{|B| : B \text{ is a minimal vertex cover of } G\}$, then $\alpha(G) = \text{height}(I)$. For vertices a and b of a graph G, the length of a shortest path from a to b is called the distance between a and b and it denoted by $\text{d}_G(a, b)$. If no such path exists between a and b, then $d_G(a, b) = \infty$. The diameter of a connected graph G is $\text{diam}(G) := \max\{\text{d}_G(a, b) : a, b \in V(G)\}$. For a detailed discussion on squarefree monomial ideals, see [17, 18] and for definitions from graph theory, see [19, 20].

Definition 2.1. *[19] For a given graph G, the line graph $L(G)$ of G is a graph whose vertex set is the edge set of G that is $V(L(G)) = E(G)$ and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex.*

The order of a graph is the cardinality of its vertex set, and size of a graph is the number of edges in it. The degree of a vertex v is denoted by $\deg(v)$, and it is the number of edges that are incident with v. The following lemma, due to Euler (1736), tells that if several people shake hands, then the number of hands shaken is even.

Lemma 2.2. *[20](Handshaking lemma) The sum of the degrees of the vertices of a graph G is twice the number of edges,*

$$\sum_{v \in V(G)} \deg(v) = 2E(G).$$

Definition 2.3. *[19] The Cartesian product of two graphs H_1 and H_2, is a graph, represented by $H_1 \Box H_2$, which has vertex set $V(H_1) \times V(H_2)$(the Cartesian product of sets), and for $(v_1, u_1), (v_2, u_2) \in V(H_1 \Box H_2)$, $(v_1, u_1)(v_2, u_2) \in E(H_1 \Box H_2)$, if either*

- $v_1v_2 \in E(H_1)$ *and* $u_1 = u_2$ *or*
- $v_1 = v_2$ *and* $u_1u_2 \in E(H_2)$.

If $n \geq 2$, then the Cartesian product of P_2 and P_n is called ladder graph. We denote this graph by $\mathcal{L}_n$ that is $\mathcal{L}_n := P_2 \square P_n$. For $n \geq 3$, the Cartesian product of P_2 and C_n is said to be a circular ladder graph. We denote this graph by $C\mathcal{L}_n$ that is $C\mathcal{L}_n := P_2 \square C_n$. Clearly $|V(P_2) \times V(P_n)| = |V(P_2) \times V(C_n)| = 2n$, thus we have $|V(\mathcal{L}_n)| = |V(C\mathcal{L}_n)| = 2n$. The graph $\mathcal{L}_n$ has four vertices of degree 2 and $2n - 4$ vertices of degree 3 so by using Lemma 2.2, we have $|E(\mathcal{L}_n)| = 3n - 2$. By definition of line graph, it follows that $|E(\mathcal{L}_n)| = |V(L(\mathcal{L}_n))| = 3n - 2$. If $n = 2$, then $\mathcal{L}_n \cong L(\mathcal{L}_n)$. Let $n \geq 3$, the graph $L(\mathcal{L}_n)$ has two vertices of degree 2, four vertices of degree 3 and $3n - 8$ vertices of degree 4, by Lemma 2.2 we have $|E(L(\mathcal{L}_n))| = 6n - 8$. Similarly, one can show that $|E(C\mathcal{L}_n)| = |V(L(C\mathcal{L}_n))| = 3n$, and $|E(L(C\mathcal{L}_n))| = 6n$. For examples of the ladder, circular ladder graphs and their corresponding line graphs see Figures 1 and 2.

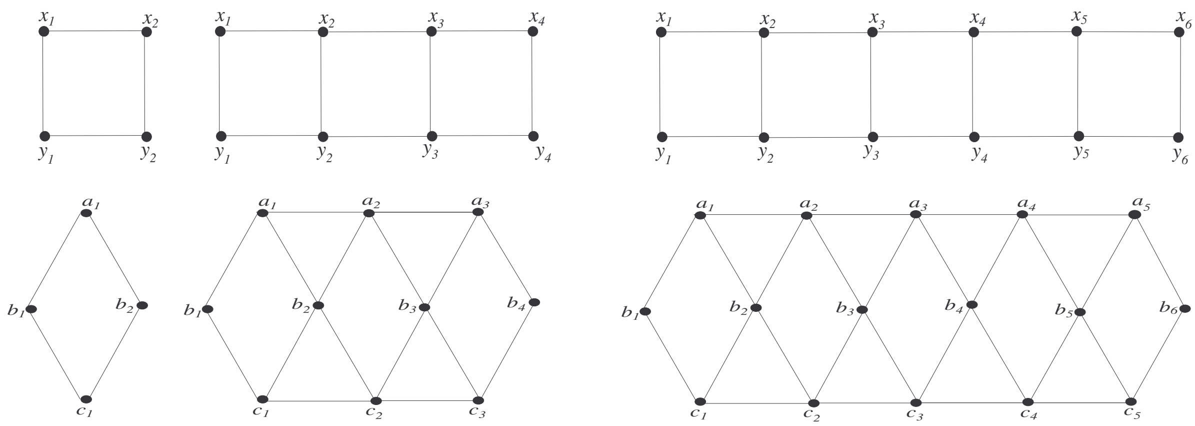

Figure 1. $\mathcal{L}_2$, $\mathcal{L}_4$, $\mathcal{L}_6$ and their line graphs $L(\mathcal{L}_2)$, $L(\mathcal{L}_4)$, $L(\mathcal{L}_6)$.

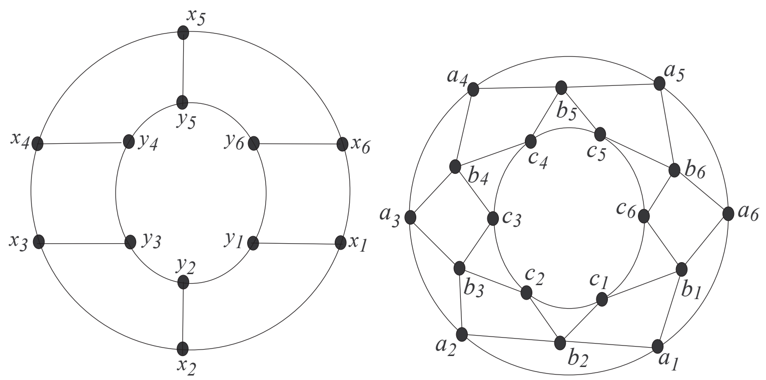

Figure 2. From left to right, $C\mathcal{L}_6$ and $L(C\mathcal{L}_6)$.

In the following, we recall several results that are used quite often in this paper.

Lemma 2.4. *[21] Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of $\mathbb{Z}^n$*-graded* S*-modules. Then*

$$\operatorname{sdepth}(B) \geq \min\{\operatorname{sdepth}(A), \operatorname{sdepth}(C)\}.$$

Lemma 2.5. *(Depth Lemma) If* $0 \to M \to N \to P \to 0$ *is a short exact sequence of modules over a local ring* S*, or a Noetherian graded ring with* S_0 *local, then*

1. $\operatorname{depth}(N) \geq \min\{\operatorname{depth}(P), \operatorname{depth}(M)\}$.
2. $\operatorname{depth}(M) \geq \min\{\operatorname{depth}(N), \operatorname{depth}(P) + 1\}$.
3. $\operatorname{depth}(P) \geq \min\{\operatorname{depth}(M) - 1, \operatorname{depth}(N)\}$.

Lemma 2.6 ([5, Lemma 3.6])**.** *Let* $J \subset S$ *be a monomial ideal, and* $\hat{S} = S[x_{n+1}, x_{n+2}, \ldots, x_{n+r}]$ *a polynomial ring of* $n + r$ *variables then*

$$\operatorname{depth}(\hat{S}/J\hat{S}) = \operatorname{depth}(S/JS) + r \quad \textit{and} \quad \operatorname{sdepth}(\hat{S}/J\hat{S}) = \operatorname{sdepth}(S/JS) + r.$$

Corollary 2.7 ([21, Corollary 1.3])**.** *Let* $J \subset S$ *be a monomial ideal. Then* $\operatorname{depth}(S/(J : v)) \geq \operatorname{depth}(S/J)$ *for all monomials* $v \notin J$*.*

Proposition 2.8 ([22, Proposition 2.7])**.** *Let* $J \subset S$ *be a monomial ideal. Then* $\operatorname{sdepth}(S/(J : v)) \geq \operatorname{sdepth}(S/J)$ *for all monomials* $v \notin J$*.*

Let $\lceil t \rceil$, $t \in Q$, denotes the smallest integer which is greater than or equal to t. Using Depth Lemma, Morey showed the following result.

Lemma 2.9 ([23, Lemma 2.8])**.** *Let* $n \geq 2$, *then* $\operatorname{depth}(S/I(P_n)) = \lceil \frac{n}{3} \rceil$.

Stefan proved a similar result for Stanley depth.

Lemma 2.10 ([24, Lemma 4])**.** *Let* $n \geq 2$, *then* $\operatorname{sdepth}(S/I(P_n)) = \lceil \frac{n}{3} \rceil$.

Cimpoeas proved the following results for depth and Stanley depth of the edge ideals of the cyclic graph.

Proposition 2.11 ([25, Proposition 1.3])**.** *Let* $n \geq 3$, *then* $\operatorname{depth}(S/I(C_n)) = \lceil \frac{n-1}{3} \rceil$.

Theorem 2.12 ([25, Theorem 1.9])**.** *Let* $n \geq 3$, *then*

(1) $\operatorname{sdepth}(S/I(C_n)) = \lceil \frac{n-1}{3} \rceil$, *if* $n \equiv 0, 2 \pmod 3$.
(2) $\operatorname{sdepth}(S/I(C_n)) \leq \lceil \frac{n}{3} \rceil$, *if* $n \equiv 1 \pmod 3$.

For the edge ideal of a graph G, Fouli and Morey gave the following lower bound for depth and Stanley depth in terms of the diameter of G.

Theorem 2.13 ([26, Theorems 3.1 and 4.18])**.** *Let* G *be a connected graph and* $I = I(G)$ *be the edge ideal of* G*. If* $d = \operatorname{diam}(G)$, *then*

$$\operatorname{depth}(S/I), \operatorname{sdepth}(S/I) \geq \lceil \frac{d+1}{3} \rceil.$$

3. Results and discussions

In this section, we find some bounds for depth and Stanley depth of the cyclic modules associated to the line graphs of $\mathcal{L}_n$ and $C\mathcal{L}_n$. We denote the edge ideals of the line graphs of $\mathcal{L}_n$ and $C\mathcal{L}_n$ with I_n and J_n respectively. We label the vertices of the line graphs of $\mathcal{L}_n$ and $C\mathcal{L}_n$ by using three sets of variables $\{a_1, a_2, \dots, a_n\}$, $\{b_1, b_2, \dots, b_n\}$ and $\{c_1, c_2, \dots, c_n\}$, see Figures 1 and 2. Let $S_n := K[a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_{n-1}]$ and $\overline{S}_n = S_n[a_n, c_n]$ be the rings of polynomials in these variables over the field K. Then I_n and J_n are squarefree monomial ideals of S_n and $\overline{S}_n$ respectively. With the labeling as shown in Figures 1 and 2, we have:

$$\mathcal{G}(I_n) = \bigcup_{i=1}^{n-1}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-2}\{a_ia_{i+1}, c_ic_{i+1}\},$$

$$\mathcal{G}(J_n) = \mathcal{G}(I_n) \bigcup \{a_1a_n, c_1c_n, a_{n-1}a_n, c_{n-1}c_n, b_1a_n, b_1c_n, a_nb_n, b_nc_n\},$$

where $\mathcal{G}(I_n)$ and $\mathcal{G}(J_n)$ stand for the minimal sets of monomial generators of monomial ideals I_n and J_n respectively.

Lemma 3.1. *For* $2 \le n \le 4$ *we have that* $\operatorname{depth}(S_n/I_n) = \operatorname{sdepth}(S_n/I_n) = n - 1$.

Proof. If $n = 2$, then $\mathcal{G}(I_2) = \{a_1b_1, b_1c_1, a_1b_2, b_2c_1\}$, which is a minimal generating set of the edge ideal of C_4. Thus by Proposition 2.11 it follows that $\operatorname{depth}(S_2/I_2) = 1$.
If $n = 3$, then $\mathcal{G}(I_3) = \mathcal{G}(I_2) \bigcup \{a_2b_2, b_2c_2, a_2b_3, b_3c_2, a_1a_2, c_1c_2\}$. Consider the following short exact sequence

$$0 \longrightarrow S_3/(I_3 : b_2) \xrightarrow{\cdot b_2} S_3/I_3 \longrightarrow S_3/(I_3, b_2) \longrightarrow 0. \tag{3.1}$$

Here $(I_3 : b_2) = (a_1, a_2, c_1, c_2)$, so we have $S_3/(I_3 : b_2) \cong K[b_1, b_2, b_3]$, thus $\operatorname{depth}(S_3/(I_3 : b_2)) = 3$. Also $(I_3, b_2) = (a_1b_1, b_1c_1, c_1c_2, c_2b_3, b_3a_2, a_2a_1, b_2)$, so we have $S_3/(I_3, b_2) \cong K[a_1, a_2, b_1, b_3, c_1, c_2]/(a_1b_1, b_1c_1, c_1c_2, c_2b_3, b_3a_2, a_2a_1) \cong K[a_1, a_2, b_1, b_3, c_1, c_2]/I(C_6)$, by Proposition 2.11, we have $\operatorname{depth}(S_3/(I_3, b_2)) = 2$. By using Depth lemma on the exact sequence (3.1), we obtain $\operatorname{depth}(S_3/I_3) \ge 2$. For the upper bound, since $b_3 \notin I_3$, by Corollary 2.7, we get $\operatorname{depth}(S_3/I_3) \le \operatorname{depth}(S_3/(I_3 : b_3))$. As $(I_3 : b_3) = (a_2, c_2, I_2)$, thus $S_3/(I_3 : b_3) \cong S_2/I_2[b_3]$, by Lemma 2.6, it follows that $\operatorname{depth}(S_3/(I_3 : b_3)) \le \operatorname{depth}(S_2/I_2) + 1 = 1 + 1 = 2$. Hence $\operatorname{depth}(S_3/I_3) = 2$. If $n = 4$, then $\mathcal{G}(I_4) = \mathcal{G}(I_3) \bigcup \{a_3b_3, b_3c_3, a_3b_4, b_4c_3, a_2a_3, c_2c_3\}$. Consider the following short exact sequence

$$0 \longrightarrow S_4/(I_4 : b_3) \xrightarrow{\cdot b_3} S_4/I_4 \longrightarrow S_4/(I_4, b_3) \longrightarrow 0. \tag{3.2}$$

Here $(I_4 : b_3) = (I_2, a_2, c_2, a_3, c_3)$, so we have $S_4/(I_4 : b_3) \cong S_2/I_2[b_3, b_4]$, thus Lemma 2.6 yields $\operatorname{depth}(S_4/(I_4 : b_3)) = \operatorname{depth}(S_2/I_2) + 2 = 1 + 2 = 3$. Let $T := (I_4, b_3) = (I_2, a_2b_2, b_2c_2, a_1a_2, a_2a_3, c_1c_2, c_2c_3, a_3b_4, c_3b_4, b_3)$. Again consider the following short exact sequence

$$0 \longrightarrow S_4/(T : b_2) \xrightarrow{\cdot b_2} S_4/T \longrightarrow S_4/(T, b_2) \longrightarrow 0. \tag{3.3}$$

Here $(T : b_2) = (a_1, a_2, c_1, c_2, b_3, a_3b_4, c_3b_4)$, so we have $S_4/(T : b_2) \cong K[a_3, b_4, c_3]/(a_3b_4, c_3b_4)[b_1, b_2]$, by Lemmas 2.6 and 2.9,

$\text{depth}(S_4/(T:b_2)) = 1+2 = 3$. Also $(T,b_2) = (a_1b_1, b_1c_1, c_1c_2, c_2c_3, c_3b_4, b_4a_3, a_3a_2, a_2a_1, b_2, b_3)$, which implies that

$$S_4/(T,b_2) \cong K[a_1,a_2,a_3,b_1,b_4,c_1,c_2,c_3]/(a_1b_1,b_1c_1,c_1c_2,c_2c_3,c_3b_4,b_4a_3,a_3a_2,a_2a_1) \cong K[a_1,a_2,a_3,b_1,b_4,c_1,c_2,c_3]/I(C_8)$$

thus Proposition 2.11 gives that $\text{depth}(S_4/(T,b_2)) = 3$. By applying [15, Lemma 3.1] on the exact sequences (3.2), and (3.3), we have $\text{depth}(S_4/I_4) = 3$.
For Stanley depth, if $n = 2$, then by Theorem 2.12, we have $\text{sdepth}(S_2/I_2) \le 1$. Also, we use [5] to show that there exist Stanley decompositions of desired Stanley depth.

$$S_2/I_2 = K[a_1] \oplus b_1K[b_1,b_2] \oplus c_1K[a_1,c_1] \oplus b_2K[b_2].$$

Thus we have $\text{sdepth}(S_2/I_2) = 1$. If $n = 3$, then by applying Lemmas 2.4, 2.10, and Theorem 2.12 on the exact sequences (3.1), we have $\text{sdepth}(S_3/I_3) \ge 2$. For upper bound, since $b_3 \notin I_3$, by Proposition 2.8, we get $\text{sdepth}(S_3/I_3) \le \text{sdepth}(S_3/(I_3:b_3))$. As $(I_3:b_3) = (a_2,c_2,I_2)$, thus $S_3/(I_3:b_3) \cong S_2/I_2[b_3]$, by Lemma 2.6, it follows that $\text{sdepth}(S_3/(I_3:b_3)) \le \text{sdepth}(S_2/I_2)+1 = 1+1 = 2$. Hence $\text{sdepth}(S_3/I_3) = 2$. If $n = 4$, by using Lemmas 2.4, 2.10, and Theorem 2.12 on the exact sequences (3.2) and (3.3), we have $\text{sdepth}(S_4/I_4) \ge 3$. For upper bound, since $b_4 \notin I_4$, by Proposition 2.8, we get $\text{sdepth}(S_4/I_4) \le \text{sdepth}(S_4/(I_4:b_4))$. As $(I_4:b_4) = (a_3,c_3,I_3)$, thus $S_4/(I_4:b_4) \cong S_3/I_3[b_4]$, by Lemma 2.6, it follows that $\text{sdepth}(S_4/(I_4:b_4)) \le \text{sdepth}(S_3/I_3)+1 = 2+1 = 3$. Hence $\text{sdepth}(S_4/I_4) = 3$. This completes the proof. □

Let $1 \le k \le n-1$ and $A_k := K[a_{n-1},a_{n-2},\dots,a_{n-k}], C_k := K[c_{n-1},c_{n-2},\dots,c_{n-k}], D_k := A_k \otimes_K C_k$ and
$\overline{D}_k := D_k \otimes_K K[b_1]$ be the subrings of S_n. Let $B_0 := (0)$, $B_j := (b_n,b_{n-1},\dots,b_{n-j+1})$, for $1 \le j \le n$, and for $3 \le j \le n-2$, $\mathcal{P}_{j-1} := (a_{n-j+1}a_{n-j+2}, a_{n-j+2}a_{n-j+3},\dots,a_{n-2}a_{n-1})$ and $\overline{\mathcal{P}}_{j-1} = (c_{n-j+1}c_{n-j+2},\dots,c_{n-2}c_{n-1})$ are the squarefree monomial ideals of S_n. In the following theorem, we give some bounds for depth and Stanley depth of S_n/I_n.

Theorem 3.2. *For $n \ge 2$ we have that* $\lceil \frac{n}{2} \rceil \le \text{depth}(S_n/I_n), \text{sdepth}(S_n/I_n) \le n-1$.

Proof. If $2 \le n \le 4$, then the result follows by Lemma 3.1. For $n \ge 5$, first we prove that $\lceil \frac{n}{2} \rceil \le \text{depth}(S_n/I_n) \le n-1$ using induction on n. For $0 \le j \le n-2$, consider the family of short exact sequences

$$0 \longrightarrow S_n/((I_n,B_0):b_n) \xrightarrow{\cdot b_n} S_n/(I_n,B_0) \longrightarrow S_n/(I_n,B_1) \longrightarrow 0 \qquad (E_1)$$

$$0 \longrightarrow S_n/((I_n,B_1):b_{n-1}) \xrightarrow{\cdot b_{n-1}} S_n/(I_n,B_1) \longrightarrow S_n/(I_n,B_2) \longrightarrow 0 \qquad (E_2)$$

$$0 \longrightarrow S_n/((I_n,B_2):b_{n-2}) \xrightarrow{\cdot b_{n-2}} S_n/(I_n,B_2) \longrightarrow S_n/(I_n,B_3) \longrightarrow 0 \qquad (E_3)$$

$$\vdots$$

$$0 \longrightarrow S_n/((I_n,B_j):b_{n-j}) \xrightarrow{\cdot b_{n-j}} S_n/(I_n,B_j) \longrightarrow S_n/(I_n,B_{j+1}) \longrightarrow 0 \qquad (E_{j+1})$$

$$\vdots$$

$$0 \longrightarrow S_n/((I_n,B_{n-2}):b_2) \xrightarrow{\cdot b_2} S_n/(I_n,B_{n-2}) \longrightarrow S_n/(I_n,B_{n-1}) \longrightarrow 0 \qquad (E_{n-1})$$

(1) If $j = 0$, then $(I_n : b_n) = (I_{n-1}, a_{n-1}, c_{n-1})$, so we have $S_n/(I_n : b_n) \cong S_{n-1}/I_{n-1}[b_n]$, the induction hypothesis and Lemma 2.6 give that $\text{depth}(S_n/(I_n : b_n)) \geq \lceil \frac{n-1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil$.

(2) If $j = 1$, then $((I_n, B_1) : b_{n-1}) = (I_{n-2}, a_{n-1}, a_{n-2}, c_{n-1}, c_{n-2}, B_1)$, so we obtain $S_n/((I_n, B_1) : b_{n-1}) \cong S_{n-2}/I_{n-2}[b_{n-1}]$, by induction and Lemma 2.6, it follows that $\text{depth}(S_n/((I_n, B_1) : b_{n-1})) \geq \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$.

(3) If $j = 2$, then $((I_n, B_2) : b_{n-2}) = (I_{n-3}, a_{n-2}, a_{n-3}, c_{n-2}, c_{n-3}, B_2)$, so we have $S_n/((I_n, B_2) : b_{n-2}) \cong S_{n-3}/I_{n-3}[a_{n-1}, b_{n-2}, c_{n-1}]$, the induction hypothesis and Lemma 2.6 give that $\text{depth}(S_n/((I_n, B_2) : b_{n-2})) \geq \lceil \frac{n-3}{2} \rceil + 3 = \lceil \frac{n+1}{2} \rceil + 1$.

(4) If $3 \leq j \leq n-3$, then $((I_n, B_j) : b_{n-j}) = \Big(I_{n-(j+1)}, (a_{n-j+1}a_{n-j+2}, a_{n-j+2}a_{n-j+3}, \dots, a_{n-2}a_{n-1}), (c_{n-j+1}c_{n-j+2}, c_{n-j+2}c_{n-j+3}, \dots, c_{n-2}c_{n-1}), a_{n-j}, a_{n-(j+1)}, c_{n-j}, c_{n-(j+1)}, B_j\Big)$, that further implies

$$S_n/((I_n, B_j) : b_{n-j}) \cong (S_{n-(j+1)}/I_{n-(j+1)}) \otimes_K (A_{j-1}/\mathcal{P}_{j-1}) \otimes_K (C_{j-1}/\overline{\mathcal{P}}_{j-1}) \otimes_K K[b_{n-j}].$$

By [18, Theorem 2.2.21], we have

$$\text{depth}(S_n/((I_n, B_j) : b_{n-j})) = \text{depth}(S_{n-(j+1)}/I_{n-(j+1)}) + \text{depth}(A_{j-1}/\mathcal{P}_{j-1}) + \text{depth}(C_{j-1}/\overline{\mathcal{P}}_{j-1}) + 1.$$

By Lemma 2.9, we get $\text{depth}(A_{j-1}/\mathcal{P}_{j-1}) = \lceil \frac{j-1}{3} \rceil = \text{depth}(C_{j-1}/\overline{\mathcal{P}}_{j-1})$ and by induction on n, $\text{depth}(S_{n-(j+1)}/I_{n-(j+1)}) \geq \lceil \frac{n-(j+1)}{2} \rceil$. Thus we have

$$\text{depth}(S_n/((I_n, B_j) : b_{n-j})) \geq \lceil \frac{n-(j+1)}{2} \rceil + \lceil \frac{j-1}{3} \rceil + \lceil \frac{j-1}{3} \rceil + 1.$$

(5) If $j = n-2$, then $((I_n, B_{n-2}) : b_2) = \Big((a_3a_4, a_4a_5, \dots, a_{n-2}a_{n-1}), a_{n-j}, a_{n-(j+1)}, c_{n-j}, c_{n-(j+1)}, (c_3c_4, \dots, c_{n-2}c_{n-1}), B_j\Big)$, so we have $S_n/((I_n, B_{n-2}) : b_2) \cong (A_{n-3}/\mathcal{P}_{n-3}) \otimes_K (C_{n-3}/\overline{\mathcal{P}}_{n-3}) \otimes_K K[b_1, b_2]$, by [18, Theorem 2.2.21], it follows that $\text{depth}(S_n/((I_n, B_{n-2}) : b_2)) = \text{depth}(A_{n-3}/\mathcal{P}_{n-3}) + \text{depth}(C_{n-3}/\overline{\mathcal{P}}_{n-3}) + 2$. By Lemma 2.9, $\text{depth}(A_{n-3}/\mathcal{P}_{n-3}) = \lceil \frac{n-3}{3} \rceil = \text{depth}(C_{n-3}/\overline{\mathcal{P}}_{n-3})$. Thus we have

$$\text{depth}(S_n/((I_n, B_{n-2}) : b_2)) = \lceil \frac{n-3}{3} \rceil + \lceil \frac{n-3}{3} \rceil + 2.$$

Also $(I_n, B_{n-1}) = (a_{n-1}a_{n-2}, \dots, a_2a_1, a_1b_1, b_1c_1, c_1c_2, c_2c_3, \dots, c_{n-2}c_{n-1}, B_{n-1})$, so we have $S_n/(I_n, B_{n-1}) \cong \overline{D}_{n-1}/I(\mathcal{P}_{2n-1})$. Thus by Lemma 2.9, it follows that $\text{depth}(S_n/(I_n, B_{n-1})) = \lceil \frac{2n-1}{3} \rceil$. By applying Depth Lemma on the above family of short exact sequences, we obtain the required lower bound for depth. Now by using induction on n, we show that $\text{depth}(S_n/I_n) \leq n-1$. For $n \geq 5$, as $b_n \notin I_n$, by Corollary 2.7, we have $\text{depth}(S_n/I_n) \leq \text{depth}(S_n/(I_n : b_n))$. Since $S_n/(I_n : b_n) \cong S_{n-1}/I_{n-1}[b_n]$, the induction hypothesis and Lemma 2.6 yield $\text{depth}(S_n/(I_n : b_n)) \leq n-1-1+1 = n-1$.

Now, it remains to show the result for Stanley depth. The required lower bound can be obtained by applying Lemmas 2.4, 2.10, and [21, Theorem 3.1] instead of Depth Lemma, Lemma 2.9, and [18, Theorem 2.2.21] respectively on above family of short exact sequences. Finally, we prove $\text{sdepth}(S_n/I_n) \leq n-1$ by using induction on n. For $n \geq 5$, as $b_n \notin I_n$, from Proposition 2.8, we get $\text{sdepth}(S_n/I_n) \leq \text{sdepth}(S_n/(I_n : b_n))$. As $S_n/(I_n : b_n) \cong S_{n-1}/I_{n-1}[b_n]$, by induction and Lemma 2.6, it follows that $\text{sdepth}(S_n/(I_n : b_n)) \leq n-1-1+1 = n-1$. This finishes the proof. □

Remark 3.3. Clearly $\text{diam}(L(\mathcal{L}_n)) = n$, then by Theorems 2.13 we have $\text{depth}(S_n/I_n), \text{sdepth}(S_n/I_n) \geq \lceil \frac{n+1}{3} \rceil$. Our Theorem 3.2 shows $\text{depth}(S_n/I_n), \text{sdepth}(S_n/I_n) \geq \lceil \frac{n}{2} \rceil$. Thus we find a better lower bound for depth and Stanley depth of these classes of edge ideals.

In order to find bounds for depth and Stanley depth of the cyclic module $\overline{S}_n/J_n$, we consider two supergraphs U_n and V_n of $L(\mathcal{L}_n)$. The vertex and edge sets of U_n are $V(U_n) = V(L(\mathcal{L}_n)) \cup \{c_n\}$ and $E(U_n) = E(L(\mathcal{L}_n)) \cup \{c_{n-1}c_n, b_nc_n\}$ respectively. The vertex and edge sets of V_n are $V(V_n) = V(U_n) \cup \{c_{n+1}\}$ and $E(V_n) = E(U_n) \cup \{c_1c_{n+1}, b_1c_{n+1}\}$ respectively. For examples of U_n and V_n, see Figure 3. We denote the edge ideals of U_n and V_n with I_n^* and I_n^{**} respectively. The minimal sets of monomial generators of I_n^* and I_n^{**} are $\mathcal{G}(I_n^*) = \mathcal{G}(I_n) \bigcup \{c_{n-1}c_n, b_nc_n\}$ and $\mathcal{G}(I_n^{**}) = \mathcal{G}(I_n^*) \bigcup \{c_1c_{n+1}, b_1c_{n+1}\}$. First, we find bounds for depth and Stanley depth of the cyclic modules S_n^*/I_n^* and S_n^{**}/I_n^{**}, where $S_n^* = S_n[c_n]$ and $S_n^{**} = S_n[c_n, c_{n+1}]$.

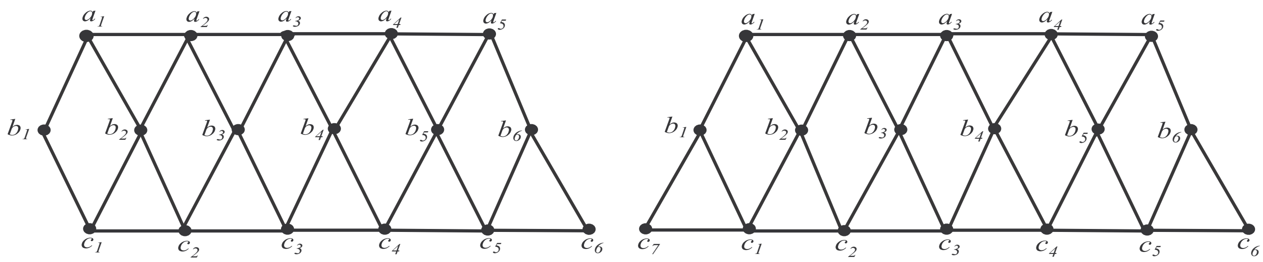

Figure 3. From left to right, supergraphs U_6 and V_6 of $L(\mathcal{L}_6)$ respectively.

Proposition 3.4. *Let $n \geq 2$. Then* $\lceil \frac{n}{2} \rceil \leq \text{depth}(S_n^*/I_n^*), \text{sdepth}(S_n^*/I_n^*) \leq n$.

Proof. If $n = 2$, then by using CoCoA, we obtain $\text{depth}(S_n^*/I_n^*) = \text{sdepth}(S_n^*/I_n^*) = 2$. For $n \geq 3$, we first prove that $\lceil \frac{n}{2} \rceil \leq \text{depth}(S_n^*/I_n^*)$ by using induction on n. For this, we assume the following short exact sequence

$$0 \longrightarrow S_n^*/(I_n^* : c_n) \xrightarrow{\cdot c_n} S_n^*/I_n^* \longrightarrow S_n^*/(I_n^*, c_n) \longrightarrow 0. \tag{3.4}$$

Here $$(I_n^* : c_n) = (\bigcup_{i=1}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, a_{n-1}a_{n-2}, a_{n-1}b_{n-1}, b_n, c_{n-1}),$$

so we have $S_n^*/(I_n^* : c_n) \cong S_{n-1}^*/I_{n-1}^*[c_n]$. By using induction and Lemma 2.6,

$$\text{depth}(S_n^*/(I_n^* : c_n)) \geq \lceil \frac{n-1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil.$$

As $$(I_n^*, c_n) = (\bigcup_{i=1}^{n-1}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-2}\{a_ia_{i+1}, c_ic_{i+1}\}, c_n) = (I_n, c_n),$$

so we obtain $S_n^*/(I_n^*, c_n) \cong S_n/I_n$. By Theorem 3.2, it follows that $\text{depth}(S_n^*/(I_n^*, c_n)) \geq \lceil \frac{n}{2} \rceil$. Therefore by applying Depth Lemma on the exact sequence (3.4), we get $\text{depth}(S_n^*/I_n^*) \geq \lceil \frac{n}{2} \rceil$. Now we prove $\text{depth}(S_n^*/I_n^*) \leq n$ by using induction on n. For $n \geq 3$, as $c_n \notin I_n^*$, from Corollary 2.7, we have $\text{depth}(S_n^*/I_n^*) \leq \text{depth}(S_n^*/(I_n^* : c_n))$. Since $S_n^*/(I_n^* : c_n) \cong S_{n-1}^*/I_{n-1}^*[c_n]$, by induction and Lemma 2.6,

$\mathrm{depth}(S_n^*/I_n^*) \leq n-1+1 = n$. It remains to show the result for Stanley depth. For $n \geq 3$, by using induction on n, and by applying Lemma 2.4 on the exact sequence (3.4), we get $\mathrm{sdepth}(S_n^*/I_n^*) \geq \lceil \frac{n}{2} \rceil$. For upper bound of Stanley depth, one can repeat the proof for depth by using Proposition 2.8 instead of Corollary 2.7. □

Proposition 3.5. *For $n \geq 2$, we have that* $\lceil \frac{n}{2} \rceil \leq \mathrm{depth}(S_n^{**}/I_n^{**}), \mathrm{sdepth}(S_n^{**}/I_n^{**}) \leq n+1$.

Proof. If $n=2$, then by using CoCoA, we obtain $\mathrm{depth}(S_n^{**}/I_n^{**}) = \mathrm{sdepth}(S_n^{**}/I_n^{**}) = 2$, and for $n=3$, $\mathrm{depth}(S_n^{**}/I_n^{**}) = \mathrm{sdepth}(S_n^{**}/I_n^{**}) = 3$. For $n \geq 4$, we first prove that $\mathrm{depth}(S_n^{**}/I_n^{**}) \geq \lceil \frac{n}{2} \rceil$ by using induction on n. Let us consider the following short exact sequence

$$0 \longrightarrow S_n^{**}/(I_n^{**} : c_n) \xrightarrow{\cdot c_n} S_n^{**}/I_n^{**} \longrightarrow S_n^{**}/(I_n^{**}, c_n) \longrightarrow 0. \tag{3.5}$$

As $$(I_n^{**}, c_n) = (\bigcup_{i=1}^{n-1} \{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-2} \{a_ia_{i+1}, c_ic_{i+1}\}, c_1c_{n+1}, b_1c_{n+1}, c_n),$$

so we have that $S_n^{**}/(I_n^{**}, c_n) \cong S_n^*/I_n^*$. Therefore by Proposition 3.4, it follows that

$$\mathrm{depth}(S_n^{**}/(I_n^{**}, c_n)) \geq \lceil \frac{n}{2} \rceil.$$

Let $$T = (I_n^{**} : c_n) = (\bigcup_{i=1}^{n-2} \{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-3} \{a_ia_{i+1}, c_ic_{i+1}\}, a_{n-1}a_{n-2}, b_n, a_{n-1}b_{n-1}, c_1c_{n+1},$$
$$b_1c_{n+1}, c_{n-1}) = (I_{n-1}^*, a_{n-1}a_{n-2}, a_{n-1}b_{n-1}, b_n, c_{n-1}).$$

Now consider another short exact sequence

$$0 \longrightarrow S_n^{**}/(T : a_{n-1}) \xrightarrow{\cdot a_{n-1}} S_n^{**}/T \longrightarrow S_n^{**}/(T, a_{n-1}) \longrightarrow 0, \tag{3.6}$$

$$(T : a_{n-1}) = (\bigcup_{i=1}^{n-3} \{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-4} \{a_ia_{i+1}, c_ic_{i+1}\}, b_{n-2}c_{n-2}, c_{n-2}c_{n-3}, b_{n-1}, c_1c_{n+1}, b_1c_{n+1},$$
$$b_n, c_{n-1}, a_{n-2}) = (I_{n-2}^{**}, b_n, c_{n-1}, a_{n-2}, b_{n-1}),$$

so we have $S_n^{**}/(T : a_{n-1}) \cong S_{n-2}^{**}/I_{n-2}^{**}[a_{n-1}, c_n]$. Thus induction on n and Lemma 2.6 give that $\mathrm{depth}(S_n^{**}/(T : a_{n-1})) \geq \lceil \frac{n-2}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil + 1$. As $(T, a_{n-1}) = (I_{n-1}^*, a_{n-1}, b_n, c_{n-1})$, which implies $S_n^{**}/(T, a_{n-1}) \cong S_{n-1}^*/I_{n-1}^*$. By Proposition 3.4 and Lemma 2.6, we obtain $\mathrm{depth}(S_n^{**}/(T, a_{n-1})) \geq \lceil \frac{n-1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil$. Therefore by applying Depth Lemma on the exact sequences (3.5) and (3.6), we get $\mathrm{depth}(S_n^{**}/I_n^{**}) \geq \lceil \frac{n}{2} \rceil$. Now we prove $\mathrm{depth}(S_n^{**}/I_n^{**}) \leq n+1$. We show this by induction on n. For $n \geq 4$, as $a_{n-1}c_n \notin I_n^{**}$, from Corollary 2.7, we have

$$\mathrm{depth}(S_n^{**}/I_n^{**}) \leq \mathrm{depth}(S_n^{**}/(I_n^{**} : a_{n-1}c_n)).$$

Since $S_n^{**}/(I_n^{**} : a_{n-1}c_n) \cong S_{n-2}^{**}/I_{n-2}^{**}[a_{n-1}, c_n]$, by induction and Lemma 2.6, $\mathrm{depth}(S_n^{**}/I_n^{**}) \leq n-2+1+2 = n+1$. It remains to prove the result for Stanley depth. For $n \geq 4$, by using induction on n, and by applying Lemma 2.4 on the exact sequences (3.5) and (3.6) we get $\mathrm{sdepth}(S_n^{**}/I_n^{**}) \geq \lceil \frac{n}{2} \rceil$. Similarly, one can obtain the required upper bound for Stanley depth by using Proposition 2.8 instead of Corollary 2.7. □

Theorem 3.6. *Let $n \geq 3$. Then* $\lceil \frac{n}{2} \rceil \leq \operatorname{depth}(\overline{S}_n/J_n) \leq n-1$, *and* $\lceil \frac{n}{2} \rceil \leq \operatorname{sdepth}(\overline{S}_n/J_n) \leq n$.

Proof. For $3 \leq n \leq 4$, by using CoCoA, (for sdepth we use SdepthLib:coc [27]), $\operatorname{depth}(\overline{S}_3/J_3) = \operatorname{sdepth}(\overline{S}_3/J_3) = 2$, $\operatorname{depth}(\overline{S}_4/J_4) = \operatorname{sdepth}(\overline{S}_4/J_4) = 3$. Now for $n \geq 5$, we first show that $\operatorname{depth}(\overline{S}_n/J_n) \geq \lceil \frac{n}{2} \rceil$. Let us consider the following short exact sequence

$$0 \longrightarrow \overline{S}_n/(J_n : a_n) \xrightarrow{\cdot a_n} \overline{S}_n/J_n \longrightarrow \overline{S}_n/(J_n, a_n) \longrightarrow \overline{S}_n/(J_n, a_n) \longrightarrow 0. \tag{3.7}$$

Let $U = (J_n, a_n) = (\bigcup_{i=1}^{n-1}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-2}\{a_ia_{i+1}, c_ic_{i+1}\}, c_1c_n, c_{n-1}c_n, b_1c_n, b_nc_n, a_n)$.

Now assume another short exact sequence

$$0 \longrightarrow \overline{S}_n/(U : c_n) \xrightarrow{\cdot c_n} \overline{S}_n/U \longrightarrow \overline{S}_n/(U, c_n) \longrightarrow 0. \tag{3.8}$$

As $(U, c_n) = (\bigcup_{i=1}^{n-1}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-2}\{a_ia_{i+1}, c_ic_{i+1}\}, c_n, a_n)$,

so we obtain $\overline{S}_n/(U, c_n) \cong S_n/I_n$. Thus Theorem 3.2 gives that $\operatorname{depth}(\overline{S}_n/(U, c_n)) \geq \lceil \frac{n}{2} \rceil$.

Also $(U : c_n) = (\bigcup_{i=2}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=2}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, a_1a_2, a_1b_2, a_{n-1}a_{n-2},$
$a_n, a_{n-1}b_{n-1}, b_1, b_n, c_1, c_{n-1}) = (I^{**}_{n-2}, a_n, b_1, b_n, c_1, c_{n-1})$,

so we get $\overline{S}_n/(U : c_n) \cong S^{**}_{n-2}/I^{**}_{n-2}[c_n]$. Thus by Proposition 3.5 and Lemma 2.6 we have

$$\operatorname{depth}(\overline{S}_n/(U : c_n)) \geq \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil.$$

Let $V = (J_n : a_n) = (\bigcup_{i=2}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=2}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, c_1b_2, c_1c_2, c_{n-1}c_{n-2},$
$c_{n-1}b_{n-1}, c_1c_n, c_nc_{n-1}, a_1, a_{n-1}, b_1, b_n)$.

Now consider the following short exact sequence

$$0 \longrightarrow \overline{S}_n/(V : c_n) \xrightarrow{\cdot c_n} \overline{S}_n/V \longrightarrow \overline{S}_n/(V, c_n) \longrightarrow 0, \tag{3.9}$$

$(V : c_n) = (\bigcup_{i=2}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=2}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, c_1, c_{n-1}, a_1, a_{n-1}, b_1, b_n)$
$= (I_{n-2}, c_1, c_{n-1}, a_1, a_{n-1}, b_1, b_n)$,

so we have $\overline{S}_n/(V : c_n) \cong S_{n-2}/I_{n-2}[a_n, c_n]$. Thus by Theorem 3.2 and Lemma 2.6, we have

$$\operatorname{depth}(\overline{S}_n/(V : c_n)) \geq \lceil \frac{n-2}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil + 1.$$

As $$(V, c_n) = (\bigcup_{i=2}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=2}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, c_1b_2, c_1c_2, c_{n-1}c_{n-2}, c_{n-1}b_{n-1}, a_1, a_{n-1}, b_1, b_n, c_n),$$

so we have that $\overline{S}_n/(V, c_n) \cong S^{**}_{n-2}/I^{**}_{n-2}$. By Proposition 3.5 and Lemma 2.6, we obtain $\operatorname{depth}(\overline{S}_n/(V, c_n)) \geq \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$. Therefore by applying Depth Lemma on the exact sequences (3.7), (3.8) and (3.9), we get $\operatorname{depth}(\overline{S}_n/J_n) \geq \lceil \frac{n}{2} \rceil$. Now we prove $\operatorname{depth}(\overline{S}_n/J_n) \leq n-1$. For $n \geq 5$, as $a_nc_n \notin J_n$, from Corollary 2.7, it follows that $\operatorname{depth}(\overline{S}_n/J_n) \leq \operatorname{depth}(\overline{S}_n/(J_n : a_nc_n))$. Since $\overline{S}_n/(J_n : a_nc_n) \cong S_{n-2}/I_{n-2}[a_n, c_n]$, by Theorem 3.2 and Lemma 2.6, we have $\operatorname{depth}(\overline{S}_n/J_n) \leq n-2-1+2 = n-1$.
It remains to show the result for Stanley depth. For $n \geq 5$, by applying Lemma 2.4 on the exact sequences (3.7), (3.8) and (3.9), we get that $\operatorname{sdepth}(\overline{S}_n/J_n) \geq \lceil \frac{n}{2} \rceil$. Similarly, one can obtain the required upper bound for Stanley depth by using Proposition 2.8 instead of Corollary 2.7. □

Remark 3.7. It is easy to see that $\operatorname{diam}(L(C\mathcal{L}_n)) = \lceil \frac{n+1}{2} \rceil$, then by Theorems 2.13, we have $\operatorname{depth}(\overline{S}_n/J_n), \operatorname{sdepth}(\overline{S}_n/J_n) \geq \lceil \frac{n+2}{6} \rceil$. Our Theorem 3.6 shows that $\operatorname{depth}(\overline{S}_n/J_n), \operatorname{sdepth}(\overline{S}_n/J_n) \geq \lceil \frac{n}{2} \rceil$. Thus we find a much better lower bound for depth and Stanley depth for these classes of edge ideals.

Proposition 3.8. *For* $n \geq 2$, *we have that* $\dim(S_n/I_n) \geq n$.

Proof. Let $E = \{a_1, a_2, \ldots, a_{n-1}, c_1, c_2, \ldots, c_{n-1}\}$ be a subset of vertex set $V(L(\mathcal{L}_n))$. The set E is a vertex cover because it covers all the edges. Now if we remove a_i for some $1 \leq i \leq n-1$ from set E then the resulting set is not a vertex cover because the edges a_ib_i and a_ib_{i+1} will not covered. Similarly by removing c_i for some $1 \leq i \leq n-1$ from set E then the resulting set is not a vertex cover because the edges c_ib_i and c_ib_{i+1} will not covered. This shows that the set E forms a minimal vertex cover of I_n. Thus we have $\operatorname{height}(I_n) \leq 2n-2$. Since S_n is a polynomial ring of dimension $3n-2$, which implies that $\dim(S_n/I_n) \geq 3n-2-(2n-2) = n$. □

Proposition 3.9. *Let* $n \geq 3$. *Then* $\dim(\overline{S}_n/J_n) \geq n$.

Proof. As in the Proposition 3.8, one can shows in a similar way that the set $F = \{a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_n\}$ forms a minimal vertex cover of J_n, therefore $\operatorname{height}(J_n) \leq 2n$. As $\overline{S}_n$ is a polynomial ring of dimension $3n$, thus $\dim(\overline{S}_n/J_n) \geq n$. □

Remark 3.10. By Theorem 3.2 and 3.6 we have that $\operatorname{depth}(S_n/I_n), \operatorname{depth}(\overline{S}_n/J_n) \leq n-1$, and by Proposition 3.8 and 3.9 we have $\dim(S_n/I_n), \dim(\overline{S}_n/J_n) \geq n$. Thus graphs $L(\mathcal{L}_n)$ and $L(C\mathcal{L}_n)$ are not Cohen-Macaulay.

Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

References

1. R. P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math., **68** (1982), 175–193.
2. B. Ichim, L. Katthän, J. J. Moyano-Fernández, *The behavior of Stanley depth under polarization*, J. Combin. Theory, Ser. A, **135** (2015), 332–347.
3. A. M. Duval, B. Goeckneker, C. J. Klivans, et al. *A non-partitionable Cohen-Macaulay simplicial complex*, Adv. Math., **299** (2016), 381–395.
4. J. Herzog, T. Hibi, *The depth of powers of an ideal*, J. Algebra, **291** (2005), 534–550.
5. J. Herzog, M. Vladoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal*, J. Algebra, **322** (2009), 3151–3169.
6. C. Biro, D. M. Howard, M. T. Keller, et al. *Interval partitions and Stanley depth*, J. Combin. Theory, Ser. A, **117** (2010), 475–482.
7. M. Cimpoeas, *Stanley depth of squarefree Veronese ideals*, An. St. Univ. Ovidius Constanta, **21** (2013), 67–71.
8. M. Ishaq, *Upper bounds for the Stanley depth*, Comm. Algebra, **40** (2012), 87–97.
9. M. Ishaq, M. I. Qureshi, *Upper and lower bounds for the Stanley depth of certain classes of monomial ideals and their residue class rings*, Comm. Algebra, **41** (2013), 1107–1116.
10. M. T. Keller, Y. Shen, N. Streib, et al. *On the Stanley Depth of Squarefree Veronese Ideals*, J. Algebr. Comb., **33** (2011), 313–324.
11. B. Ichim, L. Katthän, J. J. Moyano-Fernández, *How to compute the Stanley depth of a module*, Math. Comput., **86** (2016), 455–472.
12. B. Ichim, L. Katthän, J. J. Moyano-Fernández, *LCM lattices and Stanley depth: a first computational approach*, Exp. Math., **25** (2016), 46–53.
13. B. Ichim, L. Katthän, J. J. Moyano-Fernández, *Stanley depth and the lcm-lattice*, J. Combin. Theory, Ser. A, **150** (2017), 295–322.
14. Z. Iqbal, M. Ishaq, M. Aamir, *Depth and Stanley depth of the edge ideals of square paths and square cycles*, Comm. Algebra, **46** (2018), 1188–1198.
15. Z. Iqbal, M. Ishaq, *Depth and Stanley depth of the edge ideals of the powers of paths and cycles*, An. St. Univ. Ovidius Constanta, In press. Available from: `http://arxiv.org/pdf/1710.05996.pdf`.
16. M. R. Pournaki, S. A. Seyed Fakhari, S. Yassemi, *Stanley depth of powers of the edge ideals of a forest*, P. Am. Math. Soc., **141** (2013), 3327–3336.
17. J. Herzog, T. Hibi, *Monomial Ideals*, Springer-Verlag London Limited, 2011.

18. R. H. Villarreal, *Monomial algebras*, Monographs and Textbooks in Pure and Applied Mathematics **238**, Marcel Dekker, New York, 2001.
19. J. A. Bondy, U. S. R. Murty, *Graph Theory with applications*, Springer, 2008.
20. F. Harary, *Graph theory*, Addison-Wesley, Reading, MA, 1969.
21. A. Rauf, *Depth and Stanley depth of multigraded modules*, Comm. Algebra, **38** (2010), 773–784.
22. M. Cimpoeas, *Several inequalities regarding Stanley depth*, Romanian Journal of Mathematics and Computer Science, **2** (2012), 28–40.
23. S. Morey, *Depths of powers of the edge ideal of a tree*, Comm. Algebra, **38** (2010), 4042–4055.
24. A. Stefan, *Stanley depth of powers of path ideal*. Available from: `http://arxiv.org/pdf/1409.6072.pdf`.
25. M. Cimpoeas, *On the Stanley depth of edge ideals of line and cyclic graphs*, Romanian Journal of Mathematics and Computer Science, **5** (2015), 70–75.
26. L. Fouli, S. Morey, *A lower bound for depths of powers of edge ideals*, J. Algebr. Comb., **42** (2015), 829–848.
27. G. Rinaldo, *An algorithm to compute the Stanley depth of monomial ideals*, Le Matematiche, **63** (2008), 243–256.

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