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Research article

Applications and theorem on common fixed point in complex valued b-metric space

Khaled Berrah[1,2], **Abdelkrim Aliouche**[1] **and Taki eddine Oussaeif**[1]*

[1] Department of Mathematics and Computer and Science, Larbi ben M'hidi University, Oum el Bouaghi, Algeria

[2] Laboratory of mathematics, informatics and systems (LAMIS) Larbi Tebessi university, Tebessa, Algeria

* **Correspondence:** Email: taki_maths@live.fr; Tel: +21-367-142-0203.

Abstract: In this paper, a common fixed point theorem for four self-mappings satisfying rational contraction has been proved in complex valued b-metric space. Then, examples are provided to verify the effectiveness and usability of our main results. Finally, we validate our results by proving both the existence and the uniqueness of a common solution of the system of Urysohn integral equations and the existence of a unique solution for linear equations system.

Keywords: complex valued b-metric space; common fixed point; compatible mapping; weakly compatible mapping; integral equations; linear system
Mathematics Subject Classification: 47H10

1. Introduction and preliminaries

Banach contraction fundamental was an authority and a reference for many researchers through the last decades in the field of nonlinear analysis, it was used to establish the existence of a unique solution for a nonlinear integral equation [4]. In 1989, Bakhtin [3] initiated the motif of b-metric space after that Czerwik in [7, 8] defined it such as current structure which is considere generalization of metric spaces. The complex valued b-metric spaces concept was introduced in 2013 by Rao et al. [13], which was more general than the well-known complex valued metric spaces that were introduced in 2011 by Azam et al. [2] which proved some common fixed point theorems for mapping satisfying rational inequalities which are not worthwhile in cone metric spaces [1, 10, 11, 16]. Sundry authors have studied and proved the fixed point results for mappings with satisfying different type contraction conditions in the framework of complex valued metric (b-metric) spaces(see [5, 6, 9, 13, 17]).

The main purpose of this paper is to present common fixed point results of four self-mappings to

satisfy a rational inequality on complex valued b-metric spaces. and we establish the existence and the uniqueness of a common solution for the system of Urysohn integral equations. Also we prove the existence and the uniqueness of solution for linear system in complete complex valued b-metric space. In [2] the authors introduced the notion of complex-valued metric space and obtained a common fixed-point theorems of contraction type mappings using the partial inequality in a complex-valued metric space.
To do so, let us recall a natural relation $\leq$ on $\mathbb{C}$, the set of complex numbers as follows: let z_1, z_2 in $\mathbb{C}$

$$\begin{aligned} z_1 &\leq z_2 \Leftrightarrow \text{Re}(z_1) \leq \text{Re}(z_2) \text{ and } \text{Im}(z_1) \leq \text{Im}(z_2) \\ z_1 &< z_2 \Leftrightarrow \text{Re}(z_1) < \text{Re}(z_2) \text{ and } \text{Im}(z_1) < \text{Im}(z_2) \end{aligned}$$

In [2], the authors defined a partial order relation $z_1 \precsim z_2$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \text{Re}(z_1) \leq \text{Re}(z_2) \text{ and } \text{Im}(z_1) \leq \text{Im}(z_2).$$

As a result, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$,
(ii) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$,
(iii) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$,
(iv) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$.

In (i), (ii) and (iii) we have $|z_1| < |z_2|$. In (iv) we have $|z_1| = |z_2|$, so that, $|z_1| \leq |z_2|$ In particular, $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied. In this case $|z_1| < |z_2|$. We will write $z_1 \prec z_2$ if only (iii) is satisfied. Further,

$$\begin{aligned} 0 &\precsim z_1 \precnsim z_2 \Rightarrow | z_1 |<| z_2 |, \\ z_1 &\precsim z_2 \text{ and } z_2 \prec z_3 \Rightarrow z_1 \prec z_3. \end{aligned}$$

In [2], the authors defined the complex-valued metric space (X, d) in the following way:

Definition 1.1. *Let X be a non-empty set. A mapping $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if the following conditions are satisfied:*

(a) $0 \precsim d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0 \Leftrightarrow x = y$,

(b) $d(x, y) = d(y, x)$, *for all* $x, y \in X$,

(c) $d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a complex valued metric in X, and (X, d) is a complex valued metric space.

Example 1. *Let $X = \mathbb{C}$ define the mapping $d : X \times X \longrightarrow \mathbb{C}$ by:*

$$d(z_1, z_2) =| z_1 - z_2 | e^{i\theta}, \ \theta \in \left]0, \frac{\pi}{2}\right[.$$

Then (X, d) is a complex valued metric space.

Definition 1.2. *[13] Let X be a nonempty set and let $s \geq 1$ be given real number. A mapping $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if the following conditions are satisfied:*

(a) $0 \precsim d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0 \Leftrightarrow x = y$,

(b) $d(x, y) = d(y, x)$, *for all* $x, y \in X$,

(c) $d(x, y) \precsim s[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.

Then (X, d) *is a complex valued b-metric space.*

Example 2. *[13] Let* $X = \mathbb{C}$ *define the mapping* $d : X \times X \longrightarrow \mathbb{C}$ *by:*

$$d(z_1, z_2) = \mid z_1 - z_2 \mid^2 + i \mid z_1 - z_2 \mid^2 \ \ for\, all\, z_1, z_2 \in X$$

Then (X, d) *is a complex valued b-metric space with* $s = 2$.

Definition 1.3. *[13] Suppose that* (X, d) *is a complex valued b-metric space and* $\{z_n\}$ *is a sequence in* X *and* $z \in X$ *then*

(i) *We say that a sequence* $\{z_n\}$ *converges to an element* $z_0 \in X$ *if for every* $0 \prec c \in \mathbb{C}$, *there exists an integer* N *such that* $d(z_n, z_0) \prec c$ *for all* $n \geq N$. *In this case, we write* $z_n \longrightarrow z_0$.

(ii) *We say that* $\{z_n\}$ *is a Cauchy sequence if for every* $0 \prec c \in \mathbb{C}$, *there exists an integer* N *such that* $d(z_n, z_m) \prec c$ *for all* $n, m \geq N$.

(iii) *We say that* (X, d) *is complete, if every Cauchy sequence in* X *converges to a point in* X.

Definition 1.4. *Let* S *and* T *be self mappings of a nonemplty set* X. *If* $w = Sz = Tz$ *for some* z *in* X, *then* z *is called a coincidence points of* S *and* T *and* w *is called a point of coincidence of* S *and* T.

Definition 1.5. *[15] Let* S *and* T *be a self-mappings of a complex valued metric space* (X, d). *The mappings* S *and* T *are said to be compatible if:*

$$\lim_{n\to\infty} d\,(STz_n, TSz_n) = 0,$$

whenever $\{z_n\}$ *is a sequence in* X *such that:*

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = t \, for\, some\, t \in X.$$

Definition 1.6. *[12] Let* S *and* T *be self mappings of a nonemplty set* X. S *and* T *are said to be weakly compatible if they commute at their coincidence points, i.e,* $Sz = Tz$ *for some* z *in* X *implies that* $STz = TSz$.

Definition 1.7. *A matrix norm induced by vectors norms is given by:*

$$||A||_\infty = \max_{1\leq i\leq n} \sum_{i=1}^{n} |a_{ij}| \ \ where \ \ A = (a_{ij})_{1\leq i,j\leq n} \in \mathbb{M}_n(\mathbb{R})$$

Example 3. *Let* $X = \mathbb{C}^n$. *A vector norm in a complex valued b-metric given by:*

$$d_2(z, w) = \left[\sum_{i=1}^{n} \left(|z_i - w_i|^2 + i|z_i - w_i|^2 \right) \right]^{\frac{1}{2}},$$

where $z, w \in X$ *such that* $z = (z_1, \ldots, z_n)^t$ *and* $w = (w_1, \ldots, w_n)^t$, *then* (X, d_2) *is a complex valued b-metric space.*

Definition 1.8. *[14] defined the* max *function for the partial order relation by:*

(i) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$,

(ii) $z_1 \precsim \max\{z_1, z_3\} \Rightarrow z_1 \precsim z_2$, *or* $z_1 \precsim z_3$,

(iii) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ *or* $|z_1| \leq |z_2|$.

Using definition (8) we have the following Lemma.

Lemma 1. *[14] Let* $z_1, z_2, z_{3,...} \in \mathbb{C}$ *and the partial order relation* $\precsim$ *is defined on* $\mathbb{C}$*, the following statements are achieve:*

(i) if $z_1 \precsim \max\{z_2, z_3\}$ *then* $z_1 \precsim z_2$ *if* $z_3 \precsim z_2$
(ii) if $z_1 \precsim \max\{z_2, z_3, z_4\}$ *then* $z_1 \precsim z_2$ *if* $\max\{z_3, z_4\} \precsim z_2$,
(iii) if $z_1 \precsim \max\{z_2, z_3, z_4, z_5\}$ *then* $z_1 \precsim z_2$ *if* $\max\{z_3, z_4, z_5\} \precsim z_2$, *and so on.*

2. Main results

In this section, we prove common fixed point theorem for four mappings in a complete complex valued b-metric spaces using rational type contraction condition and we give some examples. Our first new result is the following:

Theorem 2.1. *Let* (X, d) *be a complete complex valued b-metric space and* $S, T, P, Q : X \to X$ *be a self mappings satisfying the conditions:*
C_1 $S(X) \subset Q(X)$ *and* $T(X) \subset P(X)$,
C_2 $d(Sz, Tw) \precsim \frac{\lambda}{s^2} R(z, w)$, *if* $s \geq 1$ *and* $\lambda \in (0, 1)$ *for all* $z, w \in X$ *where*

$$R(z, w) = \max\{d(Pz, Qw), d(Pz, Sz), d(Qw, Tw), \frac{1}{2}[d(Qw, Sz) + d(Pz, Tw)], \frac{d(Pz, Sz)d(Qw, Tw)}{1 + d(Pz, Qw)}\},$$

C_3 *the pair* (S, P) *is compatible and the pair* (T, Q) *is weakly compatible,*
C_4 *either* P *or* S *is continuous.*
Then S, T, P *and* Q *have a unique common fixed point in* X.

Proof. Let $z_0 \in X$ be arbitrary. From the condition C_1, there exist z_1, z_2 such that $w_0 = Qz_1 = Sz_0$ and $w_1 = Pz_2 = Tz_1$. We can construct successively the sequences $\{w_n\}$ and $\{z_n\}$ in X as follows:

$$w_{2n} = Qz_{2n+1} = Sz_{2n} \text{ and } w_{2n+1} = Pz_{2n+2} = Tz_{2n+1} \tag{2.1}$$

Using (2.1) in C_2 we get:

$$d(w_{2n}, w_{2n+1}) = d(Sz_{2n}, Tz_{2n+1}) \precsim \frac{\lambda}{s^2} R(z_{2n}, z_{2n+1}),$$

where

$$R(z_{2n}, z_{2n+1}) = \max\{d(Pz_{2n}, Qz_{2n+1}), d(Pz_{2n}, Sz_{2n}), d(Qz_{2n+1}, Tz_{2n+1}),$$

$$
\begin{aligned}
& \frac{1}{2}[d(Qz_{2n+1}, Sz_{2n}) + d(Pz_{2n}, Tz_{2n+1})], \\
& \frac{d(Pz_{2n}, Sz_{2n})d(Qz_{2n+1}, Tz_{2n+1})}{1 + d(Pz_{2n}, Qz_{2n+1})}\} \\
= \ & \max\{d(w_{2n-1}, w_{2n}), d(w_{2n-1}, w_{2n}), d(w_{2n}, w_{2n+1}), \\
& \frac{1}{2}[d(w_{2n}, w_{2n}) + d(w_{2n-1}, w_{2n+1})], \\
& \frac{d(w_{2n-1}, w_{2n})d(w_{2n}, w_{2n+1})}{1 + d(w_{2n-1}, w_{2n})}\},
\end{aligned}
$$

we have:

$$
\begin{aligned}
\frac{1}{2}d(w_{2n-1}, w_{2n+1}) & \precsim \frac{1}{2}[d(w_{2n-1}, w_{2n}) + d(w_{2n}, w_{2n+1})] \\
& \precsim \max\{d(w_{2n-1}, w_{2n}), d(w_{2n}, w_{2n+1})\}
\end{aligned} \tag{2.2}
$$

and we have

$$d(w_{2n-1}, w_{2n}) \precsim 1 + d(w_{2n-1}, w_{2n}),$$

which is implies

$$\frac{d(w_{2n-1}, w_{2n})d(w_{2n}, w_{2n+1})}{1 + d(w_{2n-1}, w_{2n})} \precsim d(w_{2n}, w_{2n+1}), \tag{2.3}$$

from (2.2) and (2.3) we get:

$$R(z_{2n}, z_{2n+1}) = \max\{d(w_{2n-1}, w_{2n}), d(w_{2n}, w_{2n+1})\}$$

with

$$d(w_{2n}, w_{2n+1}) = d(Sz_{2n}, Tz_{2n+1}) \precsim \frac{\lambda}{s^2}R(z_{2n}, z_{2n+1}).$$

If

$$R(z_{2n}, z_{2n+1}) = d(w_{2n}, w_{2n+1}),$$

then,

$$d(w_{2n}, w_{2n+1}) \precsim \frac{\lambda}{s^2}d(w_{2n}, w_{2n+1}), \text{ therefore } \left(1 - \frac{\lambda}{s^2}\right)d(w_{2n}, w_{2n+1}) \precsim 0,$$

which is a contradiction, since $\lambda \in (0, 1)$, $s \geq 1$. We conclude that $d(w_{2n}, w_{2n+1}) \precsim \frac{\lambda}{s^2}d(w_{2n-1}, w_{2n})$.
Similarly we get $d(w_{2n+1}, w_{2n+2}) \precsim \frac{\lambda}{s^2}d(w_{2n}, w_{2n+1})$.
It follows that

$$d(w_n, w_{n+1}) \precsim \frac{\lambda}{s^2}d(w_{n-1}, w_n) \precsim \cdots \precsim \left(\frac{\lambda}{s^2}\right)^n d(w_0, w_1),$$

which implies

$$|d(w_n, w_{n+1})| \leq \frac{\lambda}{s^2}|d(w_{n-1}, w_n)| \leq \cdots \leq \left(\frac{\lambda}{s^2}\right)^n |d(w_0, w_1)|,$$

for $m < n$ we have:

$$|d(w_n, w_m)| \leq s\left(\frac{\lambda}{s^2}\right)^n |d(w_0, w_1)| + s^2\left(\frac{\lambda}{s^2}\right)^{n+1} |d(w_0, w_1)| + s^3\left(\frac{\lambda}{s^2}\right)^{n+2} |d(w_0, w_1)| +$$

$$
\begin{aligned}
&\cdots + s^{m-n-1}\left(\frac{\lambda}{s^2}\right)^{m-2}|d(w_0,w_1)| + s^{m-n}\left(\frac{\lambda}{s^2}\right)^{m-1}|d(w_0,w_1)| \\
= &\sum_{i=1}^{m-n} s^i\left(\frac{\lambda}{s^2}\right)^{i+n-1}|d(w_0,w_1)|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|d(w_n,w_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}\left(\frac{\lambda}{s^2}\right)^{i+n-1}|d(w_0,w_1)| = \sum_{t=n}^{m-1} s^t\left(\frac{\lambda}{s^2}\right)^t|d(w_0,w_1)|, \\
&\leq \sum_{i=1}^{\infty}\left(\frac{\lambda}{s}\right)^t|d(w_0,w_1)| = \frac{\left(\frac{\lambda}{s}\right)^n}{\left(1-\frac{\lambda}{s}\right)}|d(w_0,w_1)|,
\end{aligned}
$$

hence,

$$
|d(w_n,w_m)| \leq \frac{\left(\frac{\lambda}{s}\right)^n}{\left(1-\frac{\lambda}{s}\right)}|d(w_0,w_1)| \to 0 \text{ as } n \to \infty.
$$

Thus, $\{w_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists some $u \in X$ such that $w_n \to u$ as $n \to \infty$. For its sub-sequences we also have $Qz_{2n+1} \to u, Sz_{2n} \to u, Pz_{2n+1} \to u$ and $Tz_{2n} \to u$
from C_4 if P is continuous
as P is continuous, then $PPz_{2n} \to Pu$ and $PSz_{2n} \to Pu$, as $n \to \infty$. Also, since the pair (S,P) is compatible, this implies that $SPz_{2n} \to Pu$. Indeed,

$$
d(SPz_{2n},Pu) \precsim s[d(SPz_{2n},PSz_{2n}) + d(PSz_{2n},Pu)].
$$

So,

$$
|d(SPz_{2n},Pu)| \leq s|d(SPz_{2n},PSz_{2n})| + s|d(PSz_{2n},Pu)| \to 0 \text{ as } n \to \infty.
$$

We prove $Pu = u$. On the contrary we suppose that $Pu \neq u$

$$
d(Pu,u) \precsim sd(Pu,SPz_{2n}) + s^2d(SPz_{2n},Tz_{2n+1}) + s^2d(Tz_{2n+1},u).
$$

using C_2 with $z = Pz_{2n}, w = z_{2n+1}$, we get:

$$
d(SPz_{2n},Tz_{2n+1}) \precsim \lambda R(Pz_{2n},z_{2n+1}),
$$

where

$$
\begin{aligned}
R(Pz_{2n},z_{2n+1}) = &\max\{d(PPz_{2n},Qz_{2n+1}), d(PPz_{2n},SPz_{2n}), d(Qz_{2n+1},Tz_{2n+1}), \\
&\frac{1}{2}[d(Pz_{2n+1},SPz_{2n}) + d(QPz_{2n},Tz_{2n+1})], \\
&\frac{d(PPz_{2n},SPz_{2n})d(Qz_{2n+1},Tz_{2n+1})}{1+d(PPz_{2n},Qz_{2n+1})}\},
\end{aligned}
$$

let $n \to \infty$ we get:

$$\begin{aligned} R(Pu,u) &= \max\{d(Pu,u), d(Pu,Pu), d(u,Pu), \\ &\quad \frac{1}{2}[d(Pu,Pu)+d(Pu,u)], \frac{d(Pu,Pu)d(u,u)}{1+d(Pu,u)}\} = d(Pu,u). \end{aligned}$$

Further,

$$|d(Pu,u)| \leq \frac{\lambda}{s^2}|d(Pu,u)|.$$

So, $(1-\frac{\lambda}{s^2})|d(Pu,u)| \leq 0$, which is a contradiction that is $|d(Pu,u)| = 0$ then $Pu = u$.
We prove $Su = u$. On the contrary we suppose that $Su \neq u$

$$d(Su,u) \precsim sd(Pu,Tz_{2n+1}) + sd(Tz_{2n+1},u).$$

Using C_2 with $z = u, w = z_{2n+1}$, we get: $d(Su,Tz_{2n+1}) \precsim \frac{\lambda}{s^2}R(u,z_{2n+1})$ where

$$\begin{aligned} R(u,z_{2n+1}) &= \max\{d(Pu,Qz_{2n+1}), d(Pu,Su), d(Qz_{2n+1},Tz_{2n+1}), \\ &\quad \frac{1}{2}[d(Qz_{2n+1},Su)+d(Pu,Tz_{2n+1})], \\ &\quad \frac{d(Pu,Su)d(Qz_{2n+1},Tz_{2n+1})}{1+d(Pu,Qz_{2n+1})}\}, \end{aligned}$$

let $n \to \infty$ we get:

$$\begin{aligned} R(u,u) &= \max\{d(u,u), d(u,Su), d(u,u), \\ &\quad \frac{1}{2}[d(u,Su)+d(u,u)], \frac{d(u,Su)d(u,u)}{1+d(u,u)}\} = d(Su,u). \end{aligned}$$

Then, $d(Su,u) \precsim \frac{\lambda}{s^2}d(Su,u)$, further, $|d(Su,u)| \leq \frac{\lambda}{s^2}|d(Su,u)|$, which is a contradiction that is $|d(Su,u)| = 0$ then, $Su = u$. We prove $Qu = Tu$, as $S(X) \subset Q(X)$, so there exists $v \in X$ such that $u = Su = Qv$. First, we shall show that $Qv = Tv$ for this we get:

$$d(Qv,Tv) = d(Su,Tv) \precsim \frac{\lambda}{s^2}R(u,v)$$

where,

$$\begin{aligned} R(u,v) &= \max\{d(Pu,Qv), d(Pu,Su), d(Qv,Tv), \\ &\quad \frac{1}{2}[d(Qv,Su)+d(Pu,Tv)], \frac{d(Pu,Su)d(Qv,Tv)}{1+d(Pu,Qv)}\}, \end{aligned}$$

then,

$$R(u,v) = \max\{d(Qv,Qv), d(u,u), d(Qv,Tv),$$

$$\frac{1}{2}[d(Qv,Qv)+d(Qv,Tv)],\frac{d(u,u)d(Qv,Tv)}{1+d(Qv,Qv)}\}.$$

Then, $d(Qv,Tv)\precsim\frac{\lambda}{s^2}d(Qv,Tv)$, further, $|d(Qv,Tv)|\leq\frac{\lambda}{s^2}|d(Qv,Tv)|$, which is a contradiction that is $|d(Qv,Tv)|=0$, then, $Qv=Tv=u$. As the pair (T,Q) is weakly compatible, so we have $TQv=QTv$,therefore $Qu=Tu$.
We prove $u=Tu$, On the contrary we suppose that $Tu\neq u$,

$$d(u,Tu)=d(Su,Tu)\precsim\frac{\lambda}{s^2}R(u,u),$$

where,

$$\begin{aligned}R(u,u) &= \max\{d(Pu,Qu),d(Pu,Su),d(Qu,Tu),\\&\quad\frac{1}{2}[d(Qu,Su)+d(Pu,Tu)],\frac{d(Pu,Su)d(Qu,Tu)}{1+d(Pu,Qu)}\},\end{aligned}$$

then,

$$\begin{aligned}R(u,v) &= \max\{d(u,Tu),d(u,u),d(Tu,Tu),\\&\quad\frac{1}{2}[d(Tu,u)+d(Tu,Tu)],\frac{d(u,u)d(Tu,Tu)}{1+d(u,Tu)}\}.\end{aligned}$$

Then, $d(u,u)\precsim\frac{\lambda}{s^2}d(u,u)$, further, $|d(u,Tu)|\leq\frac{\lambda}{s^2}|d(u,Tu)|$, which is a contradiction that is $|d(u,Tu)|=0$ then $u=Tu$.
Now we prove that $Qu=u$, On the contrary we suppose that $Qu\neq u$,, we have:

$$d(u,Qu)=d(Su,QTu)=d(Su,TQu),$$

from C_2 we get:

$$d(u,Qu)=d(Su,TQu)\precsim\frac{\lambda}{s^2}R(u,Qu)$$

where,

$$\begin{aligned}R(u,Qu) &= \max\{d(Pu,QQu),d(Pu,Su),d(QQu,TQu),\\&\quad\frac{1}{2}[d(QQu,Su)+d(Pu,TQu)],\frac{d(Pu,Su)d(QQu,TQu)}{1+d(Pu,QQu)}\}\\&= \max\{d(u,Qu),d(u,u),d(Qu,Qu),\\&\quad\frac{1}{2}[d(Qu,u)+d(u,Qu)],\frac{d(u,u)d(Qu,Qu)}{1+d(u,Qu)}\}=d(u,Qu).\end{aligned}$$

Further, $|d(u,Qu)|\leq\frac{\lambda}{s^2}|d(u,Qu)|$, which is contradiction that is $|d(u,Qu)|=0$ then $u=Qu$.
On conclude $Su=Tu=Pu=Qu=u$ when P is continuous, we get the same results when S is

continuous.
Now we prove the uniqueness, Let u^* be another common fixed point of S, T, P and Q, then

$$Su^* = Tu^* = Pu^* = Qu^* = u^*$$

Putting $z = u, w = u^*$ in C_2, we get: $d(u, u^*) = d(Su, Tu^*) \precsim \frac{\lambda}{s^2} R(u, u^*)$, where,

$$\begin{aligned} R(u, u^*) &= \max\{d(Pu, Qu^*), d(Pu, Su), d(Qu^*, Tu^*), \\ &\quad \frac{1}{2}[d(Qu^*, Su) + d(Pu, Tu^*)], \frac{d(Pu, Su)d(Qu^*, Tu^*)}{1 + d(Pu, Qu^*)}\} \\ &= \max\{d(u, u^*), d(u, u), d(u^*, u^*), \\ &\quad \frac{1}{2}[d(u^*, u) + d(u, u^*)], \frac{d(u, u)d(u^*, u^*)}{1 + d(u, u^*)}\}. \end{aligned}$$

Further, $|d(u, u^*)| \leq \frac{\lambda}{s^2} |d(u, u^*)|$, which is a contradiction that is $|d(u, u^*)| = 0$, which implies that $u = u^*$. Thus u is the unique common fixed point of S, T, P and Q in X. □

Corollary 1. *Let (X, d) be a complete complex valued b-metric space, if we put $S = T$ and $P = Q = I$ with exceeding the* max *of the rest of terms, we confirm the inequality of contraction of T in the complete complex valued b-metric space. So we get: $d(Tz, Tw) \precsim \frac{\lambda}{s^2} d(z, w)$, where, $\lambda \in (0, 1), s \geq 1$ for all $z, w \in X$. Then, T have unique fixed point in X.*

Example 4. *Let $X = [0.1]$, for all $z, w \in X$. Define $d : X \times X \to \mathbb{C}$ a complex valued b-metric with $s = 2$ by:*

$$d(z, w) = |z - w|^2 + i\,|z - w|^2 .$$

Now define the mappings $S, T, P, Q : X \to X$ by:

$$Sz = \frac{z}{32},\ Tz = \frac{z^2}{48},\ Pz = \frac{z}{2},\ Qz = \frac{z^2}{3},$$

.

$$\begin{aligned} d(Sz, Tw) &= \left[\left|\frac{z}{32} - \frac{w^2}{48}\right|^2 + i\left|\frac{z}{32} - \frac{w^2}{48}\right|^2\right] = \frac{1}{256}\left[\left|\frac{z}{2} - \frac{w^2}{3}\right|^2 + i\left|\frac{z}{2} - \frac{w^2}{3}\right|^2\right], \\ d(Pz, Qw) &= \left[\left|\frac{z}{2} - \frac{w^2}{3}\right|^2 + i\left|\frac{z}{2} - \frac{w^2}{3}\right|^2\right], \\ d(Sz, Tw) &= \frac{1}{256} d(Pz, Qw), \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied where $\lambda = \frac{1}{64}$ and $s = 2$. Then legibly '0' is the unique common fixed point of the mappings S, T, P and Q.

Example 5. *Let $X = B(0, r), r > 1$, for all $z, w \in X$. Define $d : X \times X \to \mathbb{C}$ by:*

$$d(z(u), w(u)) = \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} - \int_\Gamma \frac{w(u)}{u}\right|^2,$$

a complete complex valued b-metric where Γ is a closed path in X containing a zero. We prove that d is a complex b-metric with $s = 2$

$$\begin{aligned}
d(z(u), w(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} - \int_\Gamma \frac{w(u)}{u}\right|^2, \\
&= \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} - \int_\Gamma \frac{x(u)}{u} + \int_\Gamma \frac{x(u)}{u} - \int_\Gamma \frac{w(u)}{u}\right|^2, \\
&\precsim \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} - \int_\Gamma \frac{x(u)}{u}\right|^2 + \frac{i}{2\pi}\left|\int_\Gamma \frac{x(u)}{u} - \int_\Gamma \frac{w(u)}{u}\right|^2 + \\
&\quad 2\frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} - \int_\Gamma \frac{x(u)}{u}\right|\left|\int_\Gamma \frac{x(u)}{u} - \int_\Gamma \frac{w(u)}{u}\right|, \\
&\precsim \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} - \int_\Gamma \frac{x(u)}{u}\right|^2 + \frac{i}{2\pi}\left|\int_\Gamma \frac{x(u)}{u} - \int_\Gamma \frac{w(u)}{u}\right|^2 + \\
&\quad \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} - \int_\Gamma \frac{x(u)}{u}\right|^2 + \frac{i}{2\pi}\left|\int_\Gamma \frac{x(u)}{u} - \int_\Gamma \frac{w(u)}{u}\right|^2, \\
&\precsim 2\left\{\frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} - \int_\Gamma \frac{x(u)}{u}\right|^2 + \frac{i}{2\pi}\left|\int_\Gamma \frac{x(u)}{u} - \int_\Gamma \frac{w(u)}{u}\right|^2\right\}, \\
d(z(u), w(u)) &\precsim 2\left\{d(z(u), x(u)) + d(x(u), w(u))\right\}.
\end{aligned}$$

Now we define the mappings $S, T, P, Q : X \to X$ by:

$$Sz(u) = u, Tz(u) = e^{\frac{u}{2}}, Pz(u) = e^u - 1, Qz(u) = u^2 + \frac{1}{2}u.$$

Using the Cauchy formula when the mappings S, T, P and Q are analytics we get:

$$\begin{aligned}
d(Sz(u), Tw(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{u}{u} - \int_\Gamma \frac{e^u - 1}{u}\right|^2 = 0, \\
d(Pz(u), Qw(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{e^{\frac{u}{2}}}{u} - \int_\Gamma \frac{u^2 + \frac{1}{2}u}{u}\right|^2 = \frac{(2\pi)^2 i}{2\pi}, \\
d(Pz(u), Sz(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{e^{\frac{u}{2}}}{u} - \int_\Gamma \frac{u}{u}\right|^2 = 0, \\
d(Qw(u), Tw(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{u^2 + \frac{1}{2}u}{u} - \int_\Gamma \frac{e^u - 1}{u}\right|^2 = 0, \\
d(Qw(u), Sz(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{u^2 + \frac{1}{2}u}{u} - \int_\Gamma \frac{u}{u}\right|^2 = 0, \\
d(Pz(u), Tw(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{e^{\frac{u}{2}}}{u} - \int_\Gamma \frac{e^u - 1}{u}\right|^2 = \frac{(2\pi)^2 i}{2\pi}, \\
R(z(u), w(u)) &= \max\{2\pi i, 0\} = 2\pi i.
\end{aligned}$$

Further,

$$0 = d(Sz(u), Tw(u)) \precsim \frac{\pi\lambda i}{2}.$$

Thus all the conditions of Theorem 2.1 are satisfied then the mappings S, T, P and Q have a unique common fixed point in X.

3. Application to integral equations

Our first new results in this section is the following:

Theorem 3.1. *Let $X = C([a,b], \mathbb{R}^n), a > 0$ and $d : X \times X \to \mathbb{C}$ is defined as follows:*

$$d(z,w) = \max_{u\in[a,b]} \|z(u) - w(u)\|_\infty \sqrt{1+a^2}e^{itan^{-1}a}.$$

Consider the Urysohn integral equations

$$z(u) = \int_a^b K_1(t,s,z(u))ds + g(u), \tag{1}$$

$$z(u) = \int_a^b K_2(t,s,z(u))ds + h(u), \tag{2}$$

where $u \in [a,b] \subset \mathbb{R}$ and $z, g, h \in X$.
Assume that $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ such that $F_z, G_z \in X$ for each $z \in X$, where

$$F_z(u) = \int_a^b K_1(t,s,z(u))ds,\ G_z(u) = \int_a^b K_2(t,s,z(u))ds \text{ for all } u \in [a,b].$$

If there exist $s \geq 1,\ \lambda \in (0,1)$ such that the inequality:

$$A(z,w)(u) \precsim \frac{\lambda}{s^2} R(z,w)(u), \tag{3.1}$$

where,

$$\begin{aligned} R(z,w) \quad = \quad & \max\{D(z,w)(u), B(z,w)(u), C(z,w)(u), \\ & \frac{1}{2}[B(z,w)(u) + C(z,w)(u)], \frac{B(z,w)(u)C(z,w)(u)}{1+D(z,w)(u)}\}, \end{aligned}$$

and

$$\begin{aligned} A(z,w)(u) &= \|F_z(u) - G_w(u) + g(u) - h(u)\| \sqrt{1+a^2}e^{itan^{-1}a}, \\ B(z,w)(u) &= \|z(u) - F_z(u) - g(u)\| \sqrt{1+a^2}e^{itan^{-1}a}, \\ C(z,w)(u) &= \|w(u) - G_w(u) - h(u)\| \sqrt{1+a^2}e^{itan^{-1}a}, \\ D(z,w)(u) &= \|z(u) - w(u)\| \sqrt{1+a^2}e^{itan^{-1}a}, \end{aligned}$$

holds for all $z, w \in X$. then, the system of Urysohn integral equations has a unique common solution in X.

Proof. Define $S, T : X \to X$ by:

$$Sz = F_z + g, Tz = G_z + h.$$

Then,

$$\begin{aligned}
d(Sz, Tw) &= \max_{u\in[a,b]} \|F_z(u) - G_w(u) + g(u) - h(u)\|_\infty \sqrt{1+a^2}e^{itan^{-1}a},\\
d(z, Sz) &= \max_{u\in[a,b]} \|z(u) - F_z(u) - g(u)\|_\infty \sqrt{1+a^2}e^{itan^{-1}a},\\
d(w, Tw) &= \max_{u\in[a,b]} \|w(u) - G_w(u) - h(u)\|_\infty \sqrt{1+a^2}e^{itan^{-1}a},\\
d(z, w) &= \max_{u\in[a,b]} \|z(u) - w(u)\|_\infty \sqrt{1+a^2}e^{itan^{-1}a}.
\end{aligned}$$

From assumption 3.1, for each $u \in [a, b]$ we have:

$$\begin{aligned}
A(z,w)(u) &\precsim \frac{\lambda}{s^2} R(z,w)(u),\\
&\precsim \frac{\lambda}{s^2} \max\{D(z,w)(u), B(z,w)(u), C(z,w)(u),\\
&\quad \frac{1}{2}\left[B(z,w)(u) + C(z,w)(u)\right], \frac{B(z,w)(u)C(z,w)(u)}{1 + D(z,w)(u)}\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\max_{u\in[a,b]} A(z,w)(u) &\precsim \frac{\lambda}{s^2} \max_{u\in[a,b]} \max\{D(z,w)(u), B(z,w)(u), C(z,w)(u),\\
&\quad \frac{1}{2}\left[B(z,w)(u) + C(z,w)(u)\right], \frac{B(z,w)(u)C(z,w)(u)}{1 + D(z,w)(u)}\}\\
&\precsim \frac{\lambda}{s^2} \max\{\max_{u\in[a,b]} D(z,w)(u), \max_{u\in[a,b]} B(z,w)(u),\\
&\quad \max_{u\in[a,b]} C(z,w)(u), \frac{1}{2}[\max_{u\in[a,b]} B(z,w)(u) + \max_{u\in[a,b]} C(z,w)(u)],\\
&\quad \frac{\max_{u\in[a,b]} B(z,w)(u) \max_{u\in[a,b]} C(z,w)(u)}{1 + \max_{u\in[a,b]} D(z,w)(u)}\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(Sz, Tw) &\precsim \frac{\lambda}{s^2} \max\{d(z,w), d(z,Sz), d(w,Tw),\\
&\quad \frac{1}{2}\left[d(w,Sz) + d(z,Tw)\right], \frac{d(z,Sz)d(w,Tw)}{1 + d(z,w)}\}.
\end{aligned}$$

Thus all the conditions of Theorem 2.1 with $P = Q = I_X$ are satisfied. Therefore, the system of Urysohn integral equations has a unique common solution in X. □

4. Application to linear system

In this section we give an application using the Corollary 1 in $(X = \mathbb{C}^n, d_2)$ the complete complex valued b-metric space where,

$$d_2(z, w) = \left[\sum_{i=1}^{n}\left(|z_i - w_i|^2 + i|z_i - w_i|^2\right)\right]^{\frac{1}{2}},$$

Theorem 4.1. *Let $(X = \mathbb{C}^n, d_2)$ a complex valued b-metric space where $z = (z_1, \ldots, z_n)^t \in X$ and $w = (w_1, \ldots, w_n)^t \in X$, if $\beta < \frac{1}{n}$ where,*

$$\beta_{ij} = \begin{cases} a_{ij} & \text{if } i \neq j \\ a_{ij} + 1 & \text{if } i = j \end{cases} \qquad \text{and} \quad \beta = \max\left\{\beta_{ij}\right\}, \forall 1 \leq i, j \leq n.$$

then, the following linear system of n equations and n unknowns $AZ = B$ has a unique solution.

$$\begin{cases} a_{11}z_1 + a_{12}z_2 + \ldots + & a_{1n}z_n = b_1 \\ a_{21}z_1 + a_{22}z_2 + \ldots + & a_{2n}z_n = b_2 \\ \vdots & \\ a_{n1}z_1 + a_{n2}z_2 + \ldots + & a_{nn}z_n = b_n \end{cases} \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Where $z = (z_1, \ldots, z_n)^t \in X$ and $a_{ij} \in \mathbb{R}$ where $1 \leq i, j \leq n$ and $b_1, b_2, b_n \in \mathbb{C}$

Proof. Define $T : X \to X$ by $Tz = (A + I)Z - B$. for proving that linear system $AZ = B$ have a unique solution, its enough to prove that T is a contraction.
Since

$$\begin{aligned} d_2(Tz, Tw) &= \left[\sum_{i=1}^{n}\left(|(Tz)_i - (Tw)_i|^2 + i|(Tz)_i - (Tw)_i|^2\right)\right]^{\frac{1}{2}}, \\ &= \left[\sum_{i=1}^{n}\left(\left|\sum_{j=1}^{n}\beta_{ij}(z_j - w_j)\right|^2 + i\left|\sum_{j=1}^{n}\beta_{ij}(z_j - w_j)\right|^2\right)\right]^{\frac{1}{2}}, \end{aligned}$$

where,

$$\beta_{ij} = \begin{cases} a_{ij} & \text{if } i \neq j \\ a_{ij} + 1 & \text{if } i = j \end{cases} \qquad \text{and } \beta = \max\left\{\beta_{ij}\right\}, \forall 1 \leq i, j \leq n.$$

Then,

$$\begin{aligned} d_2(Tz, Tw) &\precsim \left[\left(\sum_{i=1}^{n}\max_{1\leq i,j\leq n}\beta_{ij}^2\right)\left(\left|\sum_{j=1}^{n}(z_j - w_j)\right|^2 + i\left|\sum_{j=1}^{n}(z_j - w_j)\right|^2\right)\right]^{\frac{1}{2}}, \\ &\precsim \left(n\beta^2\right)^{\frac{1}{2}}\left[n\left(\left|\sum_{j=1}^{n}(z_j - w_j)\right|^2 + i\left|\sum_{j=1}^{n}(z_j - w_j)\right|^2\right)\right]^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} &\precsim n\beta\left[\left(\left|\sum_{j=1}^{n}(z_j - w_j)\right|^2 + i\left|\sum_{j=1}^{n}(z_j - w_j)\right|^2\right)\right]^{\frac{1}{2}}, \\ &= n\beta d_2(z, w). \end{aligned}$$

So, we get finally that:
$d_2(Tz, Tw) \precsim n\beta d_2(z, w)$ or $\beta = \max\left\{|a_{ij}|, |a_{ij} + 1| \quad \forall 1 \leq i, j \leq n\right\}$.
We conclude that T is contraction mapping. by applying Corollary 1, the linear system has a unique solution. □

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Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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