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Research article

Identities concerning k-balancing and k-Lucas-balancing numbers of arithmetic indexes

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Abstract: In this article, we derive some identities involving k balancing and k-Lucas-balancing numbers of arithmetic indexes, say $an + p$, where a and p are some fixed integers with $0 \leq p \leq a - 1$.

Keywords: balancing numbers; Lucas-balancing numbers; k-balancing numbers; k-Lucas-balancing numbers; arithmetic index
Mathematics Subject Classification: 11B39, 11B83

1. Introduction

Balancing numbers are originally obtained from a simple Diophantine equation. They are the solutions of the Diophantine equation $1 + 2 + 3 + \ldots + (n - 1) = (n + 1) + (n + 2) + \ldots + (n + r)$, where r is a balancer corresponds to a balancing number n [1, 3]. Balancing numbers satisfy the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}, \ \ n \geq 1,$$

with $B_0 = 0$ and $B_1 = 1$, where B_n denotes the n^{th} balancing number [1]. On the other hand, Lucas-balancing numbers C_n are obtained from the formula $C_n = \sqrt{8B_n^2 + 1}$ and are the terms of the sequence $\{1, 3, 17, 99, 577, \ldots\}$ [7]. They are recursively defined same as that of balancing numbers but with different initial values, that is,

$$C_{n+1} = 6C_n - C_{n-1}, \ \ n \geq 1,$$

with $C_0 = 1$ and $C_1 = 3$ [7].

Balancing numbers are generalized in many ways. For details review of some recent works, one can go through [2, 4–6, 8–10]. One of the generalization of balancing numbers called as k-balancing numbers depending on one real parameter k, are recently introduced by Ray in [9]. The n^{th} k-balancing numbers $B_{k,n}$ are terms of the sequence $\{0, 1, 6k, 36k^2 - 1, 216k^3 - 12k, \ldots\}$ and are recursively defined by

$$B_{k,0} = 0, B_{k,1} = 1 \text{ and } B_{k,n+1} = 6kB_{k,n} - B_{k,n-1} \text{ for } k \geq 1.$$

Notice that, for $k = 1$, balancing numbers $0, 1, 6, 35, 204, \ldots$ are obtained.

On the other hand, k-Lucas-balancing numbers that are the natural extension of Lucas-balancing numbers extensively studied in [9]. The sequence of k-Lucas-balancing numbers $\{C_{k,n}\} = \{1, 3k, 18k^2 - 1, 108k^3 - 9k, \ldots\}$ satisfies the same recurrence relation as that of k-balancing numbers with different initial conditions, i.e.,

$$C_{k,0} = 1, C_{k,1} = 3k \text{ and } C_{k,n+1} = 6kC_{k,n} - C_{k,n-1} \text{ for } k \geq 1.$$

Few properties that the k-balancing numbers satisfy are summarized below.

- Binet formula for k-balancing numbers: $B_{k,n} = \frac{\lambda_k^n - \lambda_k^{-n}}{\lambda_k - \lambda_k^{-1}}$, where $\lambda_k = 3k + \sqrt{9k^2 - 1}$.
- Catalan identity for k-balancing numbers: $B_{k,n}^2 - B_{k,n-r}B_{k,n+r} = B_{k,r}^2$.
- Simson's identity for k-balancing numbers: $B_{k,n}^2 - B_{k,n-1}B_{k,n+1} = 1$.
- D' Ocagne identity for k-balancing numbers: $B_{k,m}B_{k,n+1} - B_{k,m+1}B_{k,n} = B_{k,m-n}^2$.
- For odd k-balancing numbers, $B_{k,2n+1} = B_{k,n+1}^2 - B_{k,n}^2$.
- For even k-balancing numbers, $B_{k,2n} = \frac{1}{6k}[B_{k,n+1}^2 - B_{k,n-1}^2]$.
- Generating function for k-balancing numbers: $f_k(x) = \frac{x}{1-6kx-x^2}$.
- First combinatorial formula for k-balancing numbers:

$$B_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i \binom{n-1-i}{i} (6k)^{n-2i-1}.$$

- Second combinatorial formula for k-balancing numbers:

$$B_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (6k)^{n-2i-1} (36k^2 - 4)^i.$$

2. Identities concerning k-balancing numbers of arithmetic indexes

In this section, we study different sums of k-balancing numbers of arithmetic indexes, say $an + p$ for fixed integers a and p with $0 \leq p \leq a - 1$. Several identities concerning such numbers are established straightforwardly.

The following lemmas are useful while proving the subsequent results.

Lemma 2.1. *For all integers $n \geq 1$, $\lambda_{k_1}^n + \lambda_{k_2}^n = B_{k,n+1} - B_{k,n-1}$.*

Proof. The proof of this result can be easily shown by using Binet formula for k-balancing numbers and the fact $\lambda_{k_1}^n \lambda_{k_2}^n = 1$. □

Lemma 2.2. $B_{k,a(n+2)+p} = (B_{k,a+1} - B_{k,a-1})B_{k,a(n+1)+p} - B_{k,an+p}$.

Proof. Using Binet formula for k-balancing numbers and the result from Lemma 2.1, the first term of the right hand side expression reduces

$$\begin{aligned}
(B_{k,a+1} - B_{k,a-1})B_{k,a(n+1)+p} &= \frac{1}{\lambda_{k_1} - \lambda_{k_2}} \left[\lambda_{k_1}^{a(n+2)+p} - \lambda_{k_2}^{a(n+2)+p} + \lambda_{k_1}^{an+p} - \lambda_{k_2}^{an+p}\right] \\
&= \frac{\lambda_{k_1}^{a(n+2)+p} - \lambda_{k_2}^{a(n+2)+p}}{\lambda_{k_1} - \lambda_{k_2}} + \frac{\lambda_{k_1}^{an+p} - \lambda_{k_2}^{an+p}}{\lambda_{k_1} - \lambda_{k_2}} \\
&= B_{k,a(n+2)+p} + B_{k,an+p},
\end{aligned}$$

and the result follows. □

Since $B_{k,n+1} - B_{k,n-1} = 2C_{k,n}$ [9], the previous formula reduces to an identity

$$B_{k,a(n+2)+p} = 2C_{k,a}B_{k,a(n+1)+p} - B_{k,an+p}.$$

This identity gives the general term of the sequence of k-balancing numbers $\{B_{k,an+p}\}$ as a linear combination of two preceding terms. Iterative application of this result gives the general term as a combination of first two terms as follows:

$$\begin{aligned}
B_{k,an+p} = &\left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{a+i} \begin{pmatrix} n-1-i \\ i \end{pmatrix} (2C_{k,a})^{n-1-2i}\right) B_{k,an+p}^{n-1-2i} \\
&+ \left(\sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^{(}a+1)(1+i) \begin{pmatrix} n-2-i \\ i \end{pmatrix} (2C_{k,a})^{n-2-2i}\right) B_{k,p}^{n-2-i}.
\end{aligned}$$

In particular, for $a = 1$, then $r = 0$ and we have the corresponding identity for k-balancing numbers,

$$B_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{i} \begin{pmatrix} n-1-i \\ i \end{pmatrix} (6k)^{n-2-2i}.$$

Now we will find the generating function for the sequence $\{B_{k,an+p}\}$. Let $G(k, a, p, x)$ be the generating function for the sequence $\{B_{k,an+p}\}$, where $0 \leq p \leq a-1$, then

$$G(k, a, p, x) = \sum_{n=0}^{\infty} B_{k,an+p}x^n = B_{k,p} + B_{k,a+p}x + B_{k,2a+p}x^2 + B_{k,3a+p}x^3 + \dots. \tag{2.1}$$

Multiplying $2C_{k,a}x$ and x^2 in (2.1) by turns and subtracting the first one from (2.1) and then adding the second one, we obtain

$$\begin{aligned}
(1 - 2C_{k,a}x + x^2)G(k, a, p, x) = &B_{k,p} + (B_{k,a+p} - 2B_{k,p}C_{k,a})x \\
&+ \sum_{n=2}^{\infty} (B_{k,a(n+2)+p} - (B_{k,a+1} - B_{k,a-1})B_{k,a(n+1)+p} + B_{k,an+p})x^n.
\end{aligned} \tag{2.2}$$

The expression within the summation vanishes in view of Lemma 2.2. On the other hand, using the convolution identity $B_{k,a+p} = B_{k,p}B_{k,a+1} - B_{k,p-1}B_{k,a}$ and the fact $B_{k,n+1} - B_{k,n-1} = 2C_{k,n}$, the expression $B_{k,p} + (B_{k,a+p} - 2B_{k,p}C_{k,a})$ reduces to

$$B_{k,p} + (B_{k,a+p} - 2B_{k,p}C_{k,a}) = B_{k,p}B_{k,a-1} - B_{k,a}B_{k,p-1} = -B_{k,a-p}.$$

Therefore, (2.2) gives

$$G(k,a,p,x) = \frac{B_{k,p} + B_{k,a-p}x}{1 - 2C_{k,a}x + x^2}.$$

For $a = 1$, then $p = 0$ and $G(k,1,0,x) = \frac{x}{1-6kx+x^2}$ which is indeed the generating function for k-balancing numbers. While Choosing $a = 2$, p will be 0 and 1 and we have $G(k,2,0,x) = \frac{6kx}{1-6kx+x^2}$ and $G(k,2,1,x) = \frac{1+x}{1-6kx+x^2}$.

The following theorem establishes the sum for k-balancing numbers of the type $an + p$.

Theorem 2.3. *Let a be any integer and* $0 \leq p \leq a-1$*, then*

$$\sum_{i=0}^{n} B_{k,ai+p} = \frac{B_{k,a(n+1)+p} - B_{k,an+p} - B_{k,p} - B_{k,a-p}}{B_{k,a+1} - B_{k,a-1} - 2}.$$

Proof. Using Binet formula, the formula for geometric series and the fact $\lambda_{k_1}^n \lambda_{k_2}^n = 1$, we get

$$\begin{aligned}
\sum_{i=0}^{n} B_{k,ai+p} &= \frac{1}{\lambda_{k_1} - \lambda_{k_2}}\left[\sum_{i=0}^{n} \lambda_{k_1}^{ai+p} - \sum_{i=0}^{n} \lambda_{k_2}^{ai+p}\right] \\
&= \frac{1}{\lambda_{k_1} - \lambda_{k_2}}\left[\frac{\lambda_{k_1}^{an+p+a} - \lambda_{k_1}^{p}}{\lambda_{k_1}^{a} - 1} - \frac{\lambda_{k_2}^{an+p+a} - \lambda_{k_2}^{p}}{\lambda_{k_2}^{a} - 1}\right] \\
&= \frac{1}{(\lambda_{k_1} - \lambda_{k_2})[(\lambda_{k_1}\lambda_{k_2})^a - \lambda_{k_1}^a - \lambda_{k_2}^a + 1]}[\lambda_{k_1}^{an+p}(\lambda_{k_1}\lambda_{k_2})^a \\
&\quad - \lambda_{k_1}^{an+p+a} - \lambda_{k_1}^{p}\lambda_{k_2}^{a} + \lambda_{k_1}^{p} - \lambda_{k_2}^{an+p}(\lambda_{k_1}\lambda_{k_2})^a + \lambda_{k_2}^{an+p+a} + \lambda_{k_2}^{p}\lambda_{k_1}^{a} - \lambda_{k_2}^{p}] \\
&= \frac{1}{(\lambda_{k_1} - \lambda_{k_2})[2 - (\lambda_{k_1}^a + \lambda_{k_2}^a)]}[\frac{\lambda_{k_1}^{an+p} - \lambda_{k_2}^{an+p}}{\lambda_{k_1} - \lambda_{k_2}} \\
&\quad - \frac{\lambda_{k_1}^{an+p+a} - \lambda_{k_2}^{an+p+a}}{\lambda_{k_1} - \lambda_{k_2}} \frac{\lambda_{k_1}^{p} - \lambda_{k_2}^{p}}{\lambda_{k_1} - \lambda_{k_2}} + \frac{\lambda_{k_1}^{a}\lambda_{k_2}^{p} - \lambda_{k_1}^{p}\lambda_{k_2}^{a}}{\lambda_{k_1} - \lambda_{k_2}}].
\end{aligned}$$

By virtue of Lemma 2.1 and by Binet formula, we get the desired result. □

The following results are immediate consequence of Theorem 2.3 by setting $a = 2t + 1$ and $a = 2t$ respectively.

Corollary 2.4. *The sum of odd k-balancing numbers of the kind an + p is*

$$\sum_{i=0}^{n} B_{k,(2t+1)i+p} = \frac{B_{k,(2t+1)(n+1)+p} - B_{k,(2t+1)n+p} - B_{k,p} - B_{k,(2t+1)-p}}{B_{k,2t+2} - B_{k,2t} - 2}.$$

Corollary 2.5. *The sum of even k-balancing numbers of the kind an + p is*

$$\sum_{i=0}^{n} B_{k,2ti+p} = \frac{B_{k,2t(n+1)+p} - B_{k,2tn+p} - B_{k,p} - B_{k,2t-p}}{B_{k,2t+1} - B_{k,2t-1} - 2}.$$

Obsevation 2.6. *Let $t = 0$, then $a = 1$ and $p = 0$. Therefore from Corollary 2.4, we obtain*

$$\sum_{i=0}^{n} B_{k,i} = \frac{B_{k,n+1} - B_{k,n} - 1}{6k - 2}.$$

For $k = 1$, the sum of balancing numbers is $\sum_{i=0}^{n} B_{k,i} = \frac{B_{n+1} - B_n - 1}{4}$. Similarly, for $t = 1$, $a = 3$, we have

$$\sum_{i=0}^{n} B_{k,3i+p} = \frac{B_{k,3(n+1)+p} - B_{k,3n+p} - B_{k,p} - B_{k,3-p}}{B_{k,4} - B_{k,2} - 2}.$$

For $p = 0$,

$$\sum_{i=0}^{n} B_{k,3i} = \frac{B_{k,3(n+1)} - B_{k,3n} - B_{k,3}}{B_{k,4} - B_{k,2} - 2},$$

and for $k = 0$, $\sum_{i=0}^{n} B_{3i} = \frac{B_{3n+3} - B_{3n} - 35}{196}$. For $p = 1$,

$$\sum_{i=0}^{n} B_{k,3i+1} = \frac{B_{k,3n+4} - B_{k,3n+1} - B_{k,2} - 1}{B_{k,4} - B_{k,2} - 2}.$$

For $k = 0$, the sum formula for balancing numbers is given by $\sum_{i=0}^{n} B_{3i+1} = \frac{B_{3n+4} - B_{3n+1} - 7}{196}$. Finally, for $p = 2$,

$$\sum_{i=0}^{n} B_{k,3i+2} = \frac{B_{k,3n+5} - B_{k,3n+2} - B_{k,2} - 1}{B_{k,4} - B_{k,2} - 2}.$$

Again, $k = 0$ gives $\sum_{i=0}^{n} B_{3i+2} = \frac{B_{3n+5} - B_{3n+2} - 7}{196}$. Similarly, by virtue of Corollary 2.5, for $t = 1$, then $a = 2$ implies that for $p = 0$, $\sum_{i=0}^{n} B_{k,2i} = \frac{B_{k,2n+2} - B_{k,2n} - 6k}{36k^2 - 4}$. For $p = 1$, we have $\sum_{i=0}^{n} B_{k,2i+1} = \frac{B_{k,2n+3} - B_{k,2n+1} - 2}{36k^2 - 4}$ and so on.

Now we will find the recurrence relation for the sequence $\{B_{k,an+p}\}$. For that, let us denote $\sum_{i=0}^{n} B_{k,ai+p}$ as $S_{k,an+p}$. In view of Lemma 2.2, we have

$$S_{k,an+p} = \sum_{i=0}^{n} B_{k,ai+p}$$

$$
\begin{aligned}
&= B_{k,p} + B_{k,a+p} + \sum_{i=2}^{n} B_{k,ai+p} \\
&= B_{k,p} + B_{k,a+p} + \sum_{i=2}^{n}(2C_{k,p}B_{k,a(i-1)+p} - B_{k,a(i-2)+p}) \\
&= B_{k,p} + B_{k,a+p} + 2C_{k,p}\sum_{i=1}^{n-1} B_{k,ai+p} - \sum_{i=0}^{n-2} B_{k,ai+p} \\
&= B_{k,p} + B_{k,a+p} + 2C_{k,p}(S_{k,a(n-1)+p} - B_{k,p}) - S_{k,a(n-2)+p}.
\end{aligned}
$$

It follows that,

$$S_{k,a(n+1)+p} = B_{k,p} + B_{k,a+p} + 2C_{k,p}(S_{k,an+p} - B_{k,p}) - S_{k,a(n-1)+p}.$$

Consequently,

$$S_{k,a(n+1)+p} = (2C_{k,p} + 1)S_{k,an+p} - (2C_{k,p} + 1)S_{k,a(n-1)+p} + S_{k,a(n-2)+p},$$

which is the desired recurrence relation for the sequence $\{B_{k,an+p}\}$.

3. Identities concerning *k*-Lucas-balancing numbers of arithmetic indexes

In this section, we study the k-Lucas-balancing numbers of arithmetic indexes of the form $an + p$.

A repeated application of the formula in Lemma 2.2 gives an identity that relates k-Lucas-balancing numbers with k-balancing numbers, that is, for natural numbers n and l,

$$C_{k,n} = C_{k,n-(l-1)}B_{k,p} - C_{k,n-l}B_{k,l-1}.$$

But for $l = -n$, $C_{k,n} = C_{k,2n}B_{k,n+1} - C_{k,2n+1}B_{k,n}$. Also it is observed that $C_{k,-n} = C_{k,n}$.

Lemma 3.1. *Let* $a \neq 0$ *and* $0 \leq p \leq a-1$, *then* $C_{k,a(n+1)+p} = 2C_{k,a}C_{k,an+p} - C_{k,a(n-1)+p}$.

Proof. Clearly $2C_{k,a(n+1)+p} = B_{k,a(n+1)+p+1} - B_{k,a(n+1)+p-1}$. Therefore, using Lemma 2.2, we get

$$
\begin{aligned}
2C_{k,a(n+1)+p} &= (B_{k,a+1} - B_{k,a-1})B_{k,an+p+1} - B_{k,a(n-1)+p+1} \\
&\quad -(B_{k,a+1} - B_{k,a-1})B_{k,an+p-1} + B_{k,a(n-1)+p-1} \\
&= 2C_{k,a}(B_{k,an+p+1} - B_{k,an+p-1}) - (B_{k,a(n-1)+p+1} - B_{k,a(n-1)+p-1}) \\
&= 2C_{k,a} \times 2C_{k,an+p} - 2C_{k,a(n-1)+p},
\end{aligned}
$$

and we obtain the desired result. □

In particular, for $p = 0$ the above identity reduces to $C_{k,an+a} = 2C_{k,a}C_{k,an} - C_{k,an-a}$. Applying iteratively the identity in Lemma 3.1 gives rise to the following sum formula.

$$C_{k,an+p} = \sum_{i=0}^{m}(-1)^i\binom{m}{i}(2C_{k,a})^{m-i}C_{k,a(n-m-i)+p}, \quad 0 \leq m \leq n.$$

Further, for $m = n$, this formula reduces to $C_{k,an+p} = \sum_{i=0}^{n}(-1)^i \binom{n}{i}(2C_{k,a})^{n-i}C_{k,p-ai}$.

In order to find the generating function of the sequence $\{C_{k,an+p}\}$, we proceed in the following way. Let $g(k,a,p,x)$ be the generating function for the sequence $\{C_{k,an+p}\}$, where $0 \le p \le a-1$, then

$$g(k,a,p,x) = \sum_{n=0}^{\infty} C_{k,an+p}x^n = C_{k,p} + C_{k,a+p}x + C_{k,2a+p}x^2 + C_{k,3a+p}x^3 + \dots. \tag{3.1}$$

Multiply (3.1) by $2C_{k,a}x$ and x^2 and proceed as in case of the sequence $\{B_{k,an+p}\}$, we get the generating function for the sequence $\{C_{k,an+p}\}$ as

$$g(k,a,p,x) = \frac{C_{k,p} + [C_{k,a+p} - 2C_{k,a}C_{k,p}]x}{1 - 2C_{k,a}x + x^2}.$$

It is observed that for $a = 1$, then $r = 0$ and we have the generating function for k-Lucas-balancing numbers, $g(k,x) = \frac{1-3kx}{1-6kx+x^2}$. Further, for $k = 1$, the generating function for Lucas-balancing numbers $g(x) = \frac{1-3x}{1-6x+x^2}$ is obtained.

Theorem 3.2. *Let a be any integer and* $0 \le p \le a-1$*, then*

$$\sum_{i=0}^{n} C_{k,ai+p} = \frac{C_{k,an+p} - C_{k,a(n+1)+p} + C_{k,p} - C_{k,a-p}}{2(1 - C_{k,a})}.$$

Proof. The proof of this theorem is analogous to Theorem 2.3. □

As an observation, one can see that, for $a = 1$ then $r = 0$ gives the identity $\sum_{i=0}^{n} C_i = \frac{C_{n+1} - C_n + 2}{4} = \frac{c_n + 1}{2}$, where c_n is the n^{th} Lucas-cobalancing number with $C_{n+1} - C_n = 2c_n$ [2].

We end this section by establishing an important relation between k-balancing and k-Lucas-balancing numbers.

Theorem 3.3. *For* $t \ge 1$*,*

$$\frac{B_{k,2^t n}}{B_{k,n}} = 2^t \prod_{i=0}^{t-1} C_{k,2^i n}.$$

Proof. Using Binet formula and Lemma 2.1, the left side expression becomes

$$\begin{aligned}
\frac{B_{k,2^t n}}{B_{k,n}} &= \frac{\lambda_{k_1}^{2^t n} - \lambda_{k_2}^{2^t n}}{\lambda_{k_1}^{n} - \lambda_{k_2}^{n}} \\
&= (\lambda_{k_1}^{2^{t-1} n} + \lambda_{k_2}^{2^{t-1} n})\frac{\lambda_{k_1}^{2^{t-1} n} - \lambda_{k_2}^{2^{t-1} n}}{\lambda_{k_1}^{n} - \lambda_{k_2}^{n}} \\
&= 2C_{k,2^{t-1}n}(\lambda_{k_1}^{2^{t-2} n} + \lambda_{k_2}^{2^{t-2} n})\frac{\lambda_{k_1}^{2^{t-2} n} - \lambda_{k_2}^{2^{t-2} n}}{\lambda_{k_1}^{n} - \lambda_{k_2}^{n}}
\end{aligned}$$

$$= 2^2 C_{k,2^{t-1}n} C_{k,2^{t-2}n} (\lambda_{k_1}^{2^{t-3}n} + \lambda_{k_2}^{2^{t-3}n}) \frac{\lambda_{k_1}^{2^{t-3}n} - \lambda_{k_2}^{2^{t-3}n}}{\lambda_{k_1}^{n} - \lambda_{k_2}^{n}}.$$

Continuing in this way, we finally get

$$\frac{B_{k,2^t n}}{B_{k,n}} = 2^t \prod_{i=0}^{t-1} C_{k,2^i n}.$$

This completes the proof. □

In particular, for $t = 1$, the above identity reduces to $B_{k,2n} = 2B_{k,n}C_{k,n}$, a known identity for k-balancing numbers.

Conflict of interest

The author declares no conflict of interest.

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