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Research article

Elementary properties of non-Linear Rossby-Haurwitz planetary waves revisited in terms of the underlying spherical symmetry

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Abstract: We revisit Rossby-Haurwitz planetary wave modes of a two-dimensional fluid along the surface of a rotating planet, as elements of irreducible representations of the so(3) Lie algebra. Key questions addressed are, firstly, why it is that the non-linear self-interaction of any Rossby-Haurwitz wave mode is zero, and secondly, why the phase velocity of these wave modes is insensitive to their orientation with respect to the axis of rotation of the planet, while at the same time the very rotation of the planet is a precondition for the existence of the waves. As we show, answers to both questions can be rooted in Lie group and representation theory. In our study the Rossby-Haurwitz modes emerge in a coordinate-free, as well as in a Ricci tensor rank-free manner. We find them with respect to a continuum of spherical coordinate systems, that are arbitrarily oriented with respect to the planet. Furthermore, we show that, in the same sense in which the Lie derivative of Ricci tensor fields is rank-free, the wave equation for Rossby-Haurwitz modes is rank-free. We find that, for each irreducible representation of so(3), there is a corresponding sufficient condition for existence of Rossby-Haurwitz modes as solutions that are separable with respect to space and time. This condition comes in the form of a system of equations of motion for the coordinate systems. Coordinate systems that move along with Rossby-Haurwitz modes emerge as special cases of these. In these coordinate systems the waves appear as stationary spatial fields, so that the motion of the coordinate system coincides with the wave phase propagation. The general solution of the existence condition is a continuum of moving spherical coordinate systems that precess about the axes of the Rossby-Haurwitz modes. Within a single irreducible representation of so(3), the waves are dispersionless.

Keywords: planetary wave dynamics; Rossby-Haurwitz modes; covariant representation; Lie algebra; representation theory
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1. Introduction

1.1. Key Questions

A striking property of the *non-linear* potential vorticity equation, for dynamics of a shallow fluid layer on a rotating planet, is that, *as if it were a linear wave equation*, it supports linear scaling of its normal modes, the planetary Rossby-Haurwitz [9, 20] waves. That is, the non-linear self-interaction of its normal modes vanishes identically, so that, with any finite amplitude, these modes are solutions of the *non-linear* wave equation [5].

A second remarkable feature of the Rossby-Haurwitz modes, *especially in view of the fact that their very existence has the rotation of the planet about an axis as a premise*, lies in the fact that their *orientation* with respect to the rotation axis of the planet can be any [14], to the extent that this orientation even is of no influence on their phase velocity.

These features are both well-known and straightforward to demonstrate by mere substitution and calculation, e.g. based on calculation rules for spherical harmonics [14, 22, 25]. Such a demonstration however, lucid as it is, leaves underexposed the following two challenging questions. First, the question of *why*, in this case, a *non-linear wave equation* allows for arbitrary scaling of its fundamental wave modes. Second, why, on the one hand, the wave modes apparently are a consequence of the rotation of the planet, which defines an axis and hence by itself breaks the spherical symmetry, while on the other hand the wave modes apparently are not sensitive at all to the direction of this axis of rotation.

1.2. Double contrast with beta-plane based approaches

On *local* scales the waves of interest here are commonly known-as Rossby waves, after C.G. Rossby who introduced [20] the so called β-plane model, in support of his interpretation of the mechanism of these waves. The non-linear dynamics of Rossby waves on beta-plane type models is a subject of undiminished research interest e.g. [29–31]. The approach we choose in the present manuscript, however, is fundamentally distinct from beta-plane based approaches in two ways, as follows.

As recently reviewed by Dellar [6], as an ingredient of his foundation of a range of beta-plane models, the derivation of beta-plane models involves a step that Dellar characterized as "omission of the centrifugal potential"; meant is the centrifugal potential associated with the solid body rotation of the planet, *as observed by an observer that is co-rotating with the planet*. As was explained and quantified in detail in [23], what can be described as "omission", corresponds to the fact that the component of the centrifugal acceleration that is tangent to the surface of the planet and equatorward, is exactly balanced by a tangent and poleward component of the gravitational field of the solid planet. This balance corresponds to nothing else but the assumption that the shape and the mass distribution of the planet, through the generated field of gravitation, is in equilibrium with the centrifugal acceleration that results from the rotation of the planet. This assumption was e.g. also central to Chandrasekhar's classical study of equilibrium shapes of rotating and gravitating bodies [3]. As a result, the surface of the planet would, as experienced by an observer at rest at this surface, be an equipotential surface of the potential formed by the sum of the gravitational potential and the centrifugal potential. For planets like earth, the associated deviation of the geometry of the planet from spherical geometry however, is so small that the geometry of the surface may well be approximated as spherical; this is discussed as the *Spherical Geopotential Approximation* in the study of meteorological models by White et al. [26].

It is important to emphasize that *all geopotential surfaces near the surface of the solid planet may be approximated as spherical surfaces*, so that e.g. the depth of an ocean, covering the planet and in static equilibrium, would be constant as a function of geographical latitude [23].

As a result, and Dellar [6] included this in his foundation of beta-plane models, the rotating planet may be approximated by a rotating sphere, the surface of which nevertheless coincides with an equipotential surface [23, 26] of the potential forces. We emphasize here however that this is only true from the point of view of an observer that is co-rotating with the solid planet. This is because the gravitational field on the one hand, and inertial effects such as the centrifugal potential on the other hand, transform very differently under rotational motions of reference (coordinate) systems.

As far as beta-plane models are concerned, this means that beta-plane models are not only *local* approximations, they, as part as their foundation, also pre-assume that their coordinate system is *static* with respect to the planet. In double contrast with this, the approach that we shall present in the present manuscript will explore spherical (i.e. *global*) coordinate systems that *move* (i.e. rotate) with respect to the planet.

1.3. Focus on global, planetary Rossby-Haurwitz waves

In this manuscript we shall use a global, spherical coordinate system that could rotate arbitrarily with respect to the solid planet, as a tool to investigate the dynamics of the global counterparts of Rossby waves. In the modern literature these global waves are commonly known as Rossby-Haurwitz waves [15, 17, 18]; they correspond to what Hough [12, 13] denoted as "tidal motions of the second class".

On the *global* scale, these Rossby-Haurwitz modes are of fundamental importance to atmospheric and ocean dynamics, as they are the normal modes of the barotropic vorticity equation. They are basic modes of truncated spectral models for the atmosphere [18] and their *non-linear interaction* [17] has been subject of continued research rendering conceptually clarifying models [15].

In this paper we revisit the Rossby-Haurwitz modes and we shall re-derive them in a coordinate-free form, tailor-made for the purpose. Where useful, we shall introduce some ingredients from group and representation theory. Goal is to shed light on relevant aspects that may help to understand what lies behind the challenging questions put forward in the opening paragraphs of this section and indicate paths towards further explorations of these questions.

A specific goal of this paper is to show that the existence of non-trivial sets of linearly independent, non-self-interacting fundamental wave modes, in the case at hand is actually *implied* by the near spherical symmetry of the surface of the planet, given furthermore only the algebraic structure of the non-linear interaction term and the fact that the elementary operators from which the non-linear interaction operator is composed all commute with generators of the rotation group.

1.4. Outline

The view that the absence of non-linear self-interaction of Rossby-Haurwitz modes results from the underlying spherical symmetry is established in Section 5. The result that for the same reason, the phase velocity of the modes is insensitive to their orientation, is phrased in the discussion, Section 7. Technically, this result is established in Section 6, in which the non-linear wave modes are constructed as separable solutions with respect to spherical coordinate systems that move in a suitable manner with respect to the rotating planet. The construction of this class of moving spherical coordinate systems, essentially in subsection 6.3, is one of the technical results presented. This subsection is part of Section 6, which furthermore presents the dynamics of suitably moving spherical coordinate systems, generators of rotations with respect to these and Rossby-Haurwitz modes as eigenmodes of these generators.

The results in Sections 5 and 6 are established swiftly, but only after elaborate preparations in Sections 2 to 4. In Section 2 we develop the tools supporting analysis of planetary fluid dynamics with respect to an arbitrarily moving spherical coordinate system. Kinematical tools are introduced in subsection 2.1. The Lie algebra so(3) of vectorfields that describe solid body rotations enters here. Dynamics is introduced in subsection 2.2.

A first analysis of the dynamics in terms of group theoretical concepts is presented in Section 3. Transformation properties of fundamental modes under rotations are reviewed in terms of group and representation theory. In as far as the content is no more than a review, Section 3 serves the goal to make this manuscript sufficiently self-contained as well as to make it bridge different application fields of mathematical physics. As the concept of representations of Lie algebra's plays a key role in the reasoning, a separate section, Section 4, is devoted to discussing how it enters our study.

We stress that our interest and involvement will not so much be with the mere *conclusions*, as far as these merely recover wave properties which are by themselves all very well-known, but rather with the relevant *premises* and with the theory that supports the *implications* of the conclusions, as such.

2. Formulation of the potential vorticity equation with respect to an arbitrarily rotating spherical coordinate system

2.1. An arbitrarily oriented and moving spherical coordinate system, S

2.1.1. Definition of coordinate systems C and S

In what follows we shall explore the notion of solutions that are separable, with respect to space and time, in a suitably chosen spherical coordinate system. We shall employ a spherical coordinate system S that we can orient at will. Moreover, the orientation of the coordinate system will evolve over time, ultimately in such a way so as to support separable solutions of the wave equation.

We shall exclude deformations of the coordinate system. Thus, we adopt the notion of a spherical coordinate system, S, that can rotate about the center of a spherical planet in an arbitrary way. Points that have fixed coordinates with respect to S, would move along the surface of the planet.

We shall implement this notion by rotating a standard geographical, spherical coordinate system. The rotation could e.g. be parameterized by three Euler angles λ, θ and ϕ. The evolution over time then would simply be parameterized by the time dependence of the Euler angles.

Let now C be a three-dimensional, standard Cartesian, inertial coordinate system, with coordinates (c_1, c_2, c_3) and let $R_i(\zeta)$ denote a rotation (matrix) over angle ζ about the i-th axis, here of C. We introduce the spherical coordinates (s_1, s_2) of our spherical coordinate system S as

$$c_i \ = \ a_p\, M(\lambda, \theta, \phi)\, R_3(s_1)\, R_2(s_2)\, \mathbf{e}_1 \quad , \tag{2.1}$$

where the composed rotation operator (or its representation matrix) M is defined as

$$M(\lambda, \theta, \phi) \ = \ R_3(\lambda)\, R_2(\theta)\, R_3(\phi) \quad . \tag{2.2}$$

The entity $\mathbf{e}_1$ in (2.1) is the first unit vector in C and a_p denotes the radius of the spherical surface that is coordinatized by coordinates (s_1, s_2). Our λ and θ can be interpreted as the standard geographical coordinates of the origin of S, i.e. of the point $(s_1, s_2) = (0, 0)$. Our definition of the Euler angles corresponds to the Euler angles (α, β, γ) of Rose [19] as $\lambda = \alpha, \theta = -\beta, \phi = \gamma$.

2.1.2. Vector fields instrumental to describing the motion of S and that of the planet

At each instant in time, each fixed point of coordinate system S has a velocity with respect to the surface of the planet. The whole of these velocities of points of S forms a vector field; we call this the velocity field of S, relative to the planet, and we shall denote this by the symbol $\check{v}$. We shall think of $\check{v}$ as being intrinsically represented with respect to S.

The planet, in turn, will rotate with respect to the inertial frame of reference, C. This rotation, too, can be represented as a vectorfield: it describes the velocities of fixed points of the solid planet's surface, as they move with respect to inertial frame C. We shall denote this field by the symbol ω; again, ω will be represented with respect to our moving coordinate system S. Hence, our representation of the field ω will depend on spatial coordinates (S) and it may depend on time. In this paper, we shall assume that the rotation of the planet with respect to inertial frame C is steady. Therefore, relevant time dependence of ω can only be induced by the fact that the coordinate system S moves with respect to the planet, and the following identity applies:

$$\frac{\partial}{\partial t}\omega = \mathcal{L}_{\check{v}}\,\omega \quad ; \tag{2.3}$$

the right hand side in this expression denotes the Lie derivative of ω along the vector field $\check{v}$ [7].

2.1.3. Vector fields forming an orthonormal basis for the so(3) Lie algebra, stationary in S

Let r_3 be the (contravariant) vector field of unit vectors along the curves of constant coordinate s_2; would S coincide with the standard geographical coordinate system, these would be curves of constant latitude. A flow according to r_3 therefore would simply be a solid body rotation about the polar axis of S. Such vector fields are known as *generators* of rotation.

Again, in a geographical coordinate system, the polar axis of S would correspond to the axis of the planet, and usually to a third Cartesian inertial frame of reference; hence the naming r_3 for this particular field.

As r_3 is the generator of a solid body rotation, it generates an isometry; this means that the Lie derivative along r_3 of the metric tensor, g_{ij} in Ricci notation, vanishes identically. In general, for any such generator r of an isometry,

$$\mathcal{L}_r\, g_{ij} = 0 \ . \tag{2.4}$$

Details of the metric of S are given by

$$g_{11} = a_p^2 \cos(s_2)^2,\ g_{12} = g_{21} = 0 \ \text{and}\ g_{22} = a_p^2 \ . \tag{2.5}$$

Viewed as linear differential equation for vector field r, for the given metric g_{ij}, the general solution of equation (2.4) is three-dimensional and spanned by e.g.

$$r_1 \quad = \quad (-\cos(s_1)\tan(s_2),\ \sin(s_1))\,, \tag{2.6}$$

$$r_2 \;=\; (-\sin(s_1)\tan(s_2),\, -\cos(s_1))\,, \tag{2.7}$$

$$r_3 \;=\; (1,\,0)\,, \tag{2.8}$$

in which s_1 and s_2 denote the zonal and meridional coordinates of spherical coordinate system S, respectively. With respect to the inner product

$$< v \,||\, w > = \frac{3}{8\,\pi\,a_p^4} \int_M g_{ij}\, v^i\, w^j\, \sqrt{g}\, ds_1 \wedge ds_2 \quad , \tag{2.9}$$

the set $\{r_1, r_2, r_3\}$ is orthonormal [24, App. D]. In expression (2.9) g is the determinant of the metric tensor g_{ij} and $\sqrt{g}\, ds_1 \wedge ds_2$ is the surface element.

Thus, $\{r_1, r_2, r_3\}$ forms an orthonormal basis for the general solution of equation (2.4). The fields r_1 and r_2 are generators of solid body rotations about two Cartesian axes, that together with the axis for r_3 form an orthogonal Cartesian coordinate system for the three-dimensional Euclidean space in which the spherical surface of the planet is embedded. Note that these axes are fixed in S however, so they differ from those of C and they do not form a Newtonian inertial frame, in general.

2.1.4. Vector fields associated with the planet's rotation and with variations of the Euler angles, so with the motion of S

We assume that the planet rotates steadily, with angular frequency Ω, about an axis that is stationary with respect to some inertial frame of reference. Let this axis coincide with the third Cartesian axis of inertial frame C. The velocity field of the surface of the planet, with respect to C, and represented in terms of the Cartesian coordinate system C, is then

$$\omega \;=\; \Omega\,(-c_2, c_1, 0) \;\; . \tag{2.10}$$

We write ω as the angular frequency Ω times a vector field that we shall call ω_3

$$\omega \;=\; \Omega\,\omega_3 \;\; . \tag{2.11}$$

Hence, the Cartesian representation of ω_3 is

$$\omega_3 \;=\; (-c_2, c_1, 0) \;\; ; \tag{2.12}$$

note that the index of the vector field ω_3, thus introduced, refers to the third axis of C, like the index of r_3 refers to the polar axis of S.

Expressions (2.10) and (2.12) give representations of the vectorfields ω and ω_3 in terms of the Cartesian coordinates of frame C. Throughout the manuscript however, we shall rather work with representations of these fields with respect to the moving spherical coordinate system S. Such representations can be calculated directly, by contravariant transformation e.g. [2, 7, 21]. As may be checked by direct computation however, the vector field ω_3 corresponds to the third component of the formal column vector $\boldsymbol{\omega}$, which we define as

$$\boldsymbol{\omega} \;=\; \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \;=\; M(\lambda, \theta, \phi) \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \;=\; M\,\mathbf{r} \;\; ; \tag{2.13}$$

the right hand expressions are to be understood as the matrix M, as defined in expression (2.2), acting on a formal column vector $\mathbf{r}$, the components of which are the vector fields r_1, r_2 and r_3. Combining the representations (2.6) to (2.8) of the fields r_1, r_2 and r_3 with respect to S with (2.13) provides an alternative, convenient way to obtain a particularly useful representation of ω_3 with respect to S; see also the remark about this at the end of this section.

In a similar way, we introduce a formal column vector $\boldsymbol{\varpi}$, consisting of three vectorfields ϖ_1, ϖ_2 and ϖ_3 (the use of in particular field ϖ_2 will become clear shortly):

$$\boldsymbol{\varpi} = \begin{pmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix} = M(0, \theta, \phi)\,\mathbf{r} \quad . \tag{2.14}$$

As can be verified by direct calculation, the set $\{\omega_1, \omega_2, \omega_3\}$ is orthonormal, with respect to the inner product (2.9), and so is the set $\{\varpi_1, \varpi_2, \varpi_3\}$.

The velocity field ν of S with respect to C can be decomposed into the velocity ω of the planet with respect to C and the velocity $\check{\nu}$ of S with respect to the planet:

$$\nu = \omega + \check{\nu} \quad . \tag{2.15}$$

Contravariant components of ν, and hence $\check{\nu}$, with respect to S could be directly constructed from the definition of S, (2.1), by differentiation with respect to time followed by a contravariant coordinate transformation. As could subsequently be verified by direct calculation, in terms of the fields ω_3 (2.13), ϖ_2 (2.14) and r_3 (2.8), the relative velocity field $\check{\nu}$ of S with respect to the planet can be decomposed into its components due to the variation of the subsequent Euler angles as follows

$$\check{\nu} = \dot{\lambda}\,\omega_3 - \dot{\theta}\,\varpi_2 + \dot{\phi}\,r_3 \quad , \tag{2.16}$$

in which the dots denote differentiation with respect to time. Expression (2.16) shows that variation of λ corresponds to rotation about the planet's axis, while variation of ϕ corresponds to rotation about the polar axis of S.

Remark : relations (2.13) and (2.14) are useful when it comes to calculation of the time derivative of $\check{\nu}$, as observed in S. It is then helpful to first expand $\check{\nu}$ fully with respect to the basis $\{r_1, r_2, r_3\}$, as these fields themselves do not depend on time, in S, whereas ω_3 and ϖ_2 may do so. After expansion with respect to $\{r_1, r_2, r_3\}$ however, all time dependencies are exclusively captured by the coefficients of r_1, r_2 and r_3.

2.2. Fields describing the fluid layer covering the planet; Covariant potential vorticity equation

We shall consider a two-dimensional fluid covering a spherical planet. By the velocity field $\upsilon = \upsilon(s,t)$ of the fluid we mean the velocity of the fluid at (any) point with fixed S- coordinates s, with respect to S, at the time instant t. This is known as the Eulerian velocity of subsequent, *likely different*, fluid parcels passing the point s at subsequent instances of time.

Because system S itself moves with velocity ν with respect to inertial frame C, the velocity of the fluid with respect to C is $\upsilon + \nu$.

A constraint on the fluid motion is provided by the potential vorticity equation [11, 16]. We shall express this as global conservation of absolute vorticity $*d(v + \nu)$. In this expression, the combination $*d$ of the Hodge star operator $*$ and the exterior derivative d, in combination acting on a two-dimensional vector field, renders a zero-form, which is a scalar field [7]; the result here corresponds to what conventionaly is called the component of the absolute vorticity vector [16] normal to the surface of the planet.

With respect to our moving coordinate system S and using the covariant derivative ∇_ℓ, the potential vorticity equation can be expressed as

$$\frac{\partial}{\partial t}\,(*d(v + \nu)) \;+\; \nabla_\ell\left(*d(v + \nu)\, v^\ell\right) \;=\; 0 \quad ; \tag{2.17}$$

the Einstein summation convention applies here, i.e. we sum over repeated indices, ℓ in this case. Combined with conservation of mass, say with mass density $h(s, t)$, equation (2.17) is equivalent with material conservation of potential vorticity $*d(v + \nu)/h$ [16]:

$$\left(\frac{\partial}{\partial t} \;+\; v^\ell\,\nabla_\ell\right)\frac{*d(v + \nu)}{h} \;=\; 0 \quad .$$

Because S itself moves with velocity $\check{\nu}$, as a whole, with respect to the planet, a fluid layer that would be stationary with respect to the planet would have velocity $v = -\check{\nu}$, as observed in S. Therefore, we introduce a field $\hat{\varphi}$ to describe the difference between the actually observed v and the state of a fluid layer at rest:

$$v \;=\; \hat{\varphi} - \check{\nu} \quad ; \tag{2.18}$$

the field $\hat{\varphi}$ can be thought of as describing a phenomenon in the fluid motion, i.e. a deviation from a state of rest. If the field $\hat{\varphi}$ is equal to zero, so as observed in coordinate system S, it represents a state in which the whole fluid layer is perfectly at rest, as seen by an observer which is stationary at the planet.

In terms of the fields so defined, using (2.15) and (2.18), equation (2.17) takes the form

$$\frac{\partial}{\partial t}\,(*d(\hat{\varphi} + \omega)) \;+\; \nabla_\ell(*d(\hat{\varphi} + \omega)\,(\hat{\varphi}^\ell - \check{\nu}^\ell)) \;=\; 0 \quad . \tag{2.19}$$

We rewrite this equation as

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\check{\nu}} \;+ \hat{\varphi}\cdot d + *^{-1} d * \hat{\varphi}\right) * d(\hat{\varphi} + \omega) \;=\; 0 \quad ; \tag{2.20}$$

to derive (2.20) from (2.19) we have used the fact that $\check{\nu}$ is divergence-free (i.e. $\nabla_\ell\,\check{\nu}^\ell \;=\; *^{-1} d * \check{\nu} \;=\; 0$).

Apart from the partial derivative with respect to time t, in expression (2.20) we recognize [7] the exterior derivative d, the Lie derivative $\mathcal{L}_{\check{\nu}}$ along the vectorfield $\check{\nu}$, the Hodge star operator $*$ as well as its inverse $*^{-1}$ and the local inner product $\cdot$. The latter can e.g. be implemented as the double contraction of covariant vector indices with the contravariant indices of the inverse g^{ij} of the metric g_{ij}:

$$v \cdot w \;=\; g^{pq}\, v_p\, w_q \quad ; \tag{2.21}$$

the result is a scalar field.

The operator d acting on a scalar field gives a gradient, so that $\hat{\varphi} \cdot d$ corresponds to what in the literature is known as the advection operator. The Lie derivative in equation (2.20) renders the rate of change with time of the potential vorticity, as observed in the moving coordinate system, *as far as this change is a consequence of the motion,* $\check{v}$*, of the coordinate system.* As this is in accordance to the definition of the Lie derivative with respect to general tensors, it is natural to use the Lie derivative here. In section 4.1 a further motivation for using the Lie derivatives in equation (2.20), as we do, will surface: it is that the field $\check{v}$ is a generator of rotations, and the Lie derivative provides a product which turns the set of all such generators into the Lie algebra so(3). With respect to the dynamic field $\hat{\varphi}$, no simple Lie algebra is to be expected. Therefore, and in view of (2.4), it seems more natural to write the advection operator in terms of the metric and the elementary operator d, which commutes with $\mathcal{L}_{\check{v}}$.

3. Spherical tensors: Spherical harmonics and Rossby-Haurwitz modes

3.1. Spherical harmonics as stream functions for the planetary vorticity

The field ω, being the velocity field of a solid body rotation, is divergence-free. Hence, by Poincaré's lemma, it can be represented in terms of a stream function ζ_ω; that is, there exists a scalar field ζ_ω, such that

$$\omega \;=\; *d\zeta_\omega \quad . \tag{3.1}$$

As is well known, as well as easily checked by direct calculation, a stream function for any field corresponding to a solid body rotation can be written as a linear superposition of the three spherical harmonics Y^1_{-1}, Y^1_0 and Y^1_1, i.e. the set of spherical harmonics Y^ℓ_m with $\ell \;=\; 1$. Hence, it will always be possible to represent ζ_ω for ω as such a linear combination of the spherical harmonics Y^ℓ_m with $\ell \;=\; 1$.

Since in two dimensions, with respect to any 2-form, $*^{-1} \;=\; *$, the *planetary vorticity* $*d\omega$ which occurs in equation (2.20), can be written

$$* \, d\omega \;=\; *^{-1} d * d\zeta_\omega \quad . \tag{3.2}$$

The operator on the right, $*^{-1}d * d$, can be conceived as a Laplacian operator: it can be interpreted as the divergence, $*^{-1}d*$, acting after a gradient d.

Note the sequence along which we arrived at this interpretation:

$$(*d)\,(*d) \;=\; (*^{-1}d)\,(*d) \;=\; *^{-1}d * d \;=\; (*^{-1}d\,*)\,d \quad . \tag{3.3}$$

The validity of this sequence is assured by the associativity of composition of operators, in this case the Hodge star $*$, its inverse $*^{-1}$ and the exterior derivative d, while our interpretation rests on the fact that the Laplacian,

$$\mathrm{lap} \;=\; *^{-1}\mathrm{d} * \mathrm{d} \quad , \tag{3.4}$$

and divergence,

$$\mathrm{div} \;=\; *^{-1}\mathrm{d}\, * \tag{3.5}$$

can be decomposed in terms of these more fundamental operators [7, e.g. §25], [8, §4.4].

Spherical harmonics are eigenfunctions of the Laplacian,

$$*^{-1}\, d * dY^\ell_m \;=\; -\,\ell\,(\ell+1)\,a_p^{-2}\,Y^\ell_m \quad , \tag{3.6}$$

in which a_p is the radius of the planet (a_p enters calculations in terms of any chosen coordinate system through the metric, g_{ij}, which for S is given by (2.5)). From expression (3.2) therefore it follows that the planetary vorticity $*d\omega$ is simply *proportional* to the stream function ζ_ω

$$* \, d\omega \; = \; -\frac{2}{a_p^2}\, \zeta_\omega \quad . \tag{3.7}$$

As a consequence and using (3.1), its gradient, which occurs in the vorticity balance (2.20), can be expressed as

$$d * d\omega \; = \; -\frac{2}{a_p^2}\, d\zeta_\omega \; = \; -\frac{2}{a_p^2}\; *^{-1} *d\zeta_\omega \; = \; -\frac{2}{a_p^2}\; *^{-1}\, \omega \quad . \tag{3.8}$$

Note that the precise representation of ω with respect to coordinate system S will depend on the orientation of S with respect to the planet, and so will the expansion of ζ_ω in terms of the spherical harmonics Y_m^1. In contrast to this, because these spherical harmonics all have the *same* eigenvalue, with respect to the Laplacian, expression (3.8) is independent of the *orientation* of S; in this sense expression (3.8) is coordinate-free. It only depends on the fact that ω is a generator of solid body rotation, so its stream function can be expanded in terms of the first order spherical harmonics. We express all these aspects in our notation, and yet keep it neat, by stating it in coordinate-free notation and in terms of simply the symbol ω instead of its expansion in terms of the Y_m^1. The eigenvalue $-2a_p^{-2}$ serves as a reminder about the underlying possibility of expansion in terms of spherical harmonics.

3.2. Divergence-Free flow and modes

The potential vorticity balance (2.20), by its divergence term $*^{-1}d * \hat{\varphi}$, singles out as special cases divergence-free fields, i.e. fields $\hat{\varphi}$ obeying $*^{-1}d * \hat{\varphi} \; = \; 0$. Any such divergence-free field $\hat{\varphi}$ can be expressed in terms of a stream function $\hat{\psi}$ as $\hat{\varphi} \; = \; *d\hat{\psi}$.

Analogous to (3.2), i.e. based on (3.3) and (3.4), $*d\hat{\varphi}$ can be interpreted as the Laplacian of $\hat{\psi}$. In view of (3.6), this suggests to involve in our reasoning the possibility of expansions of $\hat{\psi}$ in terms of spherical harmonics Y_m^ℓ:

$$\hat{\psi} \; = \; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{\psi}_\ell^m \, Y_m^\ell \quad . \tag{3.9}$$

We emphasize that the *coefficients* $\hat{\psi}_m^\ell$ of such an expansion would possibly depend on time, but they do not depend on spatial coordinates. There should be no confusion therefore with the usual Ricci notation of components of general vector or tensor fields, as $\hat{\psi}$ in *that* sense is a scalar field, so it doesn't *have* such components.

The sub-series of (3.9) for selected fixed values of ℓ,

$$\hat{\psi}_\ell \; = \; \sum_{m=-\ell}^{\ell} \hat{\psi}_\ell^m \, Y_m^\ell \quad , \tag{3.10}$$

are themselves still eigenfunctions of the Laplacian, with eigenvalues $-\,\ell\,(\ell+1)\,a_p^{-2}$. The subscript '$\ell$' to the symbol $\hat{\psi}_\ell$ may serve as a reminder of this.

3.3. Spherical tensors

Expansions (3.9) and (3.10) exist in any instance of S, regardless of its orientation with respect to the planet, as defined by matrix $M(\lambda, \theta, \phi)$, (2.2). Under rotation of S, according to variation of M, the coefficients $\hat{\psi}_\ell^m$ transform linearly. The matrices describing these linear transformations are known as Wigner D-matrices [27].

Let S' denote the standard geographical coordinate system, of which S is a rotated copy, with the rotation defined by the Euler angles λ, θ and φ, (2.1). That is, for $\lambda = \theta = \varphi = 0$, S' and S coincide. Denote the corresponding coordinate transform between S and S', in Ricci notation, by

$$s_{j'} = s_{j'}(\lambda, \theta, \varphi, s_i) \ . \tag{3.11}$$

Then any scalar field ϕ that would be represented with respect to S' as $\phi(s_{j'})$ can be represented with respect to S simply by substitution of relation (3.11); for the rotated version of ϕ, some authors write e.g. [19]

$$R\phi(s_i) = \phi(s_{j'}(\lambda, \theta, \varphi, s_i)) \ . \tag{3.12}$$

In the theory of general tensorfields, this is known as scalar transformation, or *transformation by invariance* [21]. Vice versa: a field is actually called a scalar field, in the sense of Ricci tensors, if and only if it does transform according to relation (3.12).

Spherical harmonics Y_m^ℓ are scalar fields in this sense, so they can be transformed by the rule (3.12). Apart from that however, they have the following remarkable property, which presents an alternative way of calculating the transformed function [19, Eqn. (4.28a)],

$$Y_m^\ell(s_{j'}(\lambda, \theta, \varphi, s_i)) = \sum_{n=-\ell}^{\ell} D_{mn}^{(\ell)}(\lambda, \theta, \varphi)\, Y_n^\ell(s_k) \ . \tag{3.13}$$

This relation furthermore defines the Wigner D-matrices $D_{nm}^{(\ell)}(\lambda, \theta, \varphi)$ the components of which are all known and explicitly documented in the literature [1, 19, 27, 28]. Objects that do transform in this way are known as *spherical tensors*, or (e.g. in quantum mechanics, where Ricci scalar fields act as operators) *irreducible tensor operators* [19].

It is important to realize that, the very fact that it is possible to give expansions (3.9) and (3.10), in exactly this form, with respect to any arbitrary instance of S, shows the coordinate-free character of the underlying object.

This supports the point of view that any scalar field $\hat{\psi}_\ell$ can be conceived as a $2\ell + 1$-dimensional vector, in a linear subspace for which the spherical harmonics that occur on the right hand side of (3.10) form a basis. The coefficients $\hat{\psi}_\ell^m$ are conceived as components of $\hat{\psi}_\ell$ with respect to this basis. A rotation of the coordinate system S then implies merely a linear transformation of basis; this transformation of basis is implemented by relation (3.13).

3.4. Ket notation; Rossby-Haurwitz modes

With respect to the vector fields associated with stream functions $\hat{\psi}_\ell$, to highlight their implied coordinate-free character and the fact that, consequently, they are characterized *merely* by the label ℓ, we introduce a ket notation

$$|\ell> = *d\hat{\psi}_\ell \ . \tag{3.14}$$

In general we shall denote by a ket such as $|\ell>$, a vector field that is an eigenfield of one or more linear operators. In this view, the field is characterized merely by its eigenvalue(s), or indices for these, ℓ in the case of expression (3.14).

We introduce the notation

$$\left| \begin{matrix} \ell \\ m \end{matrix} \right\rangle = *dY_m^\ell \tag{3.15}$$

for the indicated type of vectorfields that form the linear components of $|\ell>$ in the expansion of (3.14) – use (3.10) –

$$|\ell> = \sum_{m=-\ell}^{\ell} \hat{\psi}_\ell^m *dY_m^\ell = \sum_{m=-\ell}^{\ell} \hat{\psi}_\ell^m \left| \begin{matrix} \ell \\ m \end{matrix} \right\rangle \quad . \tag{3.16}$$

Note that the $\left| \begin{matrix} \ell \\ m \end{matrix} \right\rangle$ coincide with special cases of $|\ell>$. As we shall see in Section 6, they closely correspond to the Rossby-Haurwitz waves as commonly referred to in the GFD literature. Note that under rotation, they transform according to (3.13), so they form spherical tensors, too.

4. Representations of Lie algebras

4.1. Angular momentum operators

Direct calculation reveals that the commutators $[\,r_\lambda\,,\,r_\mu\,]$, defined by $[\,r_\lambda\,,\,r_\mu\,] = \mathcal{L}_{r_\lambda} r_\mu$, of the vector fields r_λ, expressions (2.6) to (2.8), obey commutation relations characteristic for the Lie algebra of rotations:

$$[\,r_\lambda\,,\,r_\mu\,] = -\epsilon_{\lambda\mu\nu}\, r_\nu \quad . \tag{4.1}$$

We now introduce the three differential operators

$$J_\mu = \mathcal{L}_{-ir_\mu} \quad , \quad \mu \in \{1,2,3\} \quad , \tag{4.2}$$

in which i is the imaginary unit ($i^2 = -1$).

The operators J_μ obey the commutation rules of the so(3) algebra:

$$[\,J_1\,,\,J_2\,] = iJ_3 \quad \text{(cyclic in 1, 2, 3)} \quad . \tag{4.3}$$

When the J_κ act on scalar fields, this result follows readily from the commutation relations (4.1) and a well-known property of the Lie derivative of scalar fields [7, e.g. Th.23.3.4]. When the J_κ act on vector fields, the relations (4.3) follow from again relations (4.1) and Jacobi's identity for commutators of vectorfields.

In quantum mechanics, the commutation relations (4.3) have been used to define angular momentum operators [19, §7]. Following the theory of angular momentum in quantum mechanics [19], in terms of the operators (4.2) we furthermore define the operator J^2 (read "J squared"),

$$J^2 = J_1J_1 + J_2J_2 + J_3J_3 \quad . \tag{4.4}$$

Note that with the Lie derivative, the operators J_μ and J^2 are defined for Ricci tensors of any rank.

As can be checked by straightforward calculation, the spherical harmonics Y^ℓ_m, which are scalar fields in the sense of Ricci tensors, are eigenfields * of both J^2 and J_3:

$$J^2\, Y^\ell_m \;=\; \ell\,(\ell+1)\, Y^\ell_m \;\;,\;\; J_3\, Y^\ell_m \;=\; m\, Y^\ell_m \;\;. \tag{4.5}$$

Because the fields r_μ are all generators of isometries, the operators J_μ all commute with the Hodge star operator $*$ and hence so does J^2. All of these operators also commute with the exterior derivative d, because the Lie derivative does so, fully in general. Hence the *vectorfields* (3.15) are eigenfields of these operators just the same:

$$J^2 \left| \begin{array}{c} \ell \\ m \end{array} \right\rangle = \ell\,(\ell+1) \left| \begin{array}{c} \ell \\ m \end{array} \right\rangle \;\;,\;\; J_3 \left| \begin{array}{c} \ell \\ m \end{array} \right\rangle = m \left| \begin{array}{c} \ell \\ m \end{array} \right\rangle \;\;. \tag{4.6}$$

From (4.3), bi-linearity of the commutator bracket [,], and skew symmetry of the commutator, it readily follows

$$[\, J^2\,,\, J_\mu\,] \;=\; 0 \;\;; \tag{4.7}$$

It is as a consequence of this that e.g. J^2 and J_3 have common eigenfields, (4.6).

These results are of course well-known in the theory of the group of rotations and its applications (such as angular momentum in quantum mechanics). Note however, that in the present context of the algebraic relations (4.3) and (4.7), the apparently *single* operator J_3 for example, being essentially a Lie derivative, may act on scalar fields *or* on vector fields, and its instance is rather different in both cases. In that sense, our operators J_μ denote coherent families of operators rather than individual operators; each rank of Ricci tensors then corresponds to a member of such a family. In this light, the commutation relations (4.3) and (4.7), which are the same at the two different rank levels, appear at a single higher level of abstraction.

We wish to emphasize that, whereas the commutator of vectorfields in (4.1) is defined in terms of mutual Lie derivatives of vector fields, the commutator of the differential operators in (4.3) is itself the operator that merely accounts for the difference that results from interchanged order of operation of operators K and L; that is, for any two operators K and L, by definition,

$$[\, K\,,\, L\,] \;=\; K\,L - L\,K \quad . \tag{4.8}$$

In this light, relations (4.3) show that any of the operators J_1, J_2 and J_3 can, in this sense, be viewed as a composition, involving both operator products and linear combination, of the other two and so we recognize a notion of a closed algebra of operators.

4.2. Irreducible representations

Operators J_μ, (4.2), act on the space of scalar fields or vector fields. In view of expansion (3.9), the space of scalar fields is conceived as a linear space spanned by the set of spherical harmonics Y^ℓ_m, i.e. for all values of ℓ and m. Expansion (3.16) shows a linear space of vectorfields spanned by the total of the fields (3.15).

A central concept in representation theory [4,19,27] then is the existence of non-trivial, irreducible, invariant, linear subspaces that are closed under action of all the operators that belong to the operator algebra.

* The eigenvalues of J^2 resemble those, but are not quite identical to, the eigenvalues of the Laplacian $*^{-1}d * d$; compare (3.6).

In case of the Lie algebra of rotations, so(3), for scalar fields such invariant linear subspaces happen to be spanned by the spherical harmonics Y^{ℓ}_{m}, for any fixed value of ℓ and $m = -\ell, \ell+1, \ldots 0, \ldots \ell-1, \ell$ [19].

This implies that the operators J_1, J_2 and J_3 on these subspaces can be represented by finite-dimensional matrices. The action of these operators on elements of the linear subspace is then simply implemented as matrix multiplication. This concept, i.e. a matrix representation of an algebra of operators on an irreducible linear subspace, is known as an *irreducible representation* of the Lie algebra. In the example just mentioned, the spherical harmonics Y^{ℓ}_{m}, for fixed ℓ, then form a basis for such an irreducible representation.

In the very same sense, the vector fields (3.15), for fixed value of ℓ, form a basis for irreducible representations of the operators J_1, J_2 and J_3, as acting on vector fields.

5. The vanishing of self-interaction as implied by spherical symmetry

5.1. The vanishing of self-interaction

For the quantity $|\ell >$, (3.14), (3.16), we may deduce, fully analogous to relation (3.7) and (3.8),

$$*d|\ell > = -\frac{\ell\,(\ell+1)}{a_p^2}\,\hat{\psi}_\ell \quad . \tag{5.1}$$

Hence

$$d * d|\ell > \overset{(5.1)}{=} -\frac{\ell(\ell+1)}{a_p^2} *^{-1} *d\hat{\psi}_\ell \overset{(3.14)}{=} -\frac{\ell(\ell+1)}{a_p^2} *^{-1} |\ell > \quad . \tag{5.2}$$

Because

$$\text{for 1-forms in two dimensions: } *^{-1} = -* \ , \tag{5.3}$$

and because for *any* 1-form α, the expression $\alpha \cdot (*\alpha)$ vanishes identically, relation (5.2) implies that the non-linear self-interaction $\hat{\varphi} \cdot d * d\hat{\varphi}$ in (2.20) vanishes identically for $\hat{\varphi} = |\ell >$:

$$|\ell > \cdot\, d{*}d|\ell > \ \sim \ |\ell > \cdot\, *|\ell > \ = \ 0 \ , \tag{5.4}$$

where $\sim$ denotes linear proportionality.

As has become clear from its derivation, result (5.4) is independent of orientation of coordinate system S with respect to the planet. It is a direct consequence of the algebraic properties of the operators $*$, the exterior derivative d, and the local inner product $\cdot$ and of relation (5.1).

5.2. Spherical symmetry

Relation (5.1) and hence (5.2) and (5.4), are a consequence of spherical symmetry, as follows. With (3.14) and (5.3), expression (5.1) is equivalent to an eigenvalue expression for the composite operator $*^{-1}d * d$, which in this context maps scalarfields to scalarfields. Now, the exterior derivative d commutes with the Lie derivative, and the Hodge star commutes with all Lie derivatives along Killing fields (i.e. along generators of isometries). In this case, these are vectors fields that generate rotations. Hence the composite operator $*^{-1}d * d$ commutes with $\mathcal{L}_r$, for all $r \in$ so(3), and therefore with all operators J_μ, (4.2). Schur's Lemma [4] therefore implies that, with respect to bases of irreducible representations of so(3), the operator $*^{-1}d * d$ is represented by a scalar multiple of the identity matrix; with (3.14) and (5.3), this latter statement corresponds to the form of expression (5.1).

In this sense, given the structure of the non-linear interaction operator, the vanishing of the self-interaction of Rossby-Haurwitz modes is a consequence of the underlying spherical symmetry.

Note on the use of complex spherical harmonics. Schur's lemma refers to bases of irreducible representations, of a Lie algebra in our case. It is here that it is essential to use complex spherical harmonics: the operators (4.2) involve the imaginary unit i, and therefore the real and imaginary parts of the spherical harmonics, or our ket vectorfields, each for themselves are not eigenfields of J_3; see also (4.5).

The Laplacian $*^{-1}d * d$ is a real operator however. Therefore, from the fact that the complex fields are eigenfields, it follows that the real and imaginary parts of these separately are also eigenfields of the Laplacian. Therefore, the results above are also valid for the real and imaginary parts of the Rossby-Haurwitz modes. Its explanation from the spherical symmetry rests on the complex eigenfunctions though.

6. Rossby-Haurwitz waves as separable solutions with respect to suitably moving spherical coordinates

6.1. Coordinate-Free dynamics of the mode axis

We shall now look for solutions $\hat{\varphi}$ of the potential vorticity equation (2.20) of type $\hat{\varphi} = |\ell>$, (3.14), that are furthermore separable with respect to space and time.

With reference to relation (3.8), the coupling term in (2.20) can be rewritten as

$$|\ell > \cdot\, d * d\omega \; = \; \frac{2}{a_p^2}\, \mathcal{L}_\omega\, \hat{\psi}_\ell \;\; ; \tag{6.1}$$

this relation readily follows, using (5.3) and with the help of the identity

$$*\omega \cdot *d\hat{\psi} \; = \; \mathcal{L}_\omega\, \hat{\psi} \;\; . \tag{6.2}$$

The proof of (6.2) in turn is a straightforward exercise in the calculus of differential forms and Ricci tensors.

Given relations (2.3), (5.4) and (6.1), for flow fields $\hat{\varphi}$ of type $|\ell>$ equation (2.20) reduces to

$$\frac{\partial}{\partial t} *d|\ell> \; - \mathcal{L}_{\check{v}} *d|\ell> + \frac{2}{a_p^2}\, \mathcal{L}_\omega\, \hat{\psi}_\ell \; = \; 0 \;\; . \tag{6.3}$$

Applying (5.1), we find

$$\frac{\partial}{\partial t} *d|\ell> \; - \mathcal{L}_{\check{v}} *d|\ell> \; - \frac{2}{\ell\,(\ell+1)}\, \mathcal{L}_\omega *d|\ell> \; = \; 0 \;\; , \tag{6.4}$$

or, using the fact that $\mathcal{L}_v$ is linear in v,

$$\ell\,(\ell\,+\,1)\, \frac{\partial}{\partial t} *d|\ell> \; = \; \mathcal{L}_{2\omega+\ell(\ell+1)\check{v}} *d|\ell> \;\; , \tag{6.5}$$

or, by (5.1), in terms of the coordinate free stream function (3.10),

$$\ell\,(\ell+1)\,\frac{\partial}{\partial t}\,\hat{\psi}_\ell \;=\; \mathcal{L}_{2\omega+\ell(\ell+1)\check{\nu}}\;\hat{\psi}_\ell\;\;. \tag{6.6}$$

Because both the partial derivative with respect to t and the Lie derivate along a generator or rotation commute with $*$ and d, we see from (3.14) that wave equation (6.6) also applies at the level of the corresponding vector fields:

$$\ell\,(\ell+1)\,\frac{\partial}{\partial t}\,|\ell> = \mathcal{L}_{2\omega+\ell(\ell+1)\check{\nu}}\;|\ell>\;\;; \tag{6.7}$$

this appearance of essentially the same wave equation, and in the same form, (6.6) or (6.7), at two different tensor rank levels (scalar or vectorial), is a similar phenomenon as the one we briefly discussed related to relation (4.7).

Equation (6.6) is separable, so that,

$$\hat{\psi}_\ell(s,t) \;=\; \exp(i\sigma t)\,\bar{\hat{\psi}}_\ell(s), \tag{6.8}$$

if

$$\frac{\partial}{\partial t}(2\,\omega \;+\; \ell\,(\ell+1)\,\check{\nu}) \;=\; 0\;\;, \tag{6.9}$$

that is, if the field

$$\alpha \;=\; 2\,\omega \;+\; \ell\,(\ell+1)\,\check{\nu} \tag{6.10}$$

or, with (2.11) and (2.16),

$$\alpha \;=\; \left(2\,\Omega \;+\; \ell(\ell\;+\;1)\,\dot{\lambda}\right)\,\omega_3 \;+\; \ell(\ell\;+\;1)\,\left(-\,\dot{\theta}\,\varpi_2 \;+\; \dot{\phi}\,r_3\right)\;\;, \tag{6.11}$$

is stationary in the moving coordinate system S.

6.2. Rossby-Haurwitz modes as eigenstates of a generator of rotation

Equation (6.6) describes the dynamics of the field $\hat{\psi}_\ell$ within coordinate system S. Let u_α be a normalized counterpart of α, (6.10), with positive orientation, so that $\alpha \;=\; \bar{\alpha}\,u_\alpha$. Expression (6.8) and equation (6.6) may then be combined, to an eigenvalue problem for $\mathcal{L}_{u_\alpha}$:

$$\bar{\alpha}\,\mathcal{L}_{u_\alpha}\,\bar{\hat{\psi}}_\ell(s) \;=\; i\,\ell\,(\ell+1)\,\sigma\bar{\hat{\psi}}_\ell(s)\;\;, \tag{6.12}$$

or, fully analogous to definitions (4.2) after multiplication by $-i$

$$\bar{\alpha}\,J_{u_\alpha}\bar{\hat{\psi}}_\ell(s) \;=\; \ell\,(\ell+1)\,\sigma\bar{\hat{\psi}}_\ell(s)\;\;. \tag{6.13}$$

Non-singular, scalar eigenfunctions of operator J_{u_α} are spherical harmonics, oriented about the axis of field u_α, with integer eigenvalues $m \;=\; -\ell, -\ell{+}1\ldots, \ell$. For the associated temporal angular frequencies σ, as it would be observed in the moving coordinate system S, one finds from (6.13)

$$\sigma \;=\; \frac{\bar{\alpha}m}{\ell\,(\ell+1)}\;\;. \tag{6.14}$$

6.3. Simple solution of the equation of motion for the axis

To find the motion of the coordinate system S along the planet, such that S supports separable solutions (6.8), we exploit condition (6.9). To do so we expand expression (6.11) with respect to the standard basis $\{r_1, r_2, r_3\}$ for so(3), expressions (2.6) to (2.8). Since this basis $\{r_1, r_2, r_3\}$ as a whole is fixed in S, and since the base fields r_1, r_2 and r_3 themselves are all time-independent, condition (6.9) is equivalent to the requirement that the *coefficients* of this expansion of (6.11) with respect to the basis $\{r_1, r_2, r_3\}$ are time-independent. The expansion can be done with the help of expressions (2.14) to (2.16).

Setting the time derivatives of the coefficients equal to zero renders a set of three coupled non-linear ordinary differential equations that govern the required time dependence for the Euler angles. In their turn these determine the time dependent orientation of the coordinate system S. Solving these equations thus renders coordinate systems S that support separable solutions of type (6.8). In these systems, the field α, as given by (6.10) or (6.11), will be stationary.

A hint of what this will bring lies in a simple closed form solution of (6.9) that is easily seen from (6.11), from which it is clear that α will be stationary if we choose $\dot{\theta} = 0$, keep $\dot{\phi}$ constant and choose furthermore $\dot{\lambda}$ to be constant and equal to

$$\dot{\lambda} = -\frac{2\,\Omega}{\ell\,(\ell+1)} \quad . \tag{6.15}$$

The field α (6.11) with these choices reduces to

$$\alpha = \dot{\phi}\,\ell\,(\ell+1)\,r_3 \quad , \tag{6.16}$$

which for constant $\dot{\phi}$ is time-independent indeed, while

$$\bar{\alpha} = \dot{\phi}\,\ell\,(\ell+1) \quad . \tag{6.17}$$

Factoring the spherical harmonic in terms of its harmonic zonal component and an associated Legendre function, we find, apart from a scaling factor, as a time dependent solution for $\bar{\bar{\psi}}_\ell(s,t)$

$$\bar{\bar{\psi}}_\ell(s,t) = \exp(i(\sigma t + m\phi_\alpha))\,P_\ell^m(\sin(s_2)) \quad , \tag{6.18}$$

in which we have indicated the zonal angle, in S, about the axis of α, with ϕ_α.

As, according to expression (6.14), σ is proportional to m, expressions (6.17) and (6.18) show that the *phase velocity* of the solutions (6.18) is simply

$$-\frac{\sigma}{m} \overset{(6.14)}{=} -\frac{\bar{\alpha}\,m}{\ell\,(\ell\,+\,1)}\,\frac{1}{m} = -\frac{\bar{\alpha}}{\ell\,(\ell+1)} \overset{(6.17)}{=} -\dot{\phi} \quad ; \tag{6.19}$$

above the equality signs we have indicated which relations ensure that the equalities hold.

In words: relations (6.19) show that the phase velocity $-\sigma/m$ of the wave pattern, as observed in coordinate system S, is just equal to the negative value $-\dot{\phi}$ of the angular frequency $\dot{\phi}$ with which S rotates about its polar axis (i.e. the angular frequency $\dot{\phi}$ with which S moves along vector field r_3; see also (2.16) and recall that r_3 is fixed in S).

Furthermore, relations (6.19) show that in the limit of vanishing $\dot{\phi}$ the phase velocity of the waves vanishes altogether and the waves reduce to stationary patterns in S: the coordinate system S is then simply moving along with the wave. We thus recover the classical result e.g. [10, 15] of Rossby-Haurwitz modes precessing among the axis of the planet with angular frequency $\dot{\lambda}$ given by expression (6.15).

Numerical integration of the non-linear coupled system of equations (6.9), of which we shall not present details here, renders a whole family of possible coordinate systems S each of which performs a precession like motion about the axis of the field α. In these systems α is stationary, while $\check{\nu}$ and ω are not.

7. Conclusion and discussion

The first of the two central goals of the present manuscript, i.e. to interpret the vanishing of the non-linear self-interaction of Rossby-Haurwitz modes as being a consequence of underlying spherical symmetry was achieved and discussed in Section 5.

The second goal, i.e. to explain the insensitivity of the phase velocities of these modes to their orientation with respect to the axis of rotation of the planet, while their very existence depends on the planet's rotation, was essentially reached in Section 6, as follows.

With (6.15), we have recovered the classical expression for the angular velocity of Rossby-Haurwitz modes around the axis of the planet [10]. This result was derived for fields $\hat{\varphi}$ chosen as $\hat{\varphi} = |\ell >$; we recall that this means that during our derivation $\hat{\varphi}$ was restricted to belong to an irreducible representation of so(3), characterized by ℓ, but it was not more restricted than in this way. Hence the result derived applies to the basis elements (3.15). These are vectorial wave patterns (3.14) corresponding to standard spherical harmonics Y^{ℓ}_{m}, for all $m = -\ell, -\ell + 1, \dots \ell$, with respect to coordinate system S, so, by construction, arbitrarily oriented with respect to the planet's axis of rotation. A key intermediate result in the derivation was relation (5.1). In subsection 5.2 we discussed how relation (5.1) can be perceived as a consequence of spherical symmetry, through Schur's Lemma. In the same sense then, the fact that the phase velocity (6.15) of the Rossby-Haurwitz modes does not depend on their orientation with respect to the rotation axis of the planet can also be perceived as a consequence of underlying spherical symmetry.

Combined with the transformation property (3.13) of the spherical harmonics, as spherical tensors, the above findings furthermore partially explain the well-known selection rule [18, e.g.(2.12g)], which states that Rossby-Haurwitz modes characterized by the same value of ℓ do not interact, not even by non-linear advection. Indeed, as we have seen, Rossby-Haurwitz modes, as associated with a single spherical harmonic, can have any amplitude and orientation with respect to the planet's axis. According to transformation property (3.13) however, from a rotated coordinate system, the same state of the fluid would be described as a linear superposition of a set of Rossby-Haurwitz modes, with the same value of ℓ. The superimposed modes, which could also exist and persist as individual solutions, with any amplitude, all have the same phase velocity and steadily coexist with each other in the superimposed configuration, apparently in plain linear superposition and without interaction. Longuet-Higgins [14] emphasized the implications of (3.13) in the context of the linearized wave equations, in which the possibility of superposition itself is immediate from linearity. As we have shown, the fact that the non-linear wave equation, to some extent supports linear superposition is another implication of the underlying spherical symmetry of the system.

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Conflict of interest

The author declares that there is no conflicts of interest in this paper.

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