



Research article

Fractional input stability for electrical circuits described by the Riemann-Liouville and the Caputo fractional derivatives

Ndolane Sene*

Laboratoire Lmdan, Département de Mathématiques de la Décision, Université Cheikh Anta Diop de Dakar, Faculté des Sciences Economiques et Gestion, BP 5683 Dakar Fann, Senegal

* **Correspondence:** Email: ndolane.sene@ucad.edu.sn; Tel: +221779587905.

Abstract: The fractional input stability of the electrical circuit equations described by the fractional derivative operators has been investigated. The Riemann-Liouville and the Caputo fractional derivative operators have been used. The analytical solutions of the electrical circuit equations have been developed. The Laplace transforms of the Riemann-Liouville, and the Caputo fractional derivative operators have been used. The graphical representations of the analytical solutions of the electrical circuit equations have been presented. The converging-input converging-state property of the electrical RL, RC and LC circuit equations described by the Caputo fractional derivative, and the global asymptotic stability property of the unforced electrical circuit equations have been illustrated.

Keywords: fractional derivatives; electrical circuits; fractional differential equations

Mathematics Subject Classification: 93D05, 26A33, 93D25

1. Introduction

Recently, the electrical circuit equations as Resistor-Capacitor (RC) and Resistor-Inductor (RL) have attracted many mathematicians [3, 15]. The analytical solutions of the electrical circuit equations described by a singular and a no-singular fractional derivative operators have been proposed. In [6], Aguilar et al. have introduced the electrical circuit equations considering the Caputo fractional derivative. They have proposed the analytical solutions of the electrical RC and RL equations described by the Caputo fractional derivative. In [6], the authors have proposed the graphical representations to illustrate the main results. The Caputo fractional derivative is a fractional derivative with a regular kernel. Recently, the fractional derivative operators with no-singular kernels were introduced in the literature. The Caputo-Fabrizio fractional derivative and the Atangana-Baleanu fractional derivative. In [5], Aguilar et al. have introduced electrical equations considering the fractional derivative operators with two parameters. They have proposed the solutions of the electrical

RL and RC circuit equations described by the fractional derivative operators with two parameters $\alpha, \beta \in (0, 1]$. In [5], the authors have introduced electrical circuit equations using the Atangana-Baleanu fractional derivative. They have proposed the analytical solutions using the Laplace transform. They have presented the numerical scheme of the Atangana-Baleanu fractional derivative of getting the analytical solutions of the electrical RC and RL circuit equations. In [4], Aguilar et al. have presented the electrical circuit equations using the Caputo-Fabrizio fractional derivative. They have proposed the graphical representations of the analytical solutions. In [8], Aguilar et al. have investigated of getting the analytical solution of the RLC circuit equation described by the Caputo fractional derivative. In [7], Aguilar et al. have analyzed and proved the jerk dynamics can be obtained with the electrical circuit equations. There exist many other works related to the analytical solutions, seen in [11, 20]. The stability problem interests many mathematicians. The stability problem studies the behaviors of the analytical solutions of the fractional differential equations. The analytical solution of the electrical circuit described by the singular and the no-singular fractional derivative operators is an open problem in fractional calculus and continues to receive many investigations. The analytical solutions of the electrical circuits exist. We can get them. The problem is how to analyze the behaviors of these solutions using stability notions. In [14], the authors have analyzed the stability of the electrical RLC circuit equation. The electrical RL and RC circuit equations contain source terms. The source terms have some properties. The source term of the electrical circuit equations can be constant [10]. The source term can be sinusoidal [5]. They exist some source terms which converge in the times [10]. In stability problems, there exists a novel notion of studying the behaviors the fractional differential equations with these characteristics. This notion is called the fractional input stability recently introduced in the literature by Sene in [18, 19]. In this paper, the function containing the source term will be considered as the exogenous input. The contribution of our paper is to recall the analytical solutions of the electrical RL and RC circuit equations described by the Riemann-Liouville and the Caputo fractional derivative using the Laplace transform. Secondly, we will analyze the behaviors of the obtained analytical solutions of the electrical RL and RC circuit equations using the fractional input stability. In other words, we investigate the fractional input stability of the electrical RL and RC circuit equations described by the Riemann-Liouville and the Caputo fractional derivative. The fractional input stability resumes three properties. Firstly, when the considered exogenous input converges, then the analytical solution of the considered electrical circuit converges as well. In other words, a converging input generates a converging state. That is the CICS property of the fractional input stability. Secondly, when the exogenous input is bounded, then the analytical solution of the considered electrical circuit is bounded as well. In other words, a bounded input generates a bounded state. That is the BIBS property of the fractional input stability. And at last, when the the exogenous input is null, then the trivial solution of the unforced electrical circuit is globally asymptotically stable. A fractional differential equation is said to be fractional input stable when the norm of the analytical solution at all times is bounded by a function proportional to the initial state and converging to zero in times plus a positive function depending to the norm of the exogenous input.

In stability problems in fractional calculus, the asymptotic stability is provided in many manuscripts. Using the fractional input stability, we can study the uniform global asymptotic stability of the fractional differential equations. Presently, the stability conditions for the asymptotic stability were provided in the literature. This paper offers an alternative to study the (uniform) global

asymptotic stability of the fractional differential equations. It is the main contribution of this paper. The application of the fractional input stability of the fractional differential equations is the second contribution of this paper.

The paper is described as the following form. In Section 2, we recall preliminary definitions of the fractional derivative operator and recall the definitions of the stability notions. In Section 3, we investigate the analytical solutions of the electrical RL, RC and LC circuit equations described by the Riemann-Liouville fractional derivative and the Caputo fractional derivative. We analyze the fractional input stability of the obtained analytical solutions. In Section 4, we give the graphical representations of the fractional input stability of the electrical RC and RL circuit equations described by the Caputo fractional derivative operator. In Section 5, we give the conclusions and the remarks.

Notation: \mathcal{PD} denotes the set of all continuous functions $\chi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\chi(0) = 0$ and $\chi(s) > 0$ for all $s > 0$. A class \mathcal{K} function is an increasing \mathcal{PD} function. The class \mathcal{K}_{∞} denotes the set of all unbounded \mathcal{K} function. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be class \mathcal{KL} if $\beta(\cdot, t) \in \mathcal{K}$ for any $t \geq 0$ and $\beta(s, \cdot)$ is non increasing and tends to zero as its arguments tends to infinity. Let $x \in \mathbb{R}^n$, $\|x\|$ stands for its Euclidean norm: $\|x\| := \sqrt{x_1^2 + \dots + x_n^2}$.

2. Preliminary definitions

In this section, we recall the definitions of the fractional derivative operators and the definition of the fractional input stability introduced in the literature [18, 19] of fractional calculus. We recall the definition of the Caputo fractional derivative.

Definition 1. [9, 16, 17] Consider a function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Caputo fractional derivative of the function f of order α is formulated as the following form

$$D_{\alpha}^c f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds \quad (2.1)$$

for all $t > 0$, we consider $\alpha \in (0, 1)$ and $\Gamma(\cdot)$ is the gamma function.

The Riemann-Liouville fractional derivative is recalled in the following definition.

Definition 2. [9, 13] Consider a function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Riemann-Liouville fractional derivative of the function f of order α is formulated as the following form

$$D_{\alpha}^{RL} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\alpha}} ds \quad (2.2)$$

for all $t > 0$, we consider $\alpha \in (0, 1)$ and $\Gamma(\cdot)$ is the gamma function.

The Riemann-Liouville fractional integral is recalled in the following definition.

Definition 3. [12, 18, 19] Consider a function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Riemann-Liouville fractional integral of the function f of order α is represented as the following form

$$I_{\alpha}^{RL} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad (2.3)$$

for all $t > 0$, we consider $\alpha \in (0, 1)$ and $\Gamma(\cdot)$ is the gamma function.

Definition 4. [1, 2] The MittagLeffler function with two parameters is expressed as the following form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (2.4)$$

where $\alpha > 0, \beta \in \mathbb{R}$ and $z \in \mathbb{C}$.

We consecrate this section of recalling some stability notions introduced in [18, 19]. In general, the fractional differential equation under consideration is defined by

$$D_{\alpha}^c x = f(t, x, u) \quad (2.5)$$

where $x \in \mathbb{R}^n$, a continuous and locally Lipschitz function $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represents the exogenous input. Note that the solution of the fractional differential equation (2.5) exists under the assumption "the function f is continuous and locally Lipschitz". The existence of the solution is fundamental in the stability problems. It is not necessary to study the stability of the fractional differential equation when we are not sure its the solution exists.

Definition 5. [18, 19] The fractional differential equation (2.5) is fractional input stable if for any exogenous input $u \in \mathbb{R}^m$, there exists a function $\beta \in \mathcal{KL}$, a function $\gamma \in \mathcal{K}_{\infty}$ such that for any initial state x_0 the solution of (2.5) satisfies

$$\|x(t, x_0, u)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_{\infty}). \quad (2.6)$$

The function γ is a class \mathcal{K}_{∞} function and the function β is a class \mathcal{KL} . Thus, from equation (2.6), when the exogenous input u of the fractional differential equation (2.5) is bounded it follows its solution is bounded as well, that is the BIBS property. See the proof in the next section or in the original paper [18]. We can also notice, when the exogenous input of the fractional differential equation (2.5) converges, when t tends to infinity, then its solution converges as well, that is CICS property, see in [18] for more details. At last, we observe from equation (2.6) when the exogenous input into the fractional differential equation (2.5) is null; we recover the definition of the global asymptotic stability described in the following definition.

Definition 6. [16] The trivial solution of the unforced fractional differential equation defined by $D_{\alpha}^c x = f(t, x, 0)$ is said to be (uniform) globally asymptotically stable if there exist a function $\beta \in \mathcal{KL}$, such that for any initial condition x_0 the solution of $D_{\alpha}^c x = f(t, x, 0)$ satisfies

$$\|x(t, x_0, 0)\| \leq \beta(\|x_0\|, t). \quad (2.7)$$

The last remark is the main contribution of the fractional input stability of the fractional differential equations. We finish this section by recalling the following lemma. It will be used to introduce comparison functions.

Lemma 1. [21] Let $\alpha \in (0, 2)$, β is a real number, a matrix $A \in \mathbb{C}^{n \times n}$, μ is such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$ and $C > 0$ is a real, then it holds that

$$\|E_{\alpha,\beta}(A)\| \leq \frac{C}{1 + \|A\|} \quad (2.8)$$

where $\mu \leq \arg(\lambda(A)) \leq \pi$, $\lambda(A)$ represents the eigenvalues of the matrix A .

3. Fractional input stability of the electrical circuits equations described by certain fractional derivative operators

In this section, we address the fractional input stability of the electrical circuit equations described by the Riemann-Liouville and the Caputo fractional derivative operators. The electrical circuit equations under consideration are the electrical RL circuit equation, the electrical LC circuit equation, and the electrical RC circuit equation. We analyze the bounded input bounded state (BIBS), the converging input converging state (CICS) and the (uniform) global asymptotic stability of the unforced fractional electrical circuit equations, properties derived to the fractional input stability of the electrical circuit equations. It is important to note, in the rest of the paper, the second term of all the fractional electrical circuit equations is considered as a single function which represents the exogenous input.

3.1. Fractional input stability of the electrical RL circuit equation described by the Riemann-Liouville fractional derivative

In this section, we address the fractional input stability of the electrical RL circuit described by the Riemann-Liouville fractional derivative. We consider the electrical RL circuit represented by the following fractional differential equation

$$D_{RL}^\alpha I(t) + \frac{\sigma^{1-\alpha} R}{L} I(t) = u(t) \quad (3.1)$$

with the initial boundary condition defined by $(I^{1-\alpha} I)(a) = I_0$. I represents the current across the inductor. The parameter σ is associated with the temporal components in the system, see more details in [5, 6]. R represents the resistance, and L represents the inductance. In the second member, $u(t)$ is considered as the exogenous input. It contains the source term $E(t)$ (eventually depend on time), the function $1/L$ and the temporal coefficient $\sigma^{1-\alpha}$. Summarizing $u(t) = \frac{\sigma^{1-\alpha} E(t)}{L}$.

Theorem 1. *The electrical RL circuit defined by equation (3.1) described by the Riemann-Liouville fractional derivative is fractional input stable.*

Proof: We first determine the analytical solution of the fractional differential equation defined by equation (3.1). Applying the Laplace transform to both sides of equation (3.1), we obtain the following relationships

$$\begin{aligned} s^\alpha \mathcal{L}(I(t)) - (I^{1-\alpha} I)(0) + \frac{\sigma^{1-\alpha} R}{L} \mathcal{L}(I(t)) &= \mathcal{L}(u(t)) \\ s^\alpha \mathcal{L}(I(t)) - I_0 + \frac{\sigma^{1-\alpha} R}{L} \mathcal{L}(I(t)) &= \mathcal{L}(u(t)) \\ \mathcal{L}(I(t)) \left\{ s^\alpha + \frac{\sigma^{1-\alpha} R}{L} \right\} &= I_0 + \mathcal{L}(u(t)) \\ \mathcal{L}(I(t)) &= \frac{I_0}{s^\alpha + \frac{\sigma^{1-\alpha} R}{L}} + \frac{\mathcal{L}(u(t))}{s^\alpha + \frac{\sigma^{1-\alpha} R}{L}} \\ \mathcal{L}(I(t)) &= I_0 \mathcal{L} \left(t^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha \right) \right) + \mathcal{L} \left(t^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha \right) \right) \mathcal{L}(u(t)) \end{aligned}$$

$$\mathcal{L}(I(t)) = I_0 \mathcal{L}\left(t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha\right)\right) + \mathcal{L}\left(t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha\right) * u(t)\right) \quad (3.2)$$

where \mathcal{L} represents the usual Laplace transform and "*" represents the usual convolution product. Applying the inverse of Laplace transform to both sides of equation (3.2), we obtain the following analytical solution

$$I(t) = I_0 t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha\right) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha} R}{L} (t-s)^\alpha\right) u(s) ds. \quad (3.3)$$

From which it follows by applying the euclidean norm

$$\|I(t)\| \leq \left\| I_0 t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha\right) \right\| + \|u\| \int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha} R}{L} (t-s)^\alpha\right) \right\| ds. \quad (3.4)$$

We know the term $-\frac{\sigma^{1-\alpha} R}{L} \leq 0$, then there exists a positive constant M such that we have the following inequality [17]

$$\int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha} R}{L} (t-s)^\alpha\right) \right\| ds \leq M. \quad (3.5)$$

Using Lemma 1 and inequality (3.5), we obtain the following inequality

$$\|I(t)\| \leq \frac{I_0 C_1}{\|t^{1-\alpha}\| + \frac{\sigma^{1-\alpha} R}{L} \|t\|} + \|u\| M. \quad (3.6)$$

Let the function $\beta(I_0, t) = \frac{I_0 C_1}{\|t^{1-\alpha}\| + \frac{\sigma^{1-\alpha} R}{L} \|t\|}$. We observe the function $\beta(., t)$ is a class \mathcal{K} function. Furthermore, $\beta(s, .)$ decays with the time and converges to zero when t tends to infinity, thus $\beta(s, .)$ is a class \mathcal{L} function. In conclusion the function β is a class \mathcal{KL} function. Let the function $\gamma(\|u\|) = \|u\| M$, it is straightforward to see γ is a class \mathcal{K}_∞ function. Finally, equation (3.6) is represented as follows

$$\|I(t)\| \leq \beta(I_0, t) + \gamma(\|u\|). \quad (3.7)$$

It follows from equation (3.7) the electrical RL circuit described by the Riemann-Liouville fractional derivative is fractional input stable.

Let the exogenous input $\|u\| = 0$ then $\gamma(0) = 0$ (because γ is a class \mathcal{K}_∞ function), thus equation (3.7) becomes

$$\|I(t)\| \leq \beta(I_0, t). \quad (3.8)$$

Then the equilibrium point $I = 0$ of the electrical RL circuit defined by

$$D_{RL}^\alpha I(t) + \frac{\sigma^{1-\alpha} R}{L} I(t) = 0 \quad (3.9)$$

is (uniformly) globally asymptotically stable. Then, the fractional input stability of the electrical RL circuit equation (3.1) implies the global asymptotic stability of the unforced electrical RL circuit equation (3.9). Proving the global asymptotic stability of the equilibrium point $I = 0$ of the unforced electrical RL circuit equation (3.9) using the fractional input stability is essential. The classical tools give the asymptotic stability of the equilibrium point $I = 0$. Its don't give the global asymptotic

stability. Thus, the fractional input stability is a good compromise in the stability problems. We observe when the exogenous input of equation (3.1) converges; then from equation (3.7), we notice the analytical solution of the electrical RL circuit equation (3.1) converges as well. That is the CICS property. Furthermore, from equation when the exogenous input equation (3.1) is bounded, then the analytical solution of the electrical RL circuit equation (3.1) is bounded as well. We give the proof. Let the exogenous input $\|u\| \leq \eta$ then from the fact γ is a class \mathcal{K}_∞ , there exists ϵ such that $\gamma(\|u\|) \leq \epsilon$. Then equation (3.7) becomes

$$\|I(t)\| \leq \beta(I_0, t) + \epsilon \leq \beta(I_0, 0) + \epsilon. \quad (3.10)$$

Thus the analytical solution is bounded as well.

3.2. Fractional input stability of the electrical RL circuit described by the Caputo fractional derivative

In this section, we replace the Riemann-Liouville fractional derivative by the Caputo fractional derivative. We address the fractional input stability of the electrical RL circuit described by the Caputo fractional derivative. The following fractional differential equation defines the electrical RL circuit equation described by the Caputo fractional derivative

$$D_c^\alpha I(t) + \frac{\sigma^{1-\alpha} R}{L} I(t) = u(t) \quad (3.11)$$

with the initial boundary condition defined by $I(0) = I_0$. The term $u(t)$ represents the exogenous input as defined in the previous section.

Theorem 2. *The electrical RL circuit defined by equation (3.11) described by the Caputo fractional derivative is fractional input stable.*

Proof: We apply the Laplace transform to both sides of equation (3.11), we obtain the following relationships

$$\begin{aligned} s^\alpha \mathcal{L}(I(t)) - s^{\alpha-1} I(0) + \frac{\sigma^{1-\alpha} R}{L} \mathcal{L}(I(t)) &= \mathcal{L}(u(t)) \\ s^\alpha \mathcal{L}(I(t)) - s^{\alpha-1} I_0 + \frac{\sigma^{1-\alpha} R}{L} \mathcal{L}(I(t)) &= \mathcal{L}(u(t)) \\ \mathcal{L}(I(t)) \left\{ s^\alpha + \frac{\sigma^{1-\alpha} R}{L} \right\} &= s^{\alpha-1} I_0 + \mathcal{L}(u(t)) \\ \mathcal{L}(I(t)) &= \frac{s^{\alpha-1} I_0}{s^\alpha + \frac{\sigma^{1-\alpha} R}{L}} + \frac{\mathcal{L}(u(t))}{s^\alpha + \frac{\sigma^{1-\alpha} R}{L}} \\ \mathcal{L}(I(t)) &= I_0 \mathcal{L} \left(E_\alpha \left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha \right) \right) + \mathcal{L} \left(t^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha \right) \right) \mathcal{L}(u(t)) \\ \mathcal{L}(I(t)) &= I_0 \mathcal{L} \left(E_\alpha \left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha \right) \right) + \mathcal{L} \left(t^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha \right) * u(t) \right) \end{aligned} \quad (3.12)$$

where \mathcal{L} represents the usual Laplace transform and "*" represents the usual convolution product. We apply the inverse of Laplace transform to both sides of equation ((3.12)). We obtain the following

analytical solution

$$I(t) = I_0 E_\alpha \left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha \right) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha} R}{L} (t-s)^\alpha \right) u(s) ds. \quad (3.13)$$

We apply the usual euclidean norm to equation ((3.13). It follows the following inequality

$$\|I(t)\| \leq \left\| I_0 E_\alpha \left(-\frac{\sigma^{1-\alpha} R}{L} t^\alpha \right) \right\| + \|u\| \int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha} R}{L} (t-s)^\alpha \right) \right\| ds \quad (3.14)$$

The characteristic term $-\frac{\sigma^{1-\alpha} R}{L} \leq 0$, then there exists a positive constant M such that the following inequality is held

$$\int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha} R}{L} (t-s)^\alpha \right) \right\| ds \leq M \quad (3.15)$$

Using Lemma 1 and the inequality defined by equation (3.15), we obtain the following inequality

$$\|I(t)\| \leq \frac{I_0 C_1}{1 + \frac{\sigma^{1-\alpha} R}{L} \|t^\alpha\|} + \|u\| M \quad (3.16)$$

Let the function $\beta(I_0, t) = \frac{I_0 C_1}{1 + \frac{\sigma^{1-\alpha} R}{L} \|t^\alpha\|}$. We observe the function $\beta(., t)$ is a class \mathcal{K} function. Furthermore, $\beta(s, .)$ decays in time and converges to zero when t tends to infinity, thus the function $\beta(s, .)$ is a class \mathcal{L} function. In conclusion the function β is a class \mathcal{KL} function. Let the function $\gamma(\|u\|) = \|u\| M$. The function γ is a class \mathcal{K}_∞ function. Finally, equation (3.16) is represented as the following form

$$\|I(t)\| \leq \beta(I_0, t) + \gamma(\|u\|) \quad (3.17)$$

It follows from equation (3.17) the electrical RL circuit equation described by the Caputo fractional derivative is fractional input stable.

Let the exogenous input $u = 0$. From the fact γ is a class \mathcal{K}_∞ function, we have $\gamma(0) = 0$. Equation (3.17) is represented as the following form

$$\|I(t)\| \leq \beta(I_0, t) \quad (3.18)$$

From equation (3.18), the equilibrium point $I = 0$ of the electrical RL circuit equation described by the Caputo fractional derivative defined by

$$D_c^\alpha I(t) + \frac{\sigma^{1-\alpha} R}{L} I(t) = 0 \quad (3.19)$$

is globally asymptotically stable. Then, the fractional input stability of equation (3.11) implies the global asymptotic stability of the unforced electrical RL circuit equation (3.19). We observe when the exogenous input of equation (3.11) converges, then from equation (3.17), we notice the analytical solution of the electrical RL circuit equation (3.11) converges as well. From equation (3.17), the analytical solution of the electrical RL circuit equation (3.11) described by the Caputo fractional derivative is bounded as well when the exogenous input is bounded.

3.3. Fractional input stability of the electrical LC circuit described by the Riemann-Liouville fractional derivative

In this section, we study the fractional input stability of the electrical LC circuit described by the Riemann-Liouville fractional derivative. The electrical LC circuit equation under consideration is defined by the following fractional differential equation

$$D_{RL}^{2\alpha} I(t) + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} I(t) = u(t) \quad (3.20)$$

with the initial boundary condition defined by $(I^{1-2\alpha})(0) = I_0$ and $\alpha \leq 1/2$ (in this paper). I represents the current across the inductor. The parameter σ is associated with the temporal components in the system, see more details in [6]. In the second member, $u(t)$ represents the exogenous input. It is expressed using the source term $E(t)$ (eventually depend on time), the function C/\sqrt{LC} and the temporal coefficient $\sigma^{1-\alpha}$. Summarizing, we have $u(t) = \frac{\sigma^{1-\alpha}CE(t)}{\sqrt{LC}}$. We make the following theorem.

Theorem 3. *The electrical LC circuit equation defined by equation (3.20) described by the Riemann-Liouville fractional derivative is fractional input stable.*

Proof: Let $\beta = 2\alpha$. We apply the Laplace transform to both sides of equation (3.20), we obtain the following relationships

$$\begin{aligned} s^\beta \mathcal{L}(I(t)) - (I^{1-\beta}I)(0) + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \mathcal{L}(I(t)) &= \mathcal{L}(u(t)) \\ s^\beta \mathcal{L}(I(t)) - I_0 + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \mathcal{L}(I(t)) &= \mathcal{L}(u(t)) \\ \mathcal{L}(I(t)) \left\{ s^\beta + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \right\} &= I_0 + \mathcal{L}(u(t)) \\ \mathcal{L}(I(t)) &= \frac{I_0}{s^\beta + \frac{\sigma^{1-\alpha}}{\sqrt{LC}}} + \frac{\mathcal{L}(u(t))}{s^\beta + \frac{\sigma^{1-\alpha}}{\sqrt{LC}}} \\ \mathcal{L}(I(t)) &= I_0 \mathcal{L} \left(t^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) \right) + \mathcal{L} \left(t^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) \right) \mathcal{L}(u(t)) \\ \mathcal{L}(I(t)) &= I_0 \mathcal{L} \left(t^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) \right) + \mathcal{L} \left(t^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) * u(t) \right) \end{aligned} \quad (3.21)$$

where \mathcal{L} represents the usual Laplace transform and "*" represents the usual convolution product. We apply the inverse of Laplace transform to both sides of equation (3.21). We obtain the following analytical solution

$$I(t) = I_0 t^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} (t-s)^\beta \right) u(s) ds \quad (3.22)$$

We apply the euclidean norm to both sides of equation (3.22). We get the following relationship

$$\|I(t)\| \leq \left\| I_0 t^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) \right\| + \|u\| \int_0^t \left\| (t-s)^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} (t-s)^\beta \right) \right\| ds \quad (3.23)$$

From the assumption $-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} \leq 0$, there exists a positive constant M such that we have the following inequality.

$$\int_0^t \left\| (t-s)^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} (t-s)^\beta \right) \right\| ds \leq M \quad (3.24)$$

We use Lemma 1 and inequality (3.23), we obtain the following inequality

$$\|I(t)\| \leq \frac{I_0 C_1}{\|t^{1-\beta}\| + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \|t\|} + \|u\| M. \quad (3.25)$$

Let the function $\mu(I_0, t) = \frac{I_0 C_1}{\|t^{1-\beta}\| + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \|t\|}$. We observe the function μ is a class \mathcal{KL} function. Let the function $\gamma(\|u\|) = \|u\| M$, it is straightforward to verify the function γ is a class \mathcal{K}_∞ function. Finally, equation (3.25) is represented as follows

$$\|I(t)\| \leq \mu(I_0, t) + \gamma(\|u\|) \quad (3.26)$$

It follows from equation (3.26) the electrical LC circuit described by the Riemann-Liouville fractional derivative is fractional input stable.

Let the exogenous input $\|u\| = 0$. From the fact $\gamma(0) = 0$, equation (3.26) becomes

$$\|I(t)\| \leq \mu(I_0, t). \quad (3.27)$$

Thus the equilibrium point $I = 0$ of the electrical LC circuit defined by

$$D_{RL}^{2\alpha} I(t) + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} I(t) = 0 \quad (3.28)$$

is globally asymptotically stable. As in the previous sections, the fractional input stability of equation (3.20) implies the global asymptotic stability of the unforced electrical LC circuit equation (3.28). We observe when the exogenous input of equation (3.20) converges, it follows from equation (3.26), the analytical solution of the electrical LC circuit equation (3.20) converges as well. It follows from equation (3.26), the analytical solution of the electrical LC circuit equation (3.20) is bounded as well when its exogenous input of equation is bounded.

3.4. Fractional input stability of the electrical LC circuit described by the Caputo fractional derivative

We consider the Caputo fractional derivative in this section. We investigate the fractional input stability of the electrical LC circuit equation defined by the following fractional differential equation

$$D_c^{2\alpha} I(t) + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} I(t) = u(t) \quad (3.29)$$

with the initial boundary condition defined by $I(0) = I_0$ and $\alpha \leq 1/2$. The exogenous input is $u(t)$.

Theorem 4. *The electrical LC circuit equation defined by equation (3.29) described by the Caputo fractional derivative is fractional input stable.*

Proof: Let $\beta = 2\alpha$. We determine the analytical solution of the fractional differential equation defined by (3.29). We apply the Laplace transform to both sides of equation (3.29). We obtain the following relationships

$$\begin{aligned}
 s^\beta \mathcal{L}(I(t)) - s^{\beta-1} I(0) + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \mathcal{L}(I(t)) &= \mathcal{L}(u(t)) \\
 s^\beta \mathcal{L}(I(t)) - s^{\beta-1} I_0 + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \mathcal{L}(I(t)) &= \mathcal{L}(u(t)) \\
 \mathcal{L}(I(t)) \left\{ s^\beta + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \right\} &= s^{\beta-1} I_0 + \mathcal{L}(u(t)) \\
 \mathcal{L}(I(t)) &= \frac{s^{\beta-1} I_0}{s^\beta + \frac{\sigma^{1-\alpha}}{\sqrt{LC}}} + \frac{\mathcal{L}(u(t))}{s^\beta + \frac{\sigma^{1-\alpha}}{\sqrt{LC}}} \\
 \mathcal{L}(I(t)) &= I_0 \mathcal{L} \left(E_\beta \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) \right) + \mathcal{L} \left(t^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) \right) \mathcal{L}(u(t)) \\
 \mathcal{L}(I(t)) &= I_0 \mathcal{L} \left(E_\beta \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) \right) + \mathcal{L} \left(t^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) * u(t) \right) \quad (3.30)
 \end{aligned}$$

where \mathcal{L} represents the usual Laplace transformation and "*" represents the usual convolution product. We apply the inverse of Laplace transform to both sides of equation (3.30). We obtain the following analytical solution

$$I(t) = I_0 E_\beta \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} (t-s)^\beta \right) u(s) ds \quad (3.31)$$

We apply the euclidean norm to both sides of equation (3.31). We obtain the following inequality

$$\|I(t)\| \leq \left\| I_0 E_\beta \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} t^\beta \right) \right\| + \|u\| \int_0^t \left\| (t-s)^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} (t-s)^\beta \right) \right\| ds \quad (3.32)$$

From the assumption $-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} \leq 0$, there exists a positive constant M such that the following inequality is held

$$\int_0^t \left\| (t-s)^{\beta-1} E_{\beta,\beta} \left(-\frac{\sigma^{1-\alpha}}{\sqrt{LC}} (t-s)^\beta \right) \right\| ds \leq M \quad (3.33)$$

We use Lemma 1 and equation (3.33), we obtain the following inequality

$$\|I(t)\| \leq \frac{I_0 C_1}{1 + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \|t^\beta\|} + \|u\| M \quad (3.34)$$

Let the function $\mu(I_0, t) = \frac{I_0 C_1}{1 + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} \|t^\beta\|}$. We observe the function $\mu(., t)$ is a class \mathcal{K} function. Furthermore, $\mu(s, .)$ decays in time and converges to zero when t tends to infinity. Thus $\mu(s, .)$ is a class \mathcal{L} function. In conclusion the function μ is a class \mathcal{KL} function. Let the function $\gamma(\|u\|) = \|u\| M$, it is straightforward to see γ is a class \mathcal{K}_∞ function. Finally equation (3.34) is represented as follows

$$\|I(t)\| \leq \mu(I_0, t) + \gamma(\|u\|) \quad (3.35)$$

It follows from equation (3.35) the electrical LC circuit equation (3.29) described by the Caputo fractional derivative is fractional input stable.

Let's analyze the BIBS, the CICS properties and the global asymptotic stability of the unforced electrical LC circuit equation. Let the exogenous input $\|u\| = 0$. From the fact γ is a class \mathcal{K}_∞ function, we have $\gamma(0) = 0$. Then the equation (3.35) becomes

$$\|I(t)\| \leq \mu(I_0, t) \quad (3.36)$$

That is the trivial solution of the electrical LC circuit equation described by the Caputo fractional derivative defined by

$$D_c^{2\alpha} I(t) + \frac{\sigma^{1-\alpha}}{\sqrt{LC}} I(t) = 0 \quad (3.37)$$

is globally asymptotically stable. That is to say the fractional input stability implies the global asymptotic stability of the unforced electrical RL circuit described by the Caputo fractional derivative. From equation (3.35), when the exogenous input of equation (3.29) converges then the analytical solution of the electrical LC circuit equation (3.29) converges as well. We notice from equation (3.35), when the exogenous input of equation (3.29) is bounded, then the analytical solution of the electrical LC circuit equation (3.29) described by the Caputo fractional derivative is bounded as well.

3.5. Fractional input stability of the electrical RC circuit equation described by the Riemann-Liouville fractional derivative

In this section, we investigate the fractional input stability of the electrical RC circuit equation described by the Riemann-Liouville fractional derivative. Let the electrical RC circuit equation defined by the following fractional differential equation

$$D_{RL}^\alpha V(t) + \frac{\sigma^{1-\alpha}}{RC} V(t) = u(t) \quad (3.38)$$

with the initial boundary condition defined by $(I^{1-\alpha}V)(0) = V_0$. C represents the capacitance and R represents the resistance. V represents the voltage across the capacitor. The parameter σ is associated to the temporal components in the system, see more details in [5]. The second member, $u(t)$ is considered as the exogenous input. It contains the source term $E(t)$ (eventually depend on time), the function $1/RC$ and the temporal coefficient $\sigma^{1-\alpha}$. Summarizing $u(t) = \frac{\sigma^{1-\alpha}E(t)}{RC}$. We make the following theorem.

Theorem 5. *The electrical RC circuit defined by equation (3.38) described by the Riemann-Liouville fractional derivative is fractional input stable.*

Proof: Applying the Laplace transform to both sides of equation (3.38), we get the following relationships

$$\begin{aligned} s^\alpha \mathcal{L}(V(t)) - (I^{1-\alpha}V)(0) + \frac{\sigma^{1-\alpha}}{RC} \mathcal{L}(V(t)) &= \mathcal{L}(u(t)) \\ s^\alpha \mathcal{L}(V(t)) - V_0 + \frac{\sigma^{1-\alpha}}{RC} \mathcal{L}(V(t)) &= \mathcal{L}(u(t)) \\ \mathcal{L}(V(t)) \left\{ s^\alpha + \frac{\sigma^{1-\alpha}}{RC} \right\} &= V_0 + \mathcal{L}(u(t)) \end{aligned}$$

$$\begin{aligned}\mathcal{L}(V(t)) &= \frac{V_0}{s^\alpha + \frac{\sigma^{1-\alpha}}{RC}} + \frac{\mathcal{L}(u(t))}{s^\alpha + \frac{\sigma^{1-\alpha}}{RC}} \\ \mathcal{L}(V(t)) &= I_0 \mathcal{L}\left(t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha\right)\right) + \mathcal{L}\left(t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha\right)\right) \mathcal{L}(u(t)) \\ \mathcal{L}(V(t)) &= I_0 \mathcal{L}\left(t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha\right)\right) + \mathcal{L}\left(t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha\right) * u(t)\right)\end{aligned}\quad (3.39)$$

where \mathcal{L} represents the usual Laplace transform and "*" represents the usual convolution product. Applying the inverse of Laplace transform to both sides of equation (3.39), we obtain the following analytical solution

$$V(t) = V_0 t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha\right) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha}}{RC} (t-s)^\alpha\right) u(s) ds \quad (3.40)$$

Applying the euclidean norm on equation (3.40), we have

$$\|V(t)\| \leq \left\| V_0 t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha\right) \right\| + \|u\| \int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha}}{RC} (t-s)^\alpha\right) \right\| ds \quad (3.41)$$

From the assumption $-\frac{\sigma^{1-\alpha}}{RC} \leq 0$, there exists a positive constant M such that we have the following inequality

$$\int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\sigma^{1-\alpha}}{RC} (t-s)^\alpha\right) \right\| ds \leq M \quad (3.42)$$

We use Lemma 1 and the inequality (3.42), we obtain the following inequality

$$\|V(t)\| \leq \frac{V_0 C_1}{\|t^{1-\alpha}\| + \frac{\sigma^{1-\alpha}}{RC} \|t\|} + \|u\| M \quad (3.43)$$

Let the function $\beta(V_0, t) = \frac{V_0 C_1}{\|t^{1-\alpha}\| + \frac{\sigma^{1-\alpha}}{RC} \|t\|}$. We observe the function β is a class \mathcal{KL} function. Let the function $\gamma(\|u\|) = \|u\| M$. It is straightforward to see γ is a class \mathcal{K}_∞ function. Finally, equation (3.43) is represented as the following form

$$\|V(t)\| \leq \beta(V_0, t) + \gamma(\|u\|) \quad (3.44)$$

It follows from equation (3.44) the electrical RC circuit equation described by the Riemann-Liouville fractional derivative is fractional input stable.

Let the exogenous input $\|u\| = 0$, then we have $\gamma(0) = 0$. Thus the equation (3.44) becomes

$$\|V(t)\| \leq \beta(I_0, t) \quad (3.45)$$

Then the equilibrium point $V = 0$ of the electrical RC circuit equation defined by

$$D_{RL}^\alpha V(t) + \frac{\sigma^{1-\alpha}}{RC} V(t) = 0 \quad (3.46)$$

is (uniformly) globally asymptotically stable. Thus the fractional input stability implies the global asymptotic stability of the unforced electrical RC circuit described by the Riemann-Liouville fractional derivative. We observe from (3.44) when the exogenous input of equation (3.38) converges, we notice the analytical solution of the electrical RC circuit equation (3.38) converges as well. From (3.44), when the exogenous input term of equation (3.38) is bounded, then the analytical solution of the electrical RC circuit equation (3.38) is bounded as well.

3.6. Fractional input stability of the electrical RC circuit equation described by the Caputo fractional derivative

In this section, we investigate the fractional input stability of the electrical RC circuit equation described by the Caputo fractional derivative defined by the following fractional differential equation

$$D_c^\alpha V(t) + \frac{\sigma^{1-\alpha}}{RC} V(t) = u(t) \quad (3.47)$$

with the initial boundary condition defined by $V(0) = V_0$. The term $u(t)$ represents the exogenous input as defined in the previous section.

Theorem 6. *The electrical RC circuit defined by equation (3.47) described by the Caputo fractional derivative is fractional input stable.*

Proof: Applying the Laplace transform to both sides of equation (3.47), we obtain the following relationships

$$\begin{aligned} s^\alpha \mathcal{L}(V(t)) - s^{\alpha-1} V(0) + \frac{\sigma^{1-\alpha}}{RC} \mathcal{L}(V(t)) &= \mathcal{L}(u(t)) \\ s^\alpha \mathcal{L}(V(t)) - s^{\alpha-1} V_0 + \frac{\sigma^{1-\alpha}}{RC} \mathcal{L}(V(t)) &= \mathcal{L}(u(t)) \\ \mathcal{L}(V(t)) \left\{ s^\alpha + \frac{\sigma^{1-\alpha}}{RC} \right\} &= s^{\alpha-1} V_0 + \mathcal{L}(u(t)) \\ \mathcal{L}(V(t)) &= \frac{s^{\alpha-1} V_0}{s^\alpha + \frac{\sigma^{1-\alpha}}{RC}} + \frac{\mathcal{L}(u(t))}{s^\alpha + \frac{\sigma^{1-\alpha}}{RC}} \\ \mathcal{L}(V(t)) &= V_0 \mathcal{L} \left(E_\alpha \left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha \right) \right) + \mathcal{L} \left(t^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha \right) \right) \mathcal{L}(u(t)) \\ \mathcal{L}(V(t)) &= V_0 \mathcal{L} \left(E_\alpha \left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha \right) \right) + \mathcal{L} \left(t^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha \right) * u(t) \right) \end{aligned} \quad (3.48)$$

where \mathcal{L} represents the usual Laplace transform and "*" represents the usual convolution product. Applying the inverse of Laplace transform to both sides of equation (3.48), we obtain the following analytical solution

$$V(t) = V_0 E_\alpha \left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha \right) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha}}{RC} (t-s)^\alpha \right) u(s) ds \quad (3.49)$$

From which we have the following inequality

$$\|V(t)\| \leq \left\| V_0 E_\alpha \left(-\frac{\sigma^{1-\alpha}}{RC} t^\alpha \right) \right\| + \|u\| \int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha}}{RC} (t-s)^\alpha \right) \right\| ds \quad (3.50)$$

From the assumption $-\frac{\sigma^{1-\alpha}}{RC} \leq 0$, there exists a positive constant M such that the following inequality is held

$$\int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\sigma^{1-\alpha}}{RC} (t-s)^\alpha \right) \right\| ds \leq M \quad (3.51)$$

We use Lemma 1 and equation (3.51), we obtain the following relationships

$$\|V(t)\| \leq \frac{V_0 C_1}{1 + \frac{\sigma^{1-\alpha}}{RC} \|t^\alpha\|} + \|u\| M \quad (3.52)$$

Let the function $\beta(V_0, t) = \frac{V_0 C_1}{1 + \frac{\sigma^{1-\alpha}}{RC} \|t^\alpha\|}$. We observe the function β is a class \mathcal{KL} function. Let the function $\gamma(\|u\|) = \|u\| M$. It is straightforward to see γ is a class \mathcal{K}_∞ function. Finally, equation (3.52) is represented as follows

$$\|V(t)\| \leq \beta(V_0, t) + \gamma(\|u\|) \quad (3.53)$$

From equation (3.53) the electrical RC circuit equation (3.47) described by the Caputo fractional derivative is fractional input stable.

Let the exogenous input $\|u\| = 0$. From the fact γ is a class \mathcal{K}_∞ function, we have $\gamma(0) = 0$. Then equation (3.53) becomes

$$\|V(t)\| \leq \beta(V_0, t) \quad (3.54)$$

We conclude the equilibrium point $V = 0$ of the electrical RC circuit equation described by the Caputo fractional derivative defined by

$$D_c^\alpha V(t) + \frac{\sigma^{1-\alpha}}{RC} V(t) = 0 \quad (3.55)$$

is globally asymptotically stable. We observe when the input term of equation (3.47) converges; then from equation (3.53), we notice the analytical solution of the electrical RC circuit equation (3.47) described by the Caputo fractional derivative converges as well. We notice when the exogenous input of equation (3.47) is bounded, then the analytical solution of the electrical RC circuit equation (3.47) described by the Caputo fractional derivative is bounded as well.

4. Numerical simulations

In this section, we analyze the CICS and the global asymptotic stability properties obtained with the fractional input stability. Let's the electrical RL circuit equation described by the Caputo fractional derivative defined by

$$D_c^\alpha I(t) + \frac{\sigma^{1-\alpha} R}{L} I(t) = u(t) \quad (4.1)$$

with numerical values: the resistance $R = 10\Omega$ and induction $L = 10H$. Let's the exogenous input $u(t) = 0$. The current across the inductor is depicted in Figure 1. We observe all the analytical solutions I decay everywhere except at the equilibrium point itself, thus the equilibrium point $I = 0$ of the electrical RL circuit equation (4.1) is (uniformly) globally asymptotically stable.

In Figure 2, we observe the behavior of the current in the inductor when the electrical RL circuit equation is fractional input stable.

Let's the electrical RC circuit equation described by the Caputo fractional derivative with numerical values defined by

$$D_c^\alpha V(t) + \frac{\sigma^{1-\alpha}}{RC} V(t) = u(t) \quad (4.2)$$

with the resistance $R = 10k\Omega$ and the capacitance $C = 1000\mu F$. Let's the exogenous input $u(t) = 0$, we can, observe all the analytical solutions V decay everywhere except at the equilibrium point itself,

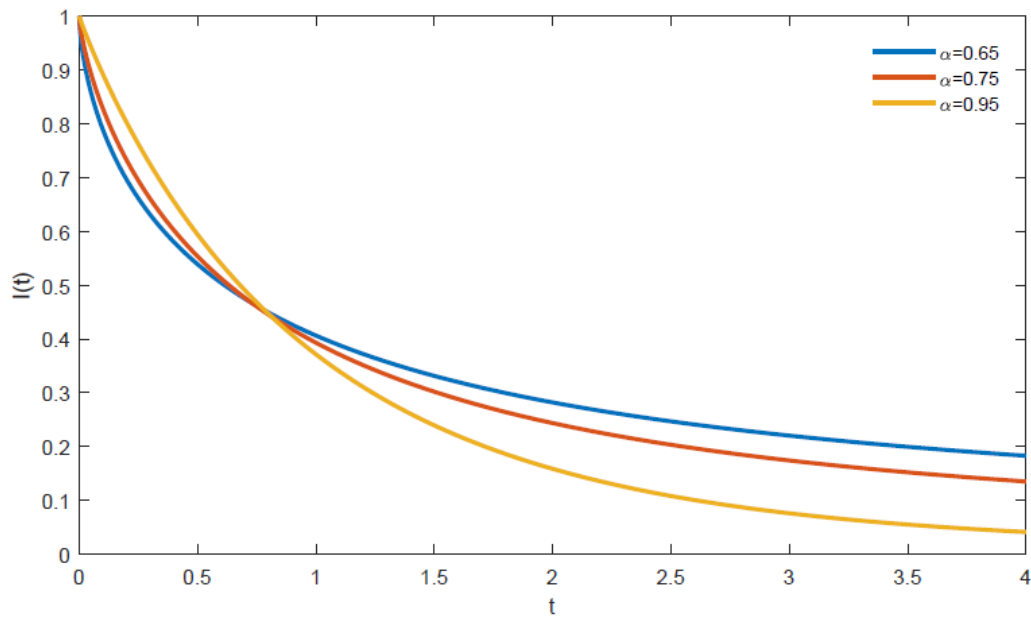


Figure 1. $I = 0$ the electrical RL circuit equation (4.1) is globally asymptotically stable.

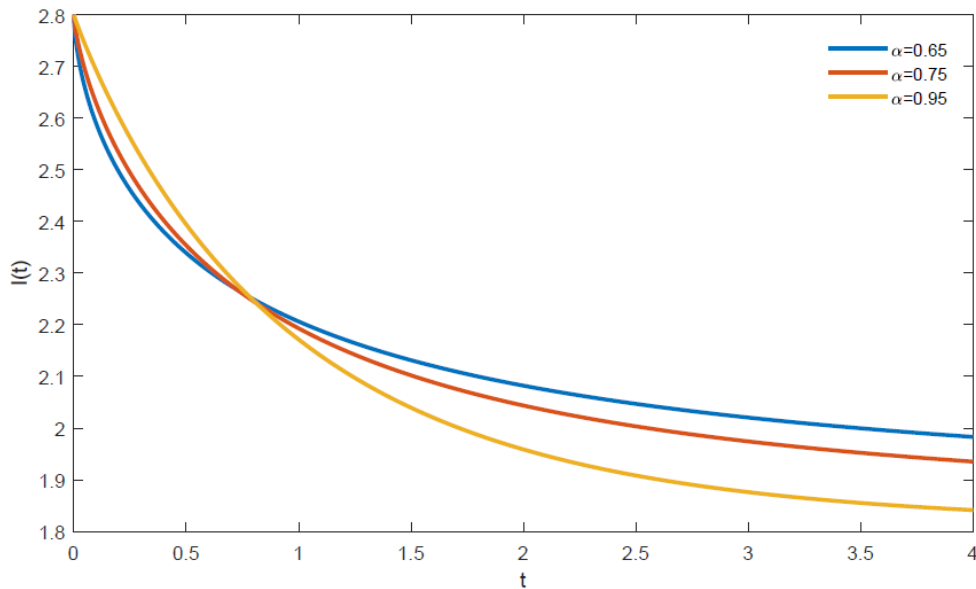


Figure 2. Fractional input stability of the electrical RL circuit equation.

thus the equilibrium point $V = 0$ of the electrical RL circuit equation (4.2) is globally asymptotically stable, see figure 3.

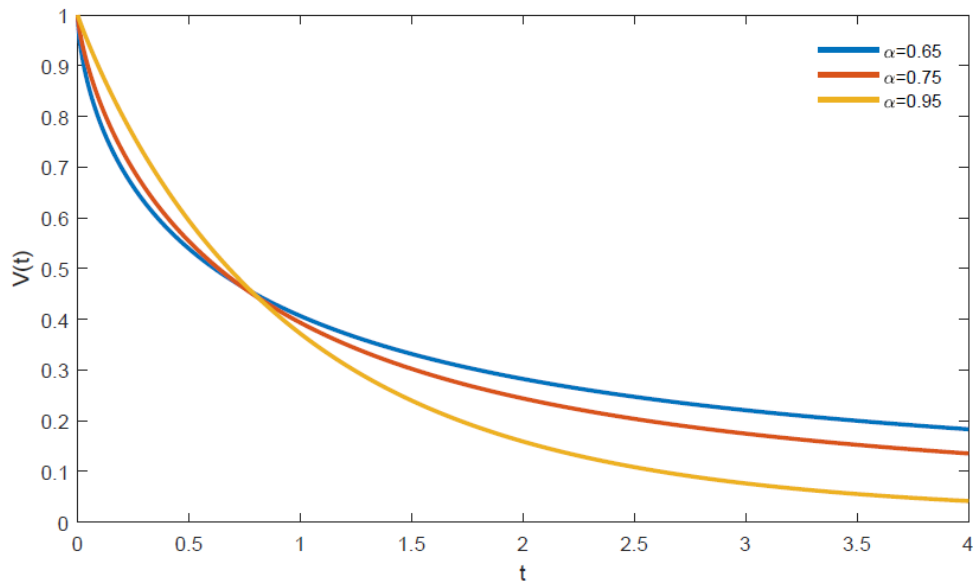


Figure 3. $V = 0$ of the electrical RC circuit equation is globally asymptotically stable.

In Figure 4, we observe the behavior of the analytical solution of the electrical RC circuit equation when it is fractional input stability.

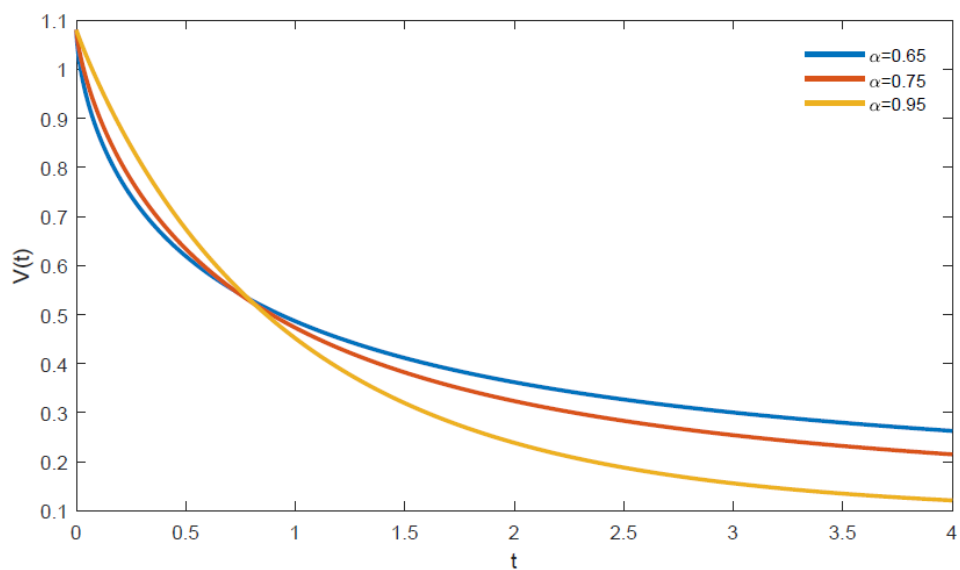


Figure 4. Fractional input stability of the electrical RC circuit.

5. Conclusion

In this paper, we have discussed the fractional input stability of the electrical circuit equation described by the Riemann-Liouville and the Caputo fractional derivative operators. This paper is the application of the fractional input stability in the electrical circuit equations. And we have noticed the fractional input stability is an excellent compromise to study the behavior of the analytical solution of the electrical RL, RC and LC circuit equations.

Conflict of interest

The author declare that there is no conflict of interest.

References

1. Y. Adjabi, F. Jarad, T. Abdeljawad, *On generalized fractional operators and a Gronwall type inequality with applications*, *Filomat*, **31** (2017), 17.
2. E. F. D. Goufo, *Chaotic processes using the two-parameter derivative with non-singular and non-local kernel: Basic theory and applications*, *Chaos*, **26** (2016), 084305.
3. J. F. Gómez-Aguilar, *Behavior characteristics of a cap-resistor, memcapacitor, and a memristor from the response obtained of RC and RL electrical circuits described by fractional differential equations*, *Turk. J. Elec. Eng. Comp. Sci.*, **24** (2016), 1421–1433.
4. J. F. Gómez-Aguilar, T. Cordova-Fraga, J. E. Escalante-Martínez, et al. *Electrical circuits described by a fractional derivative with regular kernel*, *Rev. Mex. Fis.*, **62** (2016), 144–154.
5. J. F. Gómez-Aguilar, R. F. Escobar-Jiménez, V. H. Olivares-Peregrino, et al. *Electrical circuits RC and RL involving fractional operators with bi-order*, *Adva. Mech. Eng.*, **9** (2017).
6. J. F. Gómez-Aguilar, R. G. Juan, G. C. Manuel, et al. *Fractional RC and LC electrical circuits*, *Ing., Invest. Tecno.*, **15** (2014), 311–319.
7. J. F. Gómez-Aguilar, J. Rosales-García, R. F. Escobar-Jiménez, et al. *On the possibility of the jerk derivative in electrical circuits*, *Adv. Math. Phys.*, **2016** (2016), 9740410.
8. J. F. Gómez-Aguilar, V. F. Morales-Delgado, M. A. Taneco-Hernández, et al. *Analytical solutions of the electrical RLC circuit via Liouville-Caputo operators with local and non-local kernels*, *Entropy*, **18** (2016), 402.
9. F. Jarad, T. Abdeljawad, *A modified Laplace transform for certain generalized fractional operators*, *Results Nonlinear Anal.*, **2018** (2018), 88–98.
10. V. F. Morales-Delgado, J. F. Gómez-Aguilar, M. A. Taneco-Hernandez, *Analytical solutions of electrical circuits described by fractional conformable derivatives in Liouville-Caputo sense*, *AEU-Int. J. Electron. C.*, **85** (2018), 108–117.
11. M. A. Moreles, R. Lainez, *Mathematical modelling of fractional order circuits*, (2016).
12. S. Priyadharsini, *Stability of fractional neutral and integrodifferential systems*, *J. Fract. Calc. Appl.*, **7** (2016), 87–102.

13. D. Qian, C. Li, R. P. Agarwal, et al. *Stability analysis of fractional differential system with Riemann-Liouville derivative*, Math. Comp. Model., **52** (2010), 862–874.
14. A. G. Radwan, A. S. Elwakil, A. M. Soliman, *On the generalization of second-order filters to the fractional-order domain*, J. Circuit. Syst. Comp., **18** (2009), 361–386.
15. A. G. Radwan, K. N. Salama, *Fractional-order RC and RL circuits*, Circuits, Syst., Signal Process., **31** (2012), 1901–1915.
16. N. Sene, *Exponential form for Lyapunov function and stability analysis of the fractional differential equations*, J. Math. Computer Sci., **18** (2018), 388–397.
17. N. Sene, *Lyapunov characterization of the fractional nonlinear systems with exogenous input*, Fractal. Fract., **2** (2018).
18. N. Sene, *Fractional input stability and its application to neural network*, Discrete Contin. Dyn. Syst. Ser. S, **13** (2020), in press.
19. N. Sene, *Mittag-Leffler input stability of fractional differential equations and its applications*, Discrete Contin. Dyn. Syst. Ser. S, **13** (2020), in press.
20. P.V. Shah, A. D. Patel, I. A. Salehbbhai, et al. *Analytic solution for the RL electric circuit model in fractional order*, In Abst. Appli. Analy., **2014** (2014), 343814.
21. R. Zhang, G. Tian, S. Yang, et al. *Stability analysis of a class of fractional order nonlinear systems with order lying in $(0, 2)$* , ISA T., **56** (2015), 102–110.



AIMS Press

©2019 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)