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Research article

A note on the Liouville type theorem for the smooth solutions of the stationary Hall-MHD system

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Abstract: The main result of this work is to study the Liouville type theorem for the stationary Hall-MHD system on $\mathbb{R}^3$. Specificaly, we show that if (u, B) is a smooth solutions to Hall-MHD equations satisfying $(u, B) \in L^{\frac{9}{2}}(\mathbb{R}^3)$, then we have $u = B = 0$. This improves a recent result of Chae et al. [2] and Zujin et al. [14].

Keywords: Stationary Hall-MHD equations; Liouville type theorem

1. Introduction and main result

We consider the following stationary Hall-MHD system on $\mathbb{R}^3$:

$$\begin{cases} (u.\nabla)u - (\nabla \times B) \times B - \Delta u + \nabla \pi = 0, \\ -\nabla \times (u \times B) + \nabla \times [(\nabla \times B) \times B] - \Delta B = 0, \\ \nabla.u = \nabla.B = 0, \\ (u, B)(x, 0) = (u_0(x), B_0(x)), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^3$. Here $u = u(x, t) \in \mathbb{R}^3$, $B = B(x, t) \in \mathbb{R}^3$ and $\pi = \pi(x, t)$ are non-dimensional quantities corresponding to the flow velocity, the magnetic field and the pressure at the point (x, t), while $u_0(x)$ and $B_0(x)$ are the given initial velocity and initial magnetic field with $\nabla.u_0 = 0$ and $\nabla.B_0 = 0$, respectively. An explanation of the mathematical and physical background of equations (1.1) is given for example in [1] (see also [4, 5, 6, 7, 9, 10, 11, 12, 13] and the references therein).

In their famous paper [2], Chae-Degond-Liu proved (Theorem 2.5, p. 558) (see also [14]) the following Liouville-type theorem for the smooth solutions of (1.1) :

Theorem 1.1. *Let $(u, B) \in C^2\left(\mathbb{R}^3\right)$ be a smooth solution of the stationary Hall-MHD system (1.1) such that*

(i) $(u, B) \in L^{\frac{9}{2}}(\mathbb{R}^3)$,

(ii) $(u, B) \in L^{\infty}(\mathbb{R}^3)$,

(iii) *the (weak and then by classical) solution* $(u, B) : \mathbb{R}^3 \to \mathbb{R}^3$ *is of finite energy in the sense that*

$$\int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{\mathbb{R}^3} |\nabla B|^2 \, dx < \infty.$$

Then,

$$u = B = 0.$$

The purpose of this note is to get rid of hypothesis (ii) and (iii) in theorem 1.1. More precisely, we shall prove the following result.

Theorem 1.2. *Let* $(u, B) \in C^2\left(\mathbb{R}^3\right)$ *be a smooth solution of the Hall-MHD equations (1.1) such that*

$$(u, B) \in L^{\frac{9}{2}}(\mathbb{R}^3) \ \ \textit{and} \ \ \int_{\mathbb{R}^3} |\nabla B|^2 \, dx < \infty.$$

Then,

$$u = B = 0 \ \ \textit{in} \ \ \mathbb{R}^3.$$

Remark 1.1. *As mentioned in [3], if we set* $B = 0$ *in the Hall-MHD system, the above theorem reduces to the well-known Galdi result [8] for the Navier–Stokes equations (see Theorem X.9.5, pp.729-730).*

2. Proof of Theorem 1.2

In order to prove our main result, we introduce some basic identifies in the fluid dynamic.

Lemma 2.1.

$$\begin{aligned}
\Delta u &= \nabla divu - \nabla \times (\nabla \times u), \\
u \times (\nabla \times u) &= \frac{1}{2}\nabla |u|^2 - (u.\nabla)u, \\
\nabla \times (u \times B) &= (B.\nabla)u - (u.\nabla)B + udivB - Bdivu.
\end{aligned}$$

Remark 2.1. *Based on* $\nabla.B = 0$ *and Lemma 2.1, we get*

$$(\nabla \times B) \times B = div(B \otimes B - \frac{1}{2} |B|^2 I) = -\nabla |B|^2 - (B.\nabla)B, \tag{2.1}$$

where I is the identical matrix.

We are now in a position to the proof of our main result.

Proof: Let $(u, B) \in C^2\left(\mathbb{R}^3\right)$ be a smooth solution of the Hall-MHD equations (1.1) satisfies

$$(u, B) \in L^{\frac{9}{2}}(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla B|^2 \, dx < \infty.$$

We shall first estimate the pressure in $(1.1)_1$. Taking the divergence of $(1.1)_1$ and using the identity (2.1), we have

$$\Delta\left(\pi + \frac{|B|^2}{2}\right) = -\sum_{j,k=1}^{3} \partial_j\partial_k(u_ju_k - B_jB_k),$$

from which we have the representation formula of the pressure, using the Riesz transforms in $\mathbb{R}^3$:

$$\pi = \sum_{j,k=1}^{3} \mathcal{R}_j\mathcal{R}_k(u_ju_k - B_jB_k) - \frac{|B|^2}{2}. \tag{2.2}$$

Using (2.2) and Calderòn–Zygmund estimate, one has that

$$\|\pi\|_{L^q} \le C(\|u\|_{L^{2q}}^2 + \|B\|_{L^{2q}}^2), \;\; 1 < q < \infty. \tag{2.3}$$

For $\tau > 0$, let φ_τ be a real nonincreasing smooth function defined in $\mathbb{R}^3$ such that

$$\varphi_\tau(x) = \begin{cases} 1 \;\text{ for } \;\; |x| \le \tau, \\ 0 \;\text{ for } \;\; |x| \ge 2\tau, \end{cases}$$

and satisfying

$$\left\|\nabla^k\varphi_\tau\right\|_{L^\infty} \le C\tau^{-k} \;\text{ for } k = 0, 1, 2, 3,$$

for some positive constant C independent of $x \in \mathbb{R}^3$.

Multiplying $(1.1)_1$ by $u\varphi_\tau$ and $(1.1)_2$ by $B\varphi_\tau$, respectively, integrating by parts over $\mathbb{R}^3$ and taking into acount $(1.1)_3$, add the result together, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u|^2 \varphi_\tau dx + \int_{\mathbb{R}^3} |\nabla B|^2 \varphi_\tau dx \\ =\; & \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 (u.\nabla)\varphi_\tau dx + \int_{\mathbb{R}^3} \pi(u.\nabla)\varphi_\tau dx - \int_{\mathbb{R}^3} (u \times B).(\nabla\varphi_\tau \times B)dx \\ & + \int_{\mathbb{R}^3} [(\nabla \times B) \times B].(\nabla\varphi_\tau \times B)dx + \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 \Delta\varphi_\tau dx + \frac{1}{2}\int_{\mathbb{R}^3} |B|^2 \Delta\varphi_\tau dx \\ =\; & \sum_{k=1}^{6} A_k, \end{aligned} \tag{2.4}$$

where we have used the fact

$$\begin{aligned} -\int_{\mathbb{R}^3} (\Delta w)w\varphi_\tau dx &= \int_{\mathbb{R}^3} |\nabla w|^2 \varphi_\tau dx + \int_{\mathbb{R}^3} (w\nabla w).\nabla\varphi_\tau dx \\ &= \int_{\mathbb{R}^3} |\nabla w|^2 \varphi_\tau dx + \frac{1}{2}\int_{\mathbb{R}^3} \nabla w^2.\nabla\varphi_\tau dx \\ &= \int_{\mathbb{R}^3} |\nabla w|^2 \varphi_\tau dx - \frac{1}{2}\int_{\mathbb{R}^3} w^2\Delta\varphi_\tau dx. \end{aligned}$$

In the following, we will estimate all the terms on the right-hand side of (2.4). For the first integral A_1, Hölder's inequality yields

$$|A_1| \;\; \le \;\; C\int_{\tau \le |x| \le 2\tau} |u|^3 \,|\nabla\varphi_\tau|\, dx$$

$$\begin{aligned}
&\leq \frac{1}{2\tau}\,\|\nabla\varphi\|_{L^\infty}\left(\int_{\tau\leq|x|\leq 2\tau}|u|^{\frac{9}{2}}\,dx\right)^{\frac{2}{3}}\left(\int_{\tau\leq|x|\leq 2\tau}dx\right)^{\frac{1}{3}}\\
&\leq C\,\|u\|^3_{L^{\frac{9}{2}}(\tau\leq|x|\leq 2\tau)}\to 0\ \text{ as }\ \tau\to+\infty.
\end{aligned}$$

As for A_2, using the Hölder inequality, it follows according to (2.3) that

$$\begin{aligned}
|A_2| &\leq \int_{\tau\leq|x|\leq 2\tau}|\pi|\,|u|\,|\nabla\varphi_\tau|\,dx\\
&\leq \frac{1}{\tau}\,\|\nabla\varphi\|_{L^\infty}\left(\int_{\mathbb{R}^3}|\pi|^{\frac{9}{4}}\,dx\right)^{\frac{4}{9}}\left(\int_{\tau\leq|x|\leq 2\tau}|u|^{\frac{9}{2}}\,dx\right)^{\frac{2}{9}}\left(\int_{\tau\leq|x|\leq 2\tau}dx\right)^{\frac{1}{3}}\\
&\leq C(\|u\|^2_{L^{\frac{9}{2}}}+\|B\|^2_{L^{\frac{9}{2}}})\,\|u\|_{L^{\frac{9}{2}}(\tau\leq|x|\leq 2\tau)}\to 0\ \text{ as }\ \tau\to+\infty.
\end{aligned}$$

Analogously to A_1, an application of the Hölder inequality shows that

$$\begin{aligned}
|A_3| &\leq \int_{\tau\leq|x|\leq 2\tau}|u|\,|B|^2\,|\nabla\varphi_\tau|\,dx\\
&\leq \frac{1}{\tau}\,\|\nabla\varphi\|_{L^\infty}\left(\int_{\tau\leq|x|\leq 2\tau}|B|^{\frac{9}{2}}\,dx\right)^{\frac{4}{9}}\left(\int_{\tau\leq|x|\leq 2\tau}|u|^{\frac{9}{2}}\,dx\right)^{\frac{2}{9}}\left(\int_{\tau\leq|x|\leq 2\tau}dx\right)^{\frac{1}{3}}\\
&\leq C\,\|B\|^2_{L^{\frac{9}{2}}(\tau\leq|x|\leq 2\tau)}\,\|u\|_{L^{\frac{9}{2}}(\tau\leq|x|\leq 2\tau)}\to 0\ \text{ as }\ \tau\to+\infty.
\end{aligned}$$

Similar to the treatment of A_3, A_4 can be estimated as

$$\begin{aligned}
|A_4| &\leq \int_{\tau\leq|x|\leq 2\tau}|\nabla B|\,|B|^2\,|\nabla\varphi_\tau|\,dx\\
&\leq \frac{1}{\tau}\,\|\nabla\varphi\|_{L^\infty}\left(\int_{\tau\leq|x|\leq 2\tau}|\nabla B|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\tau\leq|x|\leq 2\tau}|B|^6\,dx\right)^{\frac{1}{3}}\left(\int_{\tau\leq|x|\leq 2\tau}dx\right)^{\frac{1}{6}}\\
&\leq \frac{C}{\sqrt{\tau}}\,\|\nabla\varphi\|_{L^\infty}\,\|\nabla B\|_{L^2}\,\|B\|^2_{L^6}\\
&\leq \frac{C}{\sqrt{\tau}}\,\|\nabla\varphi\|_{L^\infty}\,\|\nabla B\|^3_{L^2}\to 0\ \text{ as }\ \tau\to+\infty.
\end{aligned}$$

Finally, calculating A_5+A_6 we obtain

$$\begin{aligned}
|A_5|+|A_6| &\leq C\int_{\tau\leq|x|\leq 2\tau}(|u|^2+|B|^2)\,|\Delta\varphi_\tau|\,dx\\
&\leq C\frac{1}{\tau^2}\,\|\Delta\varphi\|_{L^\infty}\left(\int_{\tau\leq|x|\leq 2\tau}(|u|^2+|B|^2)^{\frac{9}{4}}dx\right)^{\frac{4}{9}}\left(\int_{\tau\leq|x|\leq 2\tau}dx\right)^{\frac{5}{9}}\\
&\leq C\frac{1}{\tau^{\frac{1}{3}}}\,\|\Delta\varphi\|_{L^\infty}\left(\int_{\tau\leq|x|\leq 2\tau}(|u|^{\frac{9}{2}}+|B|^{\frac{9}{2}})dx\right)^{\frac{4}{9}}\\
&\leq C\frac{1}{\tau^{\frac{1}{3}}}\,\|\Delta\varphi\|_{L^\infty}\left(\|u\|^2_{L^{\frac{9}{2}}}+\|B\|^2_{L^{\frac{9}{2}}}\right)\to 0\ \text{ as }\ \tau\to+\infty.
\end{aligned}$$

Here we have used the Cauchy inequality. Consequently, letting $\tau \to +\infty$ in (2.4), we obtain

$$\lim_{\tau\to+\infty}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\varphi_\tau dx+\int_{\mathbb{R}^3}|\nabla B|^2\,\varphi_\tau dx\right)=0$$

On the other hand, by means of the monotone convergence theorem, we deduce

$$\int_{\mathbb{R}^3}|\nabla u|^2\,dx+\int_{\mathbb{R}^3}|\nabla B|^2\,dx=\lim_{\tau\to+\infty}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\varphi_\tau dx+\int_{\mathbb{R}^3}|\nabla B|^2\,\varphi_\tau dx\right)=0,$$

and thus u =const and B =const. Since $(u,B)\in L^{\frac{9}{2}}(\mathbb{R}^3)$, this latter condition delivers

$$u=B=0.$$

This completes the proof of Theorem 1.2. □

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Conflict of Interest

We declare no conflicts of interest in this paper.

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