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Research article

On a fractional alternating Poisson process

Antonio Di Crescenzo*and Alessandra Meoli

Department of Mathematics, University of Salerno, Via Giovanni Paolo II, 132, 84084 Fisciano (SA), Italy

* **Correspondence:** Email: adicrescenzo@unisa.it; Tel: +39 089 96 3349;
Fax: +39 089 96 3303.

Abstract: We propose a generalization of the alternating Poisson process from the point of view of fractional calculus. We consider the system of differential equations governing the state probabilities of the alternating Poisson process and replace the ordinary derivative with the fractional derivative (in the Caputo sense). This produces a fractional 2-state point process. We obtain the probability mass function of this process in terms of the (two-parameter) Mittag-Leffler function. Then we show that it can be recovered also by means of renewal theory. We study the limit state probability, and certain proportions involving the fractional moments of the sub-renewal periods of the process. In conclusion, in order to derive new Mittag-Leffler-like distributions related to the considered process, we exploit a transformation acting on pairs of stochastically ordered random variables, which is an extension of the equilibrium operator and deserves interest in the analysis of alternating stochastic processes.

Keywords: Caputo derivative; Mittag-Leffler function; fractional process; alternating process; renewal process; renewal function

1. Introduction

Extension of continuous-time point processes to the fractional case has been a major topic in literature for several years. Recently published papers consider, among others, fractional versions of the Poisson process (see Beghin and Orsingher [1], Laskin [9,10], Uchaikin and Sibatov [20]), of the pure birth process (see Orsingher and Polito [14]), of the pure death process (see Orsingher *et al.* [16]), of the birth-death process (see Orsingher and Polito [15]), of a general counting process (cf. Di Crescenzo *et al.* [4]), of branching processes (see Uchaikin *et al.* [19]). In most cases such "fractionalization" is performed by replacing the time-derivative with a fractional one in the differential equations governing the probability distribution. To this aim, one usually resorts either to the Riemann-Liouville derivative or to the Caputo one (for a comprehensive introduction to fractional calculus see [18] and [5]). A distinctive feature of these processes is the parameter, denoted by ν, describing the order of the fractional derivative, $0 < \nu \leq 1$. In fact, due to the peculiar structure of the fractional derivative, it is capable of taking into account the length of the memory of the process. For example, the Caputo derivative is a non-local operator defined as a classical derivative weighted over a time interval $[0, t]$ by means of a power-law kernel. This will be clearer in the formal definition shown in (2.2). Moreover, for some fractional processes, such as the pure birth fractional process [14], it can be seen that a decrease in the order of fractionality results in a more rapidly evolving process. Fractional processes are widely used to model real-world phenomena in science, technology and engineering systems, since they are also characterized by the presence of a heavy-tailed distribution. The probability distribution functions of such processes are given in terms of the generalized Mittag-Leffler function, which is defined as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_r \, z^r}{r! \, \Gamma(\alpha r + \beta)}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \; Re(\alpha), Re(\beta), Re(\gamma) > 0 \tag{1.1}$$

where

$$(\gamma)_r := \begin{cases} \gamma(\gamma+1)\ldots(\gamma+r-1), & r = 1, 2, \ldots, \\ 1, & r = 0 \end{cases}$$

is the Pochhammer symbol. For $\gamma = 1$, Eq. (1.1) gives the (two-parameter) Mittag-Leffler function, denoted as

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \; Re(\alpha), Re(\beta) > 0. \tag{1.2}$$

For more details on the Mittag-Leffler function see [5]. This function provides a simple but powerful generalization of the exponential function.

Inspired by the aforementioned developments in the field of fractional counting processes, in this paper we propose a fractional version of the alternating Poisson process.

1.1. Background on alternating Poisson process

Alternating renewal processes are special types of renewal processes. Specifically, an alternating renewal process is a stochastic process in which the renewal interval comprises two random subintervals that alternate cyclically. During the first one the process is in mode 1, whilst during the second one the process is in mode 0. For example, consider a repairable system which might periodically be in ON mode (running) or in OFF mode (in repair) for a random time. Another example, from the field of mathematical physics, is that of a telegraph process. A particle, in the origin at time $t = 0$, moves by turns UP (in a positive direction on a line) and DOWN (in a negative direction) during random times. Here the UP movement is one mode, and the DOWN movement is the other mode. Other examples can be taken from the fields of inventory control, finance, traffic control, etc. (cf. [21] for more details). If the system starts in state 1 and if a *cycle* consists of a mode-1 and a mode-0 interval, then the process that counts the number of cycles completed up to time t is an alternating renewal process, where returns to state 1 are the arrivals (cycle completions).

Let $\{U_k; \; k = 1, 2, \ldots\}$ and $\{D_k; \; k = 1, 2, \ldots\}$ be sequences of independent copies of two non-negative absolutely continuous random variables U, describing the duration of a mode-1 period, and D, describing the duration of a mode-0 period. Therefore, the k-th cycle is distributed as

$$X_k \stackrel{d}{=} U^{(k)} + D^{(k)}, \tag{1.3}$$

where

$$U^{(k)} = U_1 + U_2 + \cdots + U_k, \quad D^{(k)} = D_1 + D_2 + \cdots + D_k, \qquad k = 1, 2, \ldots. \tag{1.4}$$

The distribution functions of U and D are denoted respectively by F_U and F_D, whereas the corresponding complementary cumulative distribution functions are $\overline{F}_U$ and $\overline{F}_D$. If U_k and D_k are exponentially distributed with positive parameters λ and μ, the resulting counting process having interarrival times $U_1, D_1, U_2, D_2, \ldots$ is the *alternating Poisson process* (see, for instance, [8] for details). Equivalently, an alternating Poisson process is a 2-state continuous-time Markov chain, whose state occupancy probabilities satisfy, for $t \geq 0$, $\lambda, \mu > 0$, the system of equations:

$$\begin{cases} \dfrac{d\, p_{11}}{dt} = -\lambda\, p_{11}(t) + \mu\, p_{10}(t) \\ \dfrac{d\, p_{10}}{dt} = \lambda\, p_{11}(t) - \mu\, p_{10}(t). \end{cases} \tag{1.5}$$

It is well-known that the solutions of (1.5), subject to the initial conditions $p_{11}(0) = 1$, $p_{10}(0) = 0$ and normalizing condition $p_{11}(t) + p_{10}(t) = 1$, are

$$p_{11}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t} \quad \text{and} \quad p_{10}(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}. \tag{1.6}$$

Specifically, let $Y(t)$, $t \geq 0$, be a stochastic process with state space $\{0, 1\}$. If $Y(t)$ describes the state of the process at time t and $p_{ij}(t) = \mathbb{P}(Y(t) = j \,|\, Y(0) = i)$, then $p_{11}(t)$ and $p_{10}(t)$ represent respectively the probabilities of being in states 1 and 0 at time t starting from state 1 at $t = 0$. Similarly, the probabilities of being in states 1 and 0 at t starting from state 0 at $t = 0$ are found to be, for $t \geq 0$

$$p_{01}(t) = \frac{\mu}{\lambda + \mu} - \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)t} \quad \text{and} \quad p_{00}(t) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)t}.$$

If we define

$$\begin{aligned} p_j(t) &= \mathbb{P}(Y(t) = j) \\ &= p_1(0) p_{1j}(t) + p_0(0) p_{0j}(t), \qquad t \geq 0, \end{aligned} \tag{1.7}$$

then

$$p_1(t) = \frac{\mu}{\lambda + \mu} + \left[p_1(0) - \frac{\mu}{\lambda + \mu} \right] e^{-(\lambda+\mu)t} \quad \text{and} \quad p_0(t) = \frac{\lambda}{\lambda + \mu} + \left[p_0(0) - \frac{\lambda}{\lambda + \mu} \right] e^{-(\lambda+\mu)t}.$$

1.2. Plan of the paper

The paper is organized as follows. In Section 2, we develop the analysis of the fractional version (in the Caputo sense) of the alternating Poisson process, by determining explicitly the probability law, the renewal function and the renewal density. In Section 3, we deal with the asymptotic behaviour of the process, with special attention to the limit probability of the state 1 of the fractional alternating Poisson process, and to similar ratios involving the fractional moments of the renewal variables of the process. Finally, we exploit a suitable transformation of interest in the context of alternating renewal processes aiming to derive new Mittag-Leffler-like distributions.

2. Main results

In order to generalize the equations governing the alternating Poisson process, we now replace in (1.5) the time-derivative with the fractional derivative (in the Caputo sense) of order $\nu \in (0,1]$, thus obtaining the following system:

$$\begin{cases} \dfrac{d^\nu}{dt^\nu} p^\nu_{11}(t) = -\lambda\, p^\nu_{11}(t) + \mu\, p^\nu_{10}(t) \\ \dfrac{d^\nu}{dt^\nu} p^\nu_{10}(t) = \lambda\, p^\nu_{11}(t) - \mu\, p^\nu_{10}(t), \end{cases} \tag{2.1}$$

subject to the initial conditions $p^\nu_{11}(0) = 1$, $p^\nu_{10}(0) = 0$, $p^\nu_{11}(t) + p^\nu_{10}(t) = 1$. We recall that the definition of the fractional derivative in the sense of Caputo for $m \in \mathbb{N}$ is the following:

$$\frac{d^\nu}{dt^\nu} u(t) = \begin{cases} \dfrac{1}{\Gamma(m-\nu)} \displaystyle\int_0^t (t-s)^{m-\alpha-1} \frac{d^m}{ds^m} u(s) ds & \text{if } m-1 < \nu < m \\ \dfrac{d^m}{dt^m} u(t) & \text{if } \nu = m. \end{cases} \tag{2.2}$$

We remark that the use of the Caputo derivative permits us to avoid fractional derivatives in the initial conditions.

Proposition 2.1. *The solution of the Cauchy problem (2.1), for $t \geq 0$ and $\nu \in (0,1]$, is given by*

$$p^\nu_{11}(t) = 1 - \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \qquad \text{and} \qquad p^\nu_{10}(t) = \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu), \tag{2.3}$$

where $E_{\alpha,\beta}(t)$ is the Mittag-Leffler function (1.2).

Proof. The Laplace transform of the solution to system (2.1) becomes, for $s > (\lambda+\mu)^{1/\nu}$,

$$\begin{cases} \mathcal{L}\{p^\nu_{11}(t); s\} = \dfrac{s^{\nu-1}}{s^\nu + (\lambda+\mu)} + \mu \dfrac{s^{-1}}{s^\nu + (\lambda+\mu)} \\ \mathcal{L}\{p^\nu_{10}(t); s\} = \lambda \dfrac{s^{-1}}{s^\nu + (\lambda+\mu)}. \end{cases} \tag{2.4}$$

System (2.4) can be inverted by using formula (5.1.6) of [5], i.e.

$$\mathcal{L}\left\{t^{\gamma-1} E^\delta_{\beta,\gamma}\left(\omega t^\beta\right); s\right\} = \frac{s^{\beta\delta-\gamma}}{(s^\beta - \omega)^\delta}, \tag{2.5}$$

where $Re(\beta) > 0$, $Re(\gamma) > 0$, $Re(\delta) > 0$ and $s > |\omega|^{\frac{1}{Re(\beta)}}$. Indeed, recalling that (cf. (4.2.3) of [5])

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z E_{\alpha,\alpha+\beta}(z), \tag{2.6}$$

we recover

$$\begin{aligned} p^\nu_{11}(t) &= E_{\nu,1}(-(\lambda+\mu)t^\nu) + \mu t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \\ &= 1 - \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \end{aligned}$$

and

$$p^\nu_{10}(t) = \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu).$$

This completes the proof of (2.3). □

Solutions (2.3) can be interpreted as the probabilities of a *fractional alternating Poisson process* of being in states 1 and 0 at t starting from state 1 at time $t = 0$. Specifically, if $Y^\nu(t)$, $t \geq 0$, is a stochastic process with state space $\{0, 1\}$, describing the state of the process at time t, then

$$p^\nu_{11}(t) = \mathbb{P}\,(Y^\nu(t) = 1 \,|\, Y^\nu(0) = 1) \qquad \text{and} \qquad p^\nu_{10}(t) = \mathbb{P}\,(Y^\nu(t) = 0 \,|\, Y^\nu(0) = 1)\,.$$

Similarly to (2.3), we find that the probabilities of being in states 1 and 0 at t starting from state 0 at $t = 0$ are

$$p^\nu_{01}(t) = \mu t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu)$$

and

$$\begin{aligned} p^\nu_{00}(t) &= E_{\nu,1}(-(\lambda+\mu)t^\nu) + \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \\ &= 1 - \mu t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu). \end{aligned}$$

By analogy with the non-fractional case, we can define

$$\begin{aligned} p^\nu_j(t) &= \mathbb{P}\,(\text{state } j \text{ occupied at time } t) \\ &= p^\nu_1(0) p^\nu_{1j}(t) + p^\nu_0(0) p^\nu_{0j}(t), \end{aligned}$$

so that, if the process starts in state 1 at $t = 0$,

$$p^\nu_1(t) = p^\nu_{11}(t) \quad \text{and} \quad p^\nu_0(t) = p^\nu_{10}(t). \tag{2.7}$$

We point out that whereas the starting alternating Poisson process is Markovian, the new process $Y^\nu(t)$ is non-Markov. Indeed, similarly as for other stochastic processes, the “fractionalization” produces persistence or long memory effects.

Such state occupancy probabilities can be recovered also by a different approach. Indeed, we suppose that the the random variable U_k (D_k), describing the duration of the kth time interval during which the system is in state 1 (state 0), is equally distributed with a random variable U (D) following a Mittag-Leffler distribution with density

$$f_U(t) = \lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu), \qquad \left(f_D(t) = \mu t^{\nu-1} E_{\nu,\nu}(-\mu t^\nu)\right), \qquad t > 0,\ 0 < \nu < 1, \tag{2.8}$$

and complementary cumulative distribution function

$$\overline{F}_U(t) = E_{\nu,1}(-\lambda t^\nu), \qquad \left(\overline{F}_D(t) = E_{\nu,1}(-\mu t^\nu)\right), \qquad t > 0,\ 0 < \nu < 1. \tag{2.9}$$

We recall that densities (2.8) are characterized by fat tails, with polynomial decay, and, as a consequence, the mean time spent by such a process both in state 1 and in state 0 is infinite.

The probability density function of the first cycle X (cf. Eq. (1.3)), due to the independence of its summands, can be recovered by inverting its Laplace transform:

$$\mathcal{L}_X(s) = \mathcal{L}_U(s)\mathcal{L}_D(s) = \frac{\lambda\mu}{(s^\nu+\lambda)(s^\nu+\mu)}, \tag{2.10}$$

so that, bearing in mind formula (2.5), we recover the following generalized mixture, for $\lambda \neq \mu$:

$$f_X(t) = \frac{\mu}{\mu-\lambda}\lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu) - \frac{\lambda}{\mu-\lambda}\mu t^{\nu-1} E_{\nu,\nu}(-\mu t^\nu), \qquad t > 0. \tag{2.11}$$

In the next proposition we derive the expression of the renewal function of the considered alternating process. See Cahoy and Polito [2] and Gorenflo and Mainardi [6] for recent contributions on renewal processes related to the fractional Poisson process.

Proposition 2.2. *Let $M(t)$, $t \geq 0$, be the renewal function of an alternating process whose inter-renewal times are distributed as in (2.11). Then*

$$M(t) = \lambda\mu t^{2\nu} E_{\nu,2\nu+1}(-(\lambda+\mu)t^\nu), \qquad t > 0. \tag{2.12}$$

The corresponding renewal density is

$$m(t) = \lambda\mu t^{2\nu-1} E_{\nu,2\nu}(-(\lambda+\mu)t^\nu), \qquad t > 0. \tag{2.13}$$

Proof. With regard to (2.11), the Laplace transform of the renewal function of the considered process, which we call $M(t)$, is (cf. [13])

$$\mathcal{L}\{M(t); s\} = \frac{\mathcal{L}_X(s)}{s(1-\mathcal{L}_X(s))} = \frac{1}{s} \cdot \frac{\lambda\mu}{s^\nu(s^\nu + (\lambda+\mu))}, \tag{2.14}$$

where the last identity follows from (2.10). From Equation (2.14) we infer that the Laplace transform of the corresponding renewal density is

$$\begin{aligned}\mathcal{L}\{m(t)\} = \mathcal{L}\left\{\frac{dM(t)}{\mathrm{d}t}\right\} &= s\mathcal{L}\{M(t)\} \\ &= \frac{\lambda\mu}{s^\nu(s^\nu + (\lambda+\mu))},\end{aligned}$$

which can be inverted with the help of formula (2.5) in order to obtain

$$m(t) = \frac{\lambda\mu}{\lambda+\mu} \cdot \frac{t^{\nu-1}}{\Gamma(\nu)} - \frac{\lambda\mu}{\lambda+\mu} t^{\nu-1} E_{\nu,\nu}(-(\lambda+\mu)t^\nu),$$

this giving (2.13). In addition, the renewal function turns out to be the following:

$$\begin{aligned}M(t) &= \frac{\lambda\mu}{\lambda+\mu} \cdot \frac{t^\nu}{\Gamma(\nu+1)} - \frac{\lambda\mu}{\lambda+\mu} t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \\ &= \lambda\mu t^{2\nu} E_{\nu,2\nu+1}(-(\lambda+\mu)t^\nu),\end{aligned}$$

where the last equality is due to (2.6). The proof of (2.12) is thus complete. □

From the theory of alternating renewal processes (cf. formula (6.66) of [13]), it is known that, for $t \geq 0$,

$$\pi_1(t) = \overline{F}_U(t) + \int_0^t m(t-x)\overline{F}_U(t)dx,$$

where $\pi_1(t)$ is the probability that at time t the process is in state 1 and $m(t)$ is the renewal density. Recalling (2.13) and (2.9), we obtain

$$\begin{aligned}\pi_1(t) &= E_{\nu,1}(-\lambda t^\nu) + \lambda\mu \int_0^t (t-x)^{2\nu-1} E_{\nu,2\nu}(-(\lambda+\mu)(t-x)^\nu) E_{\nu,1}(-\lambda x^\nu)dx \\ &= E_{\nu,1}(-\lambda t^\nu) - \lambda^2 t^{2\nu} E_{\nu,2\nu+1}(-\lambda t^\nu) + \lambda(\lambda+\mu) t^{2\nu} E_{\nu,2\nu+1}(-(\lambda+\mu)t^\nu) \\ &= 1 - \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu),\end{aligned} \tag{2.15}$$

where the last equality follows from (2.6). Due to (2.3), we observe that probability (2.15) equals the first of (2.7). Therefore, the random times between consecutive events for a fractional alternating Poisson process alternate between two Mittag-Leffler distributions with parameter λ and μ, respectively. Consequently, the two approaches considered, i.e. the one based on the resolution of the fractional system of equations (2.1), and the one based on renewal theory arguments, lead to the same alternating process.

3. Asymptotic behaviour and some transformations

We begin the present section by studying the asymptotic behaviour of the process $Y^\nu(t)$, with reference to $p_1^\nu(t) = \pi_1(t)$.

Proposition 3.1. *The limiting probability that the fractional alternating Poisson process is in state 1 is given by*

$$\lim_{t\to+\infty} p_1^\nu(t) = \frac{\mu}{\lambda+\mu}.$$

Proof. From (2.7) we observe that the limiting probability of being in the first phase of the considered process is:

$$\begin{aligned}\lim_{t\to+\infty} p_1^\nu(t) &= \lim_{t\to+\infty} \left(1 - \lambda t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu)\right) \\ &= \lim_{t\to+\infty} \frac{\lambda}{\lambda+\mu}\left(\frac{\lambda+\mu}{\lambda} - (\lambda+\mu)\, t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu)\right).\end{aligned}$$

It holds that $(\lambda+\mu)\, t^\nu E_{\nu,\nu+1}(-(\lambda+\mu)t^\nu) \overset{t\to+\infty}{\to} 1$, since we are dealing with the probability distribution function of a Mittag-Leffler random variable with parameter $\lambda+\mu$. Hence

$$\begin{aligned}\lim_{t\to+\infty} p_1^\nu(t) &= \frac{\lambda}{\lambda+\mu}\left(\frac{\lambda+\mu}{\lambda} - 1\right) \\ &= \frac{\mu}{\lambda+\mu},\end{aligned}$$

this completing the proof. □

It is noteworthy to point out that the fractional alternating Poisson process displays the same long-run proportion of time spent in mode 1 as its non fractional counterpart (cf. [8]). Moreover, the result presented in Proposition 3.3 is in accordance with Theorem 5 of [12], where the limiting distribution of the spent lifetime is presented in the case of infinite mean renewal periods.

We are now concerned with other kinds of proportions involving the fractional moments of the sub-renewal periods of the process $Y^\nu(t)$.

Proposition 3.2. *Let U and D be random variables with densities (2.8). Then*

$$\frac{\mathbb{E}[U^q]}{\mathbb{E}[U^q]+\mathbb{E}[D^q]} = \frac{1}{\xi^{q/\nu}+1}, \qquad \xi = \frac{\lambda}{\mu}, \quad 0 < q < \nu \le 1.$$

Proof. By [17], the expression for the qth moment, $q < \nu$, of a random variable with density (2.8) is

$$\mathbb{E}[U^q] = \frac{q\pi}{\nu\lambda^{q/\nu}\Gamma(1-q)\sin(q\pi/\nu)}. \tag{3.1}$$

The proof follows by conveniently substituting the expression for the qth moment of D. □

To prove Proposition 3.3 below we need the following Lemma (see [11]).

Lemma 3.1. *Let X be a positive random variable with Laplace transform ϕ. Then*

$$\mathbb{E}\left[X^r\right] = \frac{r}{\Gamma(1-r)} \int_0^{+\infty} s^{-r-1}\left(1-\phi(s)\right) ds, \qquad r \in (0,1).$$

With regard to (1.4), we observe that (cf. [1])

$$f_U^k(t) = \mathbb{P}\{U^{(k)} \in dt\}/dt = \lambda^k t^{\nu k-1} E^k_{\nu,\nu k}(-\lambda t^\nu), \quad t > 0,\ 0 < \nu < 1, \tag{3.2}$$

with Laplace transform

$$\mathcal{L}\left\{f_U^k(t); s\right\} = \frac{\lambda^k}{(s^\nu + \lambda)^k}. \tag{3.3}$$

The density and the Laplace transform of $D^{(k)}$ can be obtained from (3.2) and (3.3) respectively, by replacing λ with μ. We are now ready to prove the next proposition, which gives an immediate extension of Proposition 3.2.

Proposition 3.3. *Let $U^{(k)}$ and $D^{(k)}$ be random variables defined as in (1.4). Then*

$$\frac{\mathbb{E}\left[\left(U^{(k)}\right)^q\right]}{\mathbb{E}\left[(U^{(k)})^q\right] + \mathbb{E}\left[(D^{(k)})^q\right]} = \frac{1}{\xi^{q/\nu}+1}, \qquad \xi = \frac{\lambda}{\mu}, \quad 0 < q < \nu \le 1. \tag{3.4}$$

Proof. From Lemma 3.1 and Eq. (3.3), for $q \in (0,1)$,

$$\begin{aligned}
\mathbb{E}\left[\left(U^{(k)}\right)^q\right] &= \frac{q}{\Gamma(1-q)} \int_0^{+\infty} s^{-q-1}\left(1 - \frac{\lambda^k}{(s^\nu+\lambda)^k}\right) ds \\
&= \frac{q}{\Gamma(1-q)} \sum_{i=0}^{k-1} \binom{k}{i} \lambda^i \int_0^{+\infty} \frac{s^{\nu(k-i)-q-1}}{(s^\nu+\lambda)^k} ds,
\end{aligned}$$

where the last equality is due to the binomial theorem. By applying formula 3.241-4 of [7], i.e.

$$\int_0^{+\infty} \frac{x^{\mu-1}}{(p+qx^\nu)^{n+1}} dx = \frac{1}{\nu p^{n+1}} \left(\frac{p}{q}\right)^{\mu/\nu} \frac{\Gamma(\mu/\nu)\Gamma(1+n-\mu/\nu)}{\Gamma(1+n)}, \quad 0 < \frac{\mu}{\nu} < n+1,\ p \neq 0,\ q \neq 0,$$

we obtain, for $0 < q < \nu \le 1$,

$$\begin{aligned}
\mathbb{E}\left[\left(U^{(k)}\right)^q\right] &= \frac{q}{\Gamma(1-q)} \sum_{i=0}^{k-1} \binom{k}{i} \frac{1}{\nu\lambda^{q/\nu}} \frac{\Gamma(k-i-q/\nu)\Gamma(i+q/\nu)}{\Gamma(k)} \\
&= \frac{q}{\Gamma(1-q)} \frac{1}{\nu\lambda^{q/\nu}} \sum_{i=0}^{k-1} \binom{k}{i} B\left(k-i-q/\nu, i+q/\nu\right),
\end{aligned} \tag{3.5}$$

where $B(x,y)$ denotes the Beta function. Observe that, in an analogous way, we can calculate

$$\mathbb{E}\left[\left(D^{(k)}\right)^q\right] = \frac{q}{\Gamma(1-q)} \frac{1}{\nu\mu^{q/\nu}} \sum_{i=0}^{k-1} \binom{k}{i} B\left(k-i-q/\nu, i+q/\nu\right).$$

The thesis thus follows. □

Some plots of the ratio (3.4) are provided in Figure 1.

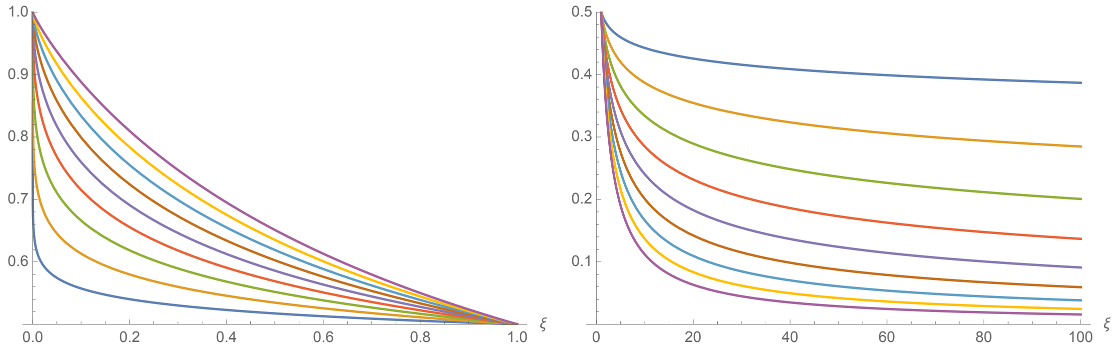

Figure 1. The ratio (3.4) is shown on the left for $0 < \xi \leq 1$ **and** $q/\nu = 0.1, 0.2, \ldots, 0.9$ **(from bottom to top), on the right for** $1 \leq \xi \leq 100$ **and** $q/\nu = 0.1, 0.2, \ldots, 0.9$ **(from top to bottom).**

Hereafter we aim to explore new stochastic models related to the fractional alternating Poisson process. Specifically, with reference to the process $Y^\nu(t)$, we now study a special transformation of the random variables involved. Such transformation, acting on pairs of non-negative random variables having unequal finite means, is an extension of the equilibrium operator. It is of interest since it arises essentially from stochastic processes characterized by two randomly alternating states. In fact, it is suitable to describe the asymptotic behaviour of the corresponding spent lifetime (cf. [3]). In general, if X and Y are non-negative random variables such that $\mathbb{E}[X] < \mathbb{E}[Y] < +\infty$, then

$$f_Z(x) = \frac{\overline{F}_Y(x) - \overline{F}_X(x)}{\mathbb{E}[Y] - \mathbb{E}[X]}, \qquad x \geq 0, \tag{3.6}$$

is the probability density function of an absolutely continuous non-negative random variable Z if and only if $X \leq_{st} Y$, where $\leq_{st}$ is the usual stochastic order (i.e., $X \leq_{st} Y$ if and only if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all x). In Eq. (3.6), $\overline{F}_X(x)$ and $\overline{F}_Y(x)$ denote the survival functions of X and Y, respectively. We write $Z \equiv \Psi(X,Y)$ to mean that Z is a random variable possessing density (3.6).

Example 3.1. Let U and D be random variables having Mittag-Leffler densities with parameters λ and μ respectively, expressed by (2.8), and fix a positive real number α, $0 < \alpha < \nu \leq 1$, such that the random variables U^α and D^α have finite means. If $\lambda < \mu$, one has $U \leq_{st} D$ and then $U^\alpha \leq_{st} D^\alpha$. From (2.9), (3.1) and (3.6), the density of $Z \equiv \Psi(U^\alpha, D^\alpha)$ is

$$f_Z(t) = \frac{\nu\lambda^{\alpha/\nu}\mu^{\alpha/\nu}\Gamma(1-\alpha)\sin(\alpha\pi/\nu)}{\alpha\pi\left(\lambda^{\alpha/\nu} - \mu^{\alpha/\nu}\right)} \left(E_{\nu,1}(-\mu t^{\nu/\alpha}) - E_{\nu,1}(-\lambda t^{\nu/\alpha})\right), \qquad t \geq 0. \tag{3.7}$$

Figure 2 shows various plots of density (3.7).

Consequently, from the probabilistic mean value theorem given in Theorem 4.1 of [3], if g is a measurable and differentiable function such that $\mathbb{E}[g(D^\alpha)]$ and $\mathbb{E}[g(U^\alpha)]$ are finite and if its derivative g' is measurable and Riemann-integrable on the interval $[x, y]$ for all $y \geq x \geq 0$, then $\mathbb{E}[g'(Z)]$ is finite and

$$\mathbb{E}[g(D^\alpha)] - \mathbb{E}[g(U^\alpha)] = \mathbb{E}[g'(Z)] (\mathbb{E}[D^\alpha] - \mathbb{E}[U^\alpha]),$$

where Z is a random variable having density (3.7).

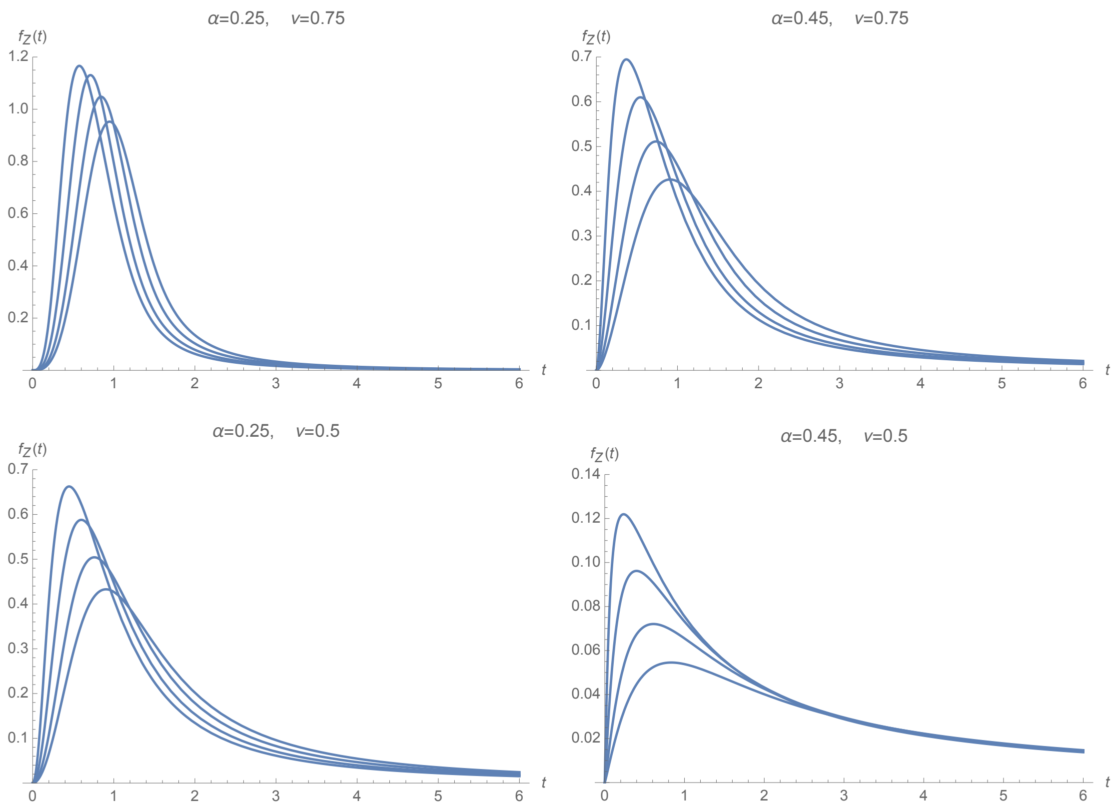

Figure 2. Density (3.7) for various choices of α and ν, with $\lambda = 1$ and $\mu = 1.01, 2, 5, 15$ (from bottom to top near the origin).

Example 3.2. Let us consider the random variables $U^{(1)}$ and $U^{(2)}$ (cf. (1.4)), with densities (2.8) and (3.2) respectively. Again, we fix a positive real number α, with $0 < \alpha < \nu \leq 1$, such that both random variables involved, i.e. $\left(U^{(1)}\right)^\alpha$ and $\left(U^{(2)}\right)^\alpha$ have finite first order moments. Clearly, $U^{(1)} \leq_{st} U^{(2)}$ and then $\left(U^{(1)}\right)^\alpha \leq_{st} \left(U^{(2)}\right)^\alpha$, so that we can study the transformation Ψ acting on $\left(U^{(1)}\right)^\alpha$ and $\left(U^{(2)}\right)^\alpha$. The complementary cumulative distribution functions of $U^{(1)}$ and $U^{(2)}$ are expressed in terms of the generalized Mittag-Leffler function (1.1), since (cf. (2.9))

$$\mathbb{P}\left(U^{(1)} > t\right) = 1 - \lambda t^\nu E_{\nu,\nu+1}(-\lambda t^\nu), \qquad t \geq 0 \tag{3.8}$$

and (cf. [1])

$$\mathbb{P}\left(U^{(2)} > t\right) = 1 - \lambda^2 t^{2\nu} E^2_{\nu,2\nu+1}(-\lambda t^\nu), \qquad t \geq 0. \tag{3.9}$$

Recalling that (cf. formula (5.1.12) of [5]) if α, β, $\gamma \in \mathbb{C}$ and $Re\ \alpha > 0$, $Re\,\beta > 0$, $Re\,\beta - \alpha > 0$

$$zE^{\gamma}_{\alpha,\beta} = E^{\gamma}_{\alpha,\beta-\alpha} - E^{\gamma-1}_{\alpha,\beta-\alpha}, \tag{3.10}$$

the following equality holds:

$$E_{\nu,\nu+1}(-\lambda t^{\nu}) = \lambda t^{\nu} E^{2}_{\nu,2\nu+1}(-\lambda t^{\nu}) + E^{2}_{\nu,\nu+1}(-\lambda t^{\nu}), \qquad t \geq 0,$$

and then

$$\lambda t^{\nu} E_{\nu,\nu+1}(-\lambda t^{\nu}) = \lambda^{2} t^{2\nu} E^{2}_{\nu,2\nu+1}(-\lambda t^{\nu}) + \lambda t^{\nu} E^{2}_{\nu,\nu+1}(-\lambda t^{\nu}), \qquad t \geq 0. \tag{3.11}$$

Owing to formula (5.1.14) of [5], i.e.

$$\alpha E^{2}_{\alpha,\beta} = E_{\alpha,\beta-1} - (1 + \alpha - \beta) E_{\alpha,\beta}$$

if α, $\beta \in \mathbb{C}$ and $Re\ \alpha > 0$, $Re\,\beta > 1$, then

$$E^{2}_{\nu,\nu+1}(-\lambda t^{\nu}) = \frac{1}{\nu} E_{\nu,\nu}(-\lambda t^{\nu}). \tag{3.12}$$

By using (3.12) into (3.11), we get

$$\lambda t^{\nu} E_{\nu,\nu+1}(-\lambda t^{\nu}) = \lambda^{2} t^{2\nu} E^{2}_{\nu,2\nu+1}(-\lambda t^{\nu}) + \frac{\lambda t^{\nu}}{\nu} E_{\nu,\nu}(-\lambda t^{\nu}), \qquad t \geq 0, \tag{3.13}$$

and the function $t^{\nu} E_{\nu,\nu}(-\lambda t^{\nu})$ is positive due to the complete monotonicity of $t^{\nu-1} E_{\nu,\nu}(-\lambda t^{\nu})$ (cf. (5.1.10) of [5]). Consequently, recalling (3.8) and (3.9), from (3.13) we obtain

$$\begin{aligned}
& 1 - \lambda^{2} t^{2\nu} E^{2}_{\nu,2\nu+1}(-\lambda t^{\nu}) \geq 1 - \lambda t^{\nu} E_{\nu,\nu+1}(-\lambda t^{\nu}) \\
\iff \quad & \mathbb{P}\left(U^{(2)} > t\right) \geq \mathbb{P}\left(U^{(1)} > t\right) \\
\iff \quad & U^{(1)} \leq_{st} U^{(2)} \\
\iff \quad & \left(U^{(1)}\right)^{\alpha} \leq_{st} \left(U^{(2)}\right)^{\alpha}.
\end{aligned}$$

Hence, if $Z \equiv \Psi\left(\left(U^{(1)}\right)^{\alpha}, \left(U^{(2)}\right)^{\alpha}\right)$, from (3.1) and (3.5) we have, for $t \geq 0$,

$$f_Z(t) = \frac{\Gamma(1-\alpha)\nu^{2}\lambda^{\alpha/\nu}\sin(\alpha\pi/\nu)}{\alpha^{2}\pi}\left(\lambda t^{\nu/\alpha} E_{\nu,\nu+1}(-\lambda t^{\nu/\alpha}) - \lambda^{2} t^{2\nu/\alpha} E^{2}_{\nu,2\nu+1}(-\lambda t^{\nu/\alpha})\right).$$

It follows that, making use of (3.10), for $0 < \alpha < \nu \leq 1$ and $\lambda > 0$ we obtain

$$f_Z(t) = \frac{\Gamma(1-\alpha)\nu^{2}\lambda^{\alpha/\nu}\sin(\alpha\pi/\nu)}{\alpha^{2}\pi}\lambda t^{\nu/\alpha} E^{2}_{\nu,\nu+1}(-\lambda t^{\nu/\alpha}), \qquad t \geq 0. \tag{3.14}$$

Again, from Theorem 4.1 of [3], if g is a suitable function and Z is a random variable with density (3.14), then

$$\mathbb{E}\left[g\left(\left(U^{(2)}\right)^{\alpha}\right)\right] - \mathbb{E}\left[g\left(\left(U^{(1)}\right)^{\alpha}\right)\right] = \mathbb{E}\left[g'(Z)\right]\left(\mathbb{E}\left[\left(U^{(2)}\right)^{\alpha}\right] - \mathbb{E}\left[\left(U^{(1)}\right)^{\alpha}\right]\right).$$

4. Conclusion

In this paper we have studied a generalization of the alternating Poisson process from the point of view of fractional calculus. In the system of differential equations governing the state occupancy probabilities for the alternating Poisson process we replace the ordinary derivative with the Caputo one, thus endowing the process with persistent memory. We obtain the probability mass function of a fractional alternating Poisson process and then show that it can be recovered also by means of renewal theory arguments. Furthermore, we provide results for the behaviour of some quantities characterizing the process under examination and derive new Mittag-Leffler-like distributions of interest in the context of alternating renewal processes.

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