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Research article

Existence of a solution to a semilinear elliptic equation

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Abstract: We consider the equation $-\Delta u = f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)d\mathbf{x}$, where the domain $\Omega = \mathbb{T}^N$, the N-dimensional torus, with $N = 2$ or $N = 3$. And f is a given smooth function of u for $u(\mathbf{x}) \in G \subset \mathbb{R}$. We prove that there exists a solution u to this equation which is unique if $|\frac{df}{du}(u_0)|$ is sufficiently small, where $u_0 \in G$ is a given constant. And we prove that the solution u is not unique if $\frac{df}{du}(u_0)$ is a simple eigenvalue of $-\Delta$.

Keywords: elliptic; existence; uniqueness; semilinear; bifurcation

1. Introduction

In this paper, we consider the following equation for u

$$-\Delta u = f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)d\mathbf{x} \tag{1.1}$$

under periodic boundary conditions. The domain $\Omega = \mathbb{T}^N$, the N-dimensional torus, with $N = 2, 3$. Here f is a given smooth function of u for $u(\mathbf{x}) \in G \subset \mathbb{R}$.

We will prove that there exists a solution u to equation (1.1) which is unique if $|\frac{df}{du}(u_0)| < \frac{1}{(C_0)^2}$, where $u_0 \in G$ is a given constant and where C_0 is the constant from Poincaré's inequality. And we will prove that the solution u is not unique if $\frac{df}{du}(u_0)$ is a simple eigenvalue of $-\Delta$.

In previous related work, many researchers have studied the equation $-\Delta u = f(u) + g$. Existence of a solution u to the equation $-\Delta u = f(u) + g$ has been proven for a Dirichlet boundary condition $u|_{\partial\Omega} = 0$ (see, e.g., [1,2,5,7]) under certain conditions on f and, g. And existence of a solution u to the equation $-\Delta, , u = f(u) + g$ has been proven for a Neumann boundary condition $\frac{\partial u}{\partial n}|_{\partial\Omega} = h$ (see, e.g., [3,4,6]) under certain conditions on f and g. We have not seen any work by other researchers on the existence of a solution u to equation (1.1) under periodic boundary conditions. And we have not seen any work by other researchers which contains the particular condition that $|\frac{df}{du}(u_0)| < \frac{1}{(C_0)^2}$, where C_0 is the constant from Poincaré's inequality and where u_0 is a given constant in the domain of the function $\frac{df}{du}$.

2. Existence theorem

In the proof that follows, we use the standard notation for the $L^2(\Omega)$ norm of a function g, namely, $\|g\|_0^2 = \int_\Omega |g|^2 d\mathbf{x}$. And we denote the inner product as $(g, h) = \int_\Omega gh d\mathbf{x}$. Also, we let Du denote the gradient of a function u. We also use the notation $|\frac{df}{du}|_{0,\overline{G}_1} = \max\{|\frac{df}{du}(u_*)| : u_* \in \overline{G}_1\}$, where $\frac{df}{du}$ is a function of u and where $\overline{G}_1 \subset \mathbb{R}$ is a closed bounded interval.

The purpose of this article is to prove the following theorem.

Theorem 2.1. *Consider the following equation for u*

$$-\Delta u = f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)d\mathbf{x} \tag{2.1}$$

where the domain $\Omega = \mathbb{T}^N$, the N-dimensional torus, with $N = 2$ or $N = 3$, and where f is a given smooth function of u for $u(\mathbf{x}) \in G \subset \mathbb{R}$. Let $u_0 \in G$ be a given constant. Then we have the following two cases:

(1) If $|\frac{df}{du}(u_0)| < \frac{1}{(C_0)^2}$, where C_0 is the constant from Poincaré's inequality, then there exists a unique classical solution $u(\mathbf{x}) \in \overline{G}_1$ to equation (2.1) which satisfies the condition $u(\mathbf{x}_0) = u_0$, where $\overline{G}_1 \subset G \subset \mathbb{R}$ and where $u_0 \in \overline{G}_1$ and where $\mathbf{x}_0 \in \Omega$ is a given point. This unique classical solution is $u = u_0$.

(2) If $\frac{df}{du}(u_0)$ is a simple eigenvalue of $-\Delta$ then there exists a solution u of equation (2.1) which is not the constant function u_0. This solution u may not necessarily satisfy the condition $u(\mathbf{x}_0) = u_0$.

Proof.

We will consider separately each of the two cases from the statement of the theorem. First, we will consider Case 1 from the statement of Theorem 2.1.

Suppose that $|\frac{df}{du}(u_0)| < \frac{1}{(C_0)^2}$, where C_0 is the constant from Poincaré's inequality and where $u_0 \in G$ is a given constant. It follows that there exists a closed bounded interval $\overline{G}_1 \subset G$ such that $u_0 \in \overline{G}_1$ and such that $|\frac{df}{du}|_{0,\overline{G}_1} < \frac{1}{(C_0)^2}$, where $|\frac{df}{du}|_{0,\overline{G}_1} = \max\{|\frac{df}{du}(u_*)| : u_* \in \overline{G}_1\}$. Suppose that u is a classical solution of equation (2.1) such that $u(\mathbf{x}) \in \overline{G}_1$ for all $\mathbf{x} \in \Omega$ and u satisfies the condition $u(\mathbf{x}_0) = u_0$, where $\mathbf{x}_0 \in \Omega$ is a given point. We will now prove that this solution is $u = u_0$.

From equation (2.1), and from using integration by parts and Poincaré's inequality, we obtain the following estimate for $\|Du\|_0^2$:

$$\begin{aligned}
\|Du\|_0^2 &= (-\Delta u, u - \frac{1}{|\Omega|}\int_\Omega u d\mathbf{x}) \\
&= (f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)d\mathbf{x}, u - \frac{1}{|\Omega|}\int_\Omega u d\mathbf{x}) \\
&\le \|f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)d\mathbf{x}\|_0 \|u - \frac{1}{|\Omega|}\int_\Omega u d\mathbf{x}\|_0 \\
&\le (C_0)^2 \|Df(u)\|_0 \|Du\|_0
\end{aligned} \tag{2.2}$$

where we used Poincaré's inequality to obtain $\|u - \frac{1}{|\Omega|}\int_\Omega u d\mathbf{x}\|_0 \le C_0\|Du\|_0$ and $\|f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)d\mathbf{x}\|_0 \le C_0\|Df(u)\|_0$.

From (2.2) we obtain the inequality

$$\begin{aligned} \|Du\|_0^2 &\leq (C_0)^4 \|Df(u)\|_0^2 \\ &\leq (C_0)^4 \Big|\frac{df}{du}\Big|^2_{L^\infty(\Omega)} \|Du\|_0^2 \\ &\leq (C_0)^4 \Big|\frac{df}{du}\Big|^2_{0,\overline{G}_1} \|Du\|_0^2 \end{aligned} \tag{2.3}$$

where we used the assumption that $u(\mathbf{x}) \in \overline{G}_1$ for all $\mathbf{x} \in \Omega$, and so it follows that $|\frac{df}{du}|_{L^\infty(\Omega)} \leq |\frac{df}{du}|_{0,\overline{G}_1}$, where $|\frac{df}{du}|_{0,\overline{G}_1} = \max\{|\frac{df}{du}(u_*)| : u_* \in \overline{G}_1\}$.

Since $\Big|\frac{df}{du}\Big|^2_{0,\overline{G}_1} < \frac{1}{(C_0)^4}$, it follows from (2.3) that $\|Du\|_0 = 0$ and so the solution u of equation (2.1) is a constant. Therefore the solution $u = u_0$ is the unique classical solution of equation (2.1) in $\overline{G}_1$ which satisfies the condition $u(\mathbf{x}_0) = u_0$. This completes the proof of Case 1 in the statement of Theorem 2.1.

Next, we consider Case 2 in the statement of Theorem 2.1. We now prove that if $\frac{df}{du}(u_0)$ is a simple eigenvalue of $-\Delta$ then there exists a solution u of equation (2.1) which is not the constant solution u_0. We remark that this solution u may not necessarily satisfy the condition that $u(\mathbf{x}_0) = u_0$, where $\mathbf{x}_0 \in \Omega$ is a given point.

We begin by letting $v = u - u_0$ and write equation (2.1) equivalently as

$$\begin{aligned} -\Delta v &= -\Delta u = f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)d\mathbf{x} \\ &= (f(u) - f(u_0)) - \frac{1}{|\Omega|}\int_\Omega (f(u) - f(u_0))d\mathbf{x} \\ &= \Big(\frac{df}{du}(u_0 + t_1(u-u_0))\Big)(u-u_0) - \frac{1}{|\Omega|}\int_\Omega \Big(\frac{df}{du}(u_0 + t_1(u-u_0))\Big)\Big(u-u_0\Big)d\mathbf{x} \\ &= \Big(\frac{df}{du}(u_0 + t_1 v)\Big)v - \frac{1}{|\Omega|}\int_\Omega \Big(\frac{df}{du}(u_0 + t_1 v)\Big)v d\mathbf{x} \end{aligned} \tag{2.4}$$

where $t_1 \in (0,1)$. Here we used the mean value theorem.

We next obtain the identity

$$\begin{aligned} \frac{df}{du}(u_0 + t_1 v) &= \frac{df}{du}(u_0 + t_1 v) - \frac{df}{du}(u_0) + \frac{df}{du}(u_0) \\ &= \Big(\frac{d^2 f}{du^2}(u_0 + t_2(t_1 v))\Big)t_1 v + \frac{df}{du}(u_0) \end{aligned} \tag{2.5}$$

where $t_2 \in (0,1)$. And we again used the mean value theorem.

Substituting (2.5) into (2.4) yields

$$-\Delta v = \frac{df}{du}(u_0)v + \Big(\frac{d^2 f}{du^2}(u_0 + t_2(t_1 v))\Big)t_1 v^2 - \frac{1}{|\Omega|}\int_\Omega \Big(\frac{df}{du}(u_0 + t_1 v)\Big)v d\mathbf{x} \tag{2.6}$$

where $v = u - u_0$, where $t_1 \in (0,1)$, and where $t_2 \in (0,1)$.

We can write equation (2.6) in the form

$$\Delta v + \lambda v = g(v) \tag{2.7}$$

where $\lambda = \frac{df}{du}(u_0)$ and where $g(v) = -\left(\frac{d^2 f}{du^2}(u_0 + t_2(t_1 v))\right)t_1 v^2 + \frac{1}{|\Omega|}\int_\Omega \left(\frac{df}{du}(u_0 + t_1 v)\right) v d\mathbf{x}$.

Let $F(v, \lambda) = \Delta v + \lambda v - g(v)$. We will apply the the implicit function theorem to the equation $F(v, \lambda) = 0$. Note that $g(0) = 0$ and $g'(0) = 0$.

If $\lambda = \frac{df}{du}(u_0)$ is not an eigenvalue of $-\Delta$, it follows from the implicit function theorem that $v = 0$ is the only small solution to the equation $F(v, \lambda) = 0$ when $F(v, \lambda) = \Delta v + \lambda v - g(v)$ and when $g(0) = 0$ and $g'(0) = 0$ (see, e.g., [7]). Therefore $u = u_0$ is the only solution of equation (2.1) in a neighborhood of u_0.

If $\lambda = \frac{df}{du}(u_0)$ is a simple eigenvalue of $-\Delta$, it follows from the implicit function theorem that there exists a non-trivial solution v to the equation $F(v, \lambda) = 0$ when $F(v, \lambda) = \Delta v + \lambda v - g(v)$ and when $g(0) = 0$ and $g'(0) = 0$ (see, e.g., [7]). Therefore there exists a solution u to equation (2.1) which is not the constant function u_0.

This completes the proof of Theorem 2.1. □

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