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Research article

Remarks on smallness of chemotactic effect for asymptotic stability in a two-species chemotaxis system

Masaaki Mizukami *

Department of Mathematics, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

* **Correspondence:** Email: masaaki.mizukami.math@gmail.com; Tel: +81-3-5228-8183

Abstract: This paper deals with the two-species chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1-u) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1-v) & \text{in } \Omega \times (0,\infty), \\ w_t = d\Delta w + h(u,v,w) & \text{in } \Omega \times (0,\infty), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \in \mathbb{N}$; h, χ_i are functions satisfying some conditions. Global existence and asymptotic stability of solutions to the above system were established under some conditions [11]. The main purpose of the present paper is to improve smallness conditions for chemotactic effect deriving asymptotic stability and to give the convergence rate in stabilization which cannot be attained in the previous work.

Keywords: Chemotaxis; Sensitivity function; Logistic term; Asymptotic behavior; Stability

1. Introduction and the main result

Nowadays, mathematics is useful in many things, for example, physics, chemistry, biology, computer, medical, architecture, and so on (see e.g., [3,7,14]). Here we focus on biology. One of the important models in biology is the logistic equation $u_t = u(1-u)$. Some of biological models have the logistic term, e.g., the Fisher–KPP equation

$$u_t = \Delta u + u(1-u).$$

On the other hand, many mathematicians study a chemotaxis system lately, which describes a part of the life cycle of cellular slime molds with chemotaxis. After the pioneering work of Keller–Segel [8], a number of variations of the chemotaxis system are proposed and investigated (see e.g., [2,4,5]). Also, multi-species chemotaxis systems have been studied by e.g., [6,15]. In this paper we focus on a two-species chemotaxis system with logistic term which describes a situation in which multi populations react on a single chemoattractant.

We consider the two-species chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1-u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1-v), & x \in \Omega,\ t > 0, \\ w_t = d\Delta w + h(u,v,w), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = \dfrac{\partial w}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$) with smooth boundary $\partial\Omega$ and n denotes the unit outer normal vector of $\partial\Omega$. The initial data u_0, v_0 and w_0 are assumed to be nonnegative functions. The unknown functions $u(x,t)$ and $v(x,t)$ represent the population densities of two species and $w(x,t)$ shows the concentration of the substance at place x and time t.

In mathematical view, global existence and behavior of solutions are fundamental theme. Recently, Negreanu–Tello [12,13] built a technical way to prove global existence and asymptotic behavior of solutions to (1.1). In [13] they dealt with (1.1) when $d = 0$, $\mu_i > 0$ under the condition

$$\exists\, \overline{w} \geq w_0;\ h(\overline{u}, \overline{v}, \overline{w}) \leq 0,$$

where $\overline{u}$, $\overline{v}$ satisfy some representations determined by $\overline{w}$. In [12] they studied (1.1) when $0 < d < 1$, $\mu_i = 0$ under similar conditions as in [13] and

$$\chi_i' + \frac{1}{1-d}\chi_i^2 \leq 0 \quad (i = 1,2). \tag{1.2}$$

They supposed in [12,13] that the functions h, χ_i for $i = 1,2$ generalize of the prototypical case $\chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}}$ ($K_i > 0, \sigma_i \geq 1$), $h(u,v,w) = u+v-w$. These days, the restriction of $0 \leq d < 1$ for global existence is completely removed and asymptotic stability of solutions to (1.1) is established for the first time under a smallness condition for the function χ_i generalizing of $\chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}}$ ($K_i > 0, \sigma_i > 1$) ([11]).

The purpose of this paper is to improve a way in [11] for obtaining asymptotic stability of solutions to (1.1) under a more general and sharp smallness condition for the sensitivity function $\chi_i(w)$. We shall suppose throughout this paper that h, χ_i ($i = 1,2$) satisfy the following conditions:

$$\chi_i \in C^{1+\omega}([0,\infty)) \cap L^1(0,\infty)\ (0 < \exists\,\omega < 1), \quad \chi_i > 0 \quad (i = 1,2), \tag{1.3}$$

$$h \in C^1([0,\infty) \times [0,\infty) \times [0,\infty)), \quad h(0,0,0) \geq 0, \tag{1.4}$$

$$\exists\,\gamma > 0;\ \frac{\partial h}{\partial u}(u,v,w) \geq 0, \quad \frac{\partial h}{\partial v}(u,v,w) \geq 0, \quad \frac{\partial h}{\partial w}(u,v,w) \leq -\gamma, \tag{1.5}$$

$$\exists\,\delta > 0,\ \exists\, M > 0;\ |h(u,v,w) + \delta w| \leq M(u+v+1), \tag{1.6}$$

$$\exists\, k_i > 0;\ -\chi_i(w)h(0,0,w) \leq k_i \quad (i = 1,2). \tag{1.7}$$

We also assume that

$$\exists\, p > N;\ 2d\chi_i'(w) + \left((d-1)p + \sqrt{(d-1)^2p^2 + 4dp}\right)[\chi_i(w)]^2 \le 0 \quad (i = 1, 2). \tag{1.8}$$

The above conditions cover the prototypical example $\chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}}$ ($K_i > 0$, $\sigma_i > 1$), $h(u, v, w) = u + v - w$. We assume that the initial data u_0, v_0, w_0 satisfy

$$0 \le u_0 \in C(\overline{\Omega}) \setminus \{0\},\ 0 \le v_0 \in C(\overline{\Omega}) \setminus \{0\},\ 0 \le w_0 \in W^{1,q}(\Omega)\ (\exists\, q > N). \tag{1.9}$$

The following result which is concerned with global existence and boundedness in (1.1) was established in [11].

Theorem 1.1 ([11, Theorem 1.1])**.** *Let $d \ge 0$, $\mu_i > 0$ $(i = 1, 2)$. Assume that h, χ_i satisfy* (1.3)–(1.8). *Then for any u_0, v_0, w_0 satisfying* (1.9) *for some $q > N$, there exists an exactly one pair (u, v, w) of nonnegative functions*

$$u,\ v,\ w \in C(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \quad \text{when } d > 0,$$
$$u,\ v,\ w \in C([0,\infty); W^{1,q}(\Omega)) \cap C^1((0,\infty); W^{1,q}(\Omega)) \quad \text{when } d = 0,$$

which satisfy (1.1). *Moreover, the solution (u, v, w) is uniformly bounded, i.e., there exists a constant $C_1 > 0$ such that*

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \le C_1 \quad \text{for all } t \ge 0.$$

Since Theorem 1.1 guarantees that u, v and w exist globally and are bounded and nonnegative, it is possible to define nonnegative numbers α, β by

$$\alpha := \max_{(u,v,w)\in I} h_u(u, v, w), \qquad \beta := \max_{(u,v,w)\in I} h_v(u, v, w), \tag{1.10}$$

where $I = (0, C_1)^3$ and C_1 is defined in Theorem 1.1.

Now the main result reads as follows. The main theorem is concerned with asymptotic stability in (1.1).

Theorem 1.2. *Let $d > 0$, $\mu_i > 0$ $(i = 1, 2)$. Under the conditions* (1.3)–(1.9) *and*

$$\alpha > 0, \quad \beta > 0, \qquad \chi_1(0)^2 < \frac{16\mu_1 d\gamma}{\alpha^2 + \beta^2 + 2\alpha\beta}, \quad \chi_2(0)^2 < \frac{16\mu_2 d\gamma}{\alpha^2 + \beta^2 + 2\alpha\beta}, \tag{1.11}$$

the unique global solution (u, v, w) of (1.1) *satisfies that there exist $C > 0$ and $\lambda > 0$ such that*

$$\|u(t) - 1\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \widetilde{w}\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \quad (t > 0),$$

where $\widetilde{w} \ge 0$ such that $h(1, 1, \widetilde{w}) = 0$.

Remark 1.1. This result improves the previous result [11, Theorem 1.2]. Indeed, the condition (1.11) is sharper than "$\chi_i(0)$ are suitably small" assumed in [11]. Moreover, this result attains to show the convergence rate which cannot be given in [11].

Remark 1.2. From (1.4)–(1.6) there exists $\widetilde{w} \geq 0$ such that $h(1,1,\widetilde{w}) = 0$. Indeed, if we choose $\overline{w} \geq 3M/\delta$, then (1.6) yields that $h(1,1,\overline{w}) \leq 3M - \delta\overline{w} \leq 0$. On the other hand, (1.4) and (1.5) imply that $h(1,1,0) \geq h(0,0,0) \geq 0$. Hence, by the intermediate value theorem there exists $\widetilde{w} \geq 0$ such that $h(1,1,\widetilde{w}) = 0$.

The strategy for the proof of Theorem 1.2 is to modify an argument in [10]. The key for this strategy is to construct the following energy estimate which was not given in [11]:

$$\frac{d}{dt}E(t) \leq -\varepsilon\left(\int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2\right)$$

with some function $E(t) \geq 0$ and some $\varepsilon > 0$. This strategy enables us to improve the conditions assumed in [11].

2. Proof of the main result

In this section we will establish asymptotic stability of solutions to (1.1). For the proof of Theorem 1.2, we shall prepare some elementary results.

Lemma 2.1 ([1, Lemma 3.1])**.** *Suppose that $f : (1,\infty) \to \mathbb{R}$ is a uniformly continuous nonnegative function satisfying $\int_1^\infty f(t)\,dt < \infty$. Then $f(t) \to 0$ as $t \to \infty$.*

Lemma 2.2. *Let $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$. Suppose that*

$$a_1 > 0, \quad a_3 > 0, \quad a_5 - \frac{a_2^2}{4a_1} - \frac{a_4^2}{4a_3} > 0. \tag{2.1}$$

Then

$$a_1x^2 + a_2xz + a_3y^2 + a_4yz + a_5z^2 \geq 0 \tag{2.2}$$

holds for all $x, y, z \in \mathbb{R}$.

Proof. From straightforward calculations we obtain

$$a_1x^2 + a_2xz + a_3y^2 + a_4yz + a_5z^2 = a_1\left(x + \frac{a_2z}{2a_1}\right)^2 + a_3\left(y + \frac{a_4z}{2a_3}\right)^2 + \left(a_5 - \frac{a_2^2}{4a_1} - \frac{a_4^2}{4a_3}\right)z^2.$$

In view of the above equation, (2.1) leads to (2.2). □

Now we will prove the key estimate for the proof of Theorem 1.2.

Lemma 2.3. *Let (u, v, w) be a solution to* (1.1). *Under the conditions* (1.3)–(1.9) *and* (1.11), *there exist $\delta_1, \delta_2 > 0$ and $\varepsilon > 0$ such that the nonnegative functions E_1 and F_1 defined by*

$$E_1(t) := \int_\Omega (u - 1 - \log u) + \delta_1\frac{\mu_1}{\mu_2}\int_\Omega (v - 1 - \log v) + \frac{\delta_2}{2}\int_\Omega (w-\widetilde{w})^2$$

and

$$F_1(t) := \int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2$$

satisfy

$$\frac{d}{dt}E_1(t) \le -\varepsilon F_1(t) \qquad (t > 0). \tag{2.3}$$

Proof. Thanks to (1.11), we can choose $\delta_1 = \frac{\beta}{\alpha} > 0$ and $\delta_2 > 0$ satisfying

$$\max\left\{\frac{\chi_1(0)^2(1+\delta_1)}{4d}, \frac{\mu_1\chi_2(0)^2(1+\delta_1)}{4\mu_2 d}\right\} < \delta_2 < \frac{4\mu_1\gamma\delta_1}{\alpha^2\delta_1 + \beta^2}. \tag{2.4}$$

We denote by $A_1(t)$, $B_1(t)$, $C_1(t)$ the functions defined as

$$A_1(t) := \int_\Omega (u - 1 - \log u), \quad B_1(t) = \int_\Omega (v - 1 - \log v), \quad C_1(t) := \frac{1}{2}\int_\Omega (w - \widetilde{w})^2,$$

and we write as

$$E_1(t) = A_1(t) + \delta_1\frac{\mu_1}{\mu_2}B_1(t) + \delta_2 C_1(t).$$

The Taylor formula applied to $H(s) = s - \log s$ $(s \ge 0)$ yields $A_1(t) = \int_\Omega (H(u) - H(1))$ is a nonnegative function for $t > 0$ (more detail, see [1, Lemma 3.2]). Similarly, we have that $B_1(t)$ is a positive function. By straightforward calculations we infer

$$\begin{aligned}
\frac{d}{dt}A_1(t) &= -\mu_1\int_\Omega (u-1)^2 - \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega \frac{\chi_1(w)}{u}\nabla u\cdot\nabla w,\\
\frac{d}{dt}B_1(t) &= -\mu_2\int_\Omega (v-1)^2 - \int_\Omega \frac{|\nabla v|^2}{v^2} + \int_\Omega \frac{\chi_2(w)}{v}\nabla v\cdot\nabla w,\\
\frac{d}{dt}C_1(t) &= \int_\Omega h_u(u-1)(w-\widetilde{w}) + \int_\Omega h_v(v-1)(w-\widetilde{w}) + \int_\Omega h_w(w-\widetilde{w})^2 - d\int_\Omega |\nabla w|^2
\end{aligned}$$

with some derivatives h_u, h_v and h_w. Hence we have

$$\frac{d}{dt}E_1(t) = I_1(t) + I_2(t), \tag{2.5}$$

where

$$\begin{aligned}
I_1(t) := &-\mu_1\int_\Omega (u-1)^2 - \delta_1\mu_1\int_\Omega (v-1)^2 + \delta_2\int_\Omega h_u(u-1)(w-\widetilde{w})\\
&+ \delta_2\int_\Omega h_v(v-1)(w-\widetilde{w}) + \delta_2\int_\Omega h_w(w-\widetilde{w})^2
\end{aligned}$$

and

$$\begin{aligned}
I_2(t) := &-\int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega \frac{\chi_1(w)}{u}\nabla u\cdot\nabla w - \delta_1\frac{\mu_1}{\mu_2}\int_\Omega \frac{|\nabla v|^2}{v^2}\\
&+ \delta_1\frac{\mu_1}{\mu_2}\int_\Omega \frac{\chi_2(w)}{v}\nabla v\cdot\nabla w - d\delta_2\int_\Omega |\nabla w|^2.
\end{aligned} \tag{2.6}$$

At first, we shall show from Lemma 2.2 that there exists $\varepsilon_1 > 0$ such that

$$I_1(t) \le -\varepsilon_1 \left(\int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2 \right). \tag{2.7}$$

To see this, we put

$$\begin{aligned} &g_1(\varepsilon) := \mu_1 - \varepsilon, \qquad g_2(\varepsilon) := \delta_1\mu_1 - \varepsilon, \\ &g_3(\varepsilon) := (-\delta_2 h_w - \varepsilon) - \frac{h_u^2}{4(\mu_1-\varepsilon)}\delta_2^2 - \frac{h_v^2}{4(\delta_1\mu_1-\varepsilon)}\delta_2^2. \end{aligned}$$

Since $\mu_1 > 0$ and $\delta_1 = \frac{\beta}{\alpha} > 0$, we have $g_1(0) = \mu_1 > 0$ and $g_2(0) = \delta_1\mu_1 > 0$. In light of (1.5) and the definitions of $\delta_2, \alpha, \beta > 0$ (see (1.10) and (2.4)) we obtain

$$\begin{aligned} g_3(0) &= \delta_2 \left(-h_w - \left(\frac{h_u^2}{4\mu_1} + \frac{h_v^2}{4\delta_1\mu_1} \right) \delta_2 \right) \\ &\ge \delta_2 \left(\gamma - \left(\frac{\alpha^2\delta_1 + \beta}{4\delta_1\mu_1} \right) \delta_2 \right) > 0. \end{aligned}$$

Combination of the above inequalities and the continuity of g_i for $i = 1, 2, 3$ yield that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ hold for $i = 1, 2, 3$. Thanks to Lemma 2.2 with

$$\begin{aligned} &a_1 = \mu_1 - \varepsilon_1, \quad a_2 = -\delta_2 h_u, \quad a_3 = \delta_1\mu_1 - \varepsilon_1, \\ &a_4 = -\delta_2 h_v, \quad a_5 = -\delta_2 h_w - \varepsilon_1, \\ &x = u(t) - 1, \quad y = v(t) - 1, \quad z = w(t) - \widetilde{w}, \end{aligned}$$

we obtain (2.7) with $\varepsilon_1 > 0$. Lastly we will prove

$$I_2(t) \le 0. \tag{2.8}$$

Noting that $\chi_i' < 0$ (from (1.8)) and then using the Young inequality, we have

$$\begin{aligned} \int_\Omega \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w &\le \chi_1(0) \int_\Omega \frac{|\nabla u \cdot \nabla w|}{u} \\ &\le \frac{\chi_1(0)^2(1+\delta_1)}{4d\delta_2} \int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{d\delta_2}{1+\delta_1} \int_\Omega |\nabla w|^2 \end{aligned}$$

and

$$\begin{aligned} \delta_1 \frac{\mu_1}{\mu_2} \int_\Omega \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w &\le \chi_2(0)\delta_1 \frac{\mu_1}{\mu_2} \int_\Omega \frac{|\nabla v \cdot \nabla w|}{v} \\ &\le \frac{\chi_2(0)^2\delta_1(1+\delta_1)}{4d\delta_2} \left(\frac{\mu_1}{\mu_2} \right)^2 \int_\Omega \frac{|\nabla v|^2}{v^2} + \frac{d\delta_1\delta_2}{1+\delta_1} \int_\Omega |\nabla w|^2. \end{aligned}$$

Plugging these into (2.6) we infer

$$I_2(t) \le -\left(1 - \frac{\chi_1(0)^2(1+\delta_1)}{4d\delta_2}\right) \int_\Omega \frac{|\nabla u|^2}{u^2} - \delta_1 \frac{\mu_1}{\mu_2} \left(1 - \frac{\mu_1\chi_2(0)^2(1+\delta_1)}{4d\mu_2\delta_2}\right) \int_\Omega \frac{|\nabla v|^2}{v^2}.$$

We note from the definition of $\delta_2 > 0$ that

$$1 - \frac{\chi_1(0)^2(1+\delta_1)}{4d\delta_2} > 0, \qquad 1 - \frac{\mu_1\chi_2(0)^2(1+\delta_1)}{4d\mu_2\delta_2} > 0.$$

Thus we have (2.8). Combination of (2.5), (2.7) and (2.8) implies the end of the proof. □

Lemma 2.4. *Let* (u, v, w) *be a solution to* (1.1). *Under the conditions* (1.3)–(1.9) *and* (1.11), (u, v, w) *has the following asymptotic behavior*:

$$\|u(t) - 1\|_{L^\infty(\Omega)} \to 0, \quad \|v(t) - 1\|_{L^\infty(\Omega)} \to 0, \quad \|w(t) - \widetilde{w}\|_{L^\infty(\Omega)} \to 0 \quad (t \to \infty).$$

Proof. Firstly the boundedness of u, v, ∇w and a standard parabolic regularity theory ([9]) yield that there exist $\theta \in (0, 1)$ and $C > 0$ such that

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[1,t])} + \|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[1,t])} + \|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[1,t])} \le C \quad \text{for all } t \ge 1.$$

Therefore in view of the Gagliardo–Nirenberg inequality

$$\|\varphi\|_{L^\infty(\Omega)} \le c\|\varphi\|_{W^{1,\infty}(\Omega)}^{\frac{N}{N+2}}\|\varphi\|_{L^2(\Omega)}^{\frac{2}{N+2}} \quad (\varphi \in W^{1,\infty}(\Omega)), \tag{2.9}$$

it is sufficient to show that

$$\|u(t) - 1\|_{L^2(\Omega)} \to 0, \quad \|v(t) - 1\|_{L^2(\Omega)} \to 0, \quad \|w(t) - \widetilde{w}\|_{L^2(\Omega)} \to 0 \quad (t \to \infty).$$

We let

$$f_1(t) := \int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2.$$

We have that $f_1(t)$ is a nonnegative function, and thanks to the regularity of u, v, w we can see that $f_1(t)$ is uniformly continuous. Moreover, integrating (2.3) over $(1, \infty)$, we infer from the positivity of $E_1(t)$ that

$$\int_1^\infty f_1(t)\, dt \le \frac{1}{\varepsilon}E_1(1) < \infty.$$

Therefore we conclude from Lemma 2.1 that $f_1(t) \to 0$ $(t \to \infty)$, which means

$$\int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2 \to 0 \quad (t \to \infty).$$

This implies the end of the proof. □

Lemma 2.5. *Let* (u, v, w) *be a solution to* (1.1). *Under the conditions* (1.3)–(1.9) *and* (1.11), *there exist* $C > 0$ *and* $\lambda > 0$ *such that*

$$\|u(t) - 1\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \widetilde{w}\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \quad (t > 0).$$

Proof. From the L'Hôpital theorem applied to $H(s) := s - \log s$ we can see

$$\lim_{s\to 1}\frac{H(s)-H(1)}{(s-1)^2} = \lim_{s\to 1}\frac{H''(s)}{2} = \frac{1}{2}. \tag{2.10}$$

In view of the combination of (2.10) and $\|u-1\|_{L^\infty(\Omega)} \to 0$ from Lemma 2.4 we obtain that there exists $t_0 > 0$ such that

$$\frac{1}{4}\int_\Omega (u-1)^2 \le A_1(t) = \int_\Omega (H(u)-H(1)) \le \int_\Omega (u-1)^2 \quad (t > t_0). \tag{2.11}$$

A similar argument, for the function v, yields that there exists $t_1 > t_0$ such that

$$\frac{1}{4}\int_\Omega (v-1)^2 \le B_1(t) \le \int_\Omega (v-1)^2 \quad (t > t_1). \tag{2.12}$$

We infer from (2.11) and the definitions of $E_1(t)$, $F_1(t)$ that

$$E_1(t) \le c_6 F_1(t)$$

for all $t > t_1$ with some $c_6 > 0$. Plugging this into (2.3), we have

$$\frac{d}{dt}E_1(t) \le -\varepsilon F_1(t) \le -\frac{\varepsilon}{c_6}E_1(t) \qquad (t > t_1),$$

which implies that there exist $c_7 > 0$ and $\ell > 0$ such that

$$E_1(t) \le c_7 e^{-\ell t} \qquad (t > t_1).$$

Thus we obtain from (2.11) and (2.12) that

$$\int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2 \le c_8 E_1(t) \le c_7 c_8 e^{-\ell t}$$

for all $t > t_1$ with some $c_8 > 0$. From the Gagliardo–Nirenberg inequality (2.9) with the regularity of u, v, w, we achieve that there exist $C > 0$ and $\lambda > 0$ such that

$$\|u(t)-1\|_{L^\infty(\Omega)} + \|v(t)-1\|_{L^\infty(\Omega)} + \|w(t)-\widetilde{w}\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \quad (t > 0).$$

This completes the proof of Lemma 2.5. □

Proof of Theorem 1.2. Theorem 1.2 follows directly from Lemma 2.5. □

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