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Research article

Existence and uniqueness of solutions for a conserved phase-field type model

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Abstract: In this paper, we study the existence and the uniqueness of solutions of a conserved phase-field model in a bounded and smooth domain.

Keywords: Conserved phase-field model; Dirichlet boundary conditions; well-posedness

1. Introduction

The Caginalp phase-field system,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = T, \tag{1.1}$$

$$\frac{\partial T}{\partial t} - \Delta T = -\frac{\partial u}{\partial t}, \tag{1.2}$$

has been proposed in [13] to model phase transition phenomena, such as melting-solidification phenomena. Here u is the order parameter, T is the relative temperature and f is the derivative of a double-well potential F. Furthermore, here and below, we set all physical parameters equal to one. This system has been much studied; we refer the reader to, e.g., [7] and [15].

These equations can be derived as follows. One introduces the (total Ginzburg-Landau) free energy

$$\Psi = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u) - uT - \frac{1}{2}T^2)\mathrm{d}x, \tag{1.3}$$

where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^n$, $n = 2$ or 3, with boundary Γ), and the enthalpy

$$H = u + T. \tag{1.4}$$

As far as the evolution equation for the order parameter is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial t} = -\frac{D\Psi}{Du}, \tag{1.5}$$

where $\frac{D}{Du}$ denotes a variational derivative with respect to u, which yields (1.1). Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -divq, \tag{1.6}$$

where q is the heat flux. Assuming finally the usual Fourier law for heat condition,

$$q = -\nabla T, \tag{1.7}$$

we obtain (1.2).

Now, one essential drawback of the Fourier law is that it predicts that thermal signals propagate at an infinite speed, which violates causality (the so-called paradox of heat conduction). To overcome this drawback, or at least to account for more realistic features, several alternatives to the Fourier law, based, e.g., on the Maxwell-Cattaneo law or recent laws from thermomechanics, have been proposed and studied, in the context of the Caginalp phase-field system, in [8], [9] and [12].

In the late 1960's, several authors proposed a heat conduction theory based on two temperatures (see [16]). More precisely, one now considers the conductive temperature T and the thermodynamic temperature θ. In particular, for simple materials, these two temperatures are shown to coincide. However, for non-simple materials, they differ and are related as follows:

$$\theta = T - \Delta T. \tag{1.8}$$

The Caginalp system, based on this two temperatures theory and the usual Fourier law, was studied in [10] and [17].

Our aim in this paper is to study a variant of the Caginalp phase-field system based on the type III thermomechanics theory (see [4]) with two temperatures recently proposed in [10].

In that case, the free energy reads, in terms of the (relative) thermodynamic temperature θ,

$$\Psi = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2)\mathrm{d}x \tag{1.9}$$

and (1.5)yields, in view of (1.8), the following evolution equation for the order parameter:

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = T - \Delta T. \tag{1.10}$$

Furthermore, the enthalpy now reads

$$H = u + \theta = u + T - \Delta T, \tag{1.11}$$

which yields, owing to (1.6), the energy equation

$$\frac{\partial T}{\partial t} - \Delta\frac{\partial T}{\partial t} + divq = -\frac{\partial u}{\partial t}. \tag{1.12}$$

Finally, the heat flux is given, in the type III theory with two temperatures, by (see [14])

$$q = -\nabla\alpha - \nabla T, \tag{1.13}$$

where

$$\alpha(t,x)=\int_0^t T(\tau,x)\mathrm{d}\tau+\alpha_0(x) \tag{1.14}$$

is the conductive thermal displacement. Noting that $T=\frac{\partial\alpha}{\partial t}$, we finally deduce from (1.10) and (1.12) − (1.13) the following variant of the Caginalp phase-field system (see [17]):

$$\frac{\partial u}{\partial t}-\Delta u+f(u)=\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}, \tag{1.15}$$

$$\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}. \tag{1.16}$$

In this paper, we consider the following conserved phase-field model:

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=-\Delta(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}), \tag{1.17}$$

$$\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}. \tag{1.18}$$

These equations are know as the conserved phase-field model (see [1], [2], [3] and [5]) based on type III heat conduction and with two temperatures (see [8]), conservative in the sens that, when endowed with Neumann boundary conditions, the spatial average of the order parameter is a conserved quantity. Indeed, in that case, integrating (1.17) over the spatial domain Ω, we have the conservation of mass,

$$\langle u(t)\rangle=\langle u(0)\rangle, t\geqslant 0, \tag{1.19}$$

where

$$\langle .\rangle=\frac{1}{vol(\Omega)}\int_\Omega .\mathrm{d}x \tag{1.20}$$

denotes the spatial average. Furthermore, integrating (1.18) over Ω, we obtain

$$\langle\alpha(t)\rangle=\langle\alpha(0)\rangle, t\geqslant 0. \tag{1.21}$$

Our aim in this paper is to study the existence and uniqueness of solution of (1.15) − (1.16). We consider here only one type of boundary condition, namely, Dirichlet (see [6]). Furthermore, we consider regular nonlinear term f (a usual choice being the cubic term $f(s)=s^3-s$).

Notation.

We denote by $\|.\|$ the usual L^2-norm (with associated scalar product $((.,.))$) and set $\|.\|_{-1}=\|(-\Delta)^{\frac{-1}{2}}.\|$, where $-\Delta$ denotes the minus Laplace operator with Dirichlet boundary conditions. More generally, $\|.\|_X$ denotes the norm in the Banach space X.

Throughout this paper, the same letters c, c' and c'' denotes (generally positive) constants which may change from line to line, or even in a same line. Similary, the same letter Q denotes monotone increasing (with respect to each argument) functions which may change from line to line, or even in a same line.

2. Setting of the problem

We consider the following initial and boundary value problem

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right), \tag{2.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t}, \tag{2.2}$$

$$u|_\Gamma = \alpha|_\Gamma = \Delta u|_\Gamma = 0, \tag{2.3}$$

$$u|_{t=0} = u_0, \alpha|_{t=0} = \alpha_0, \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1, \tag{2.4}$$

where Γ is the boundary of the spatial domain Ω.

We make the following assumptions:

$$f \quad is \quad of \quad class \quad C^2(\mathbb{R}), \quad f(0) = 0, \tag{2.5}$$

$$f'(s) \geqslant -c_0, \quad c_0 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.6}$$

$$f(s)s \geqslant c_1 F(s) - c_2 \geqslant -c_3, \quad c_1 > 0, \quad c_2, c_3 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.7}$$

where $F(s) = \int_0^s f(\tau)d\tau$. In particular, the usual cubic nonlinear term $f(s) = s^3 - s$ satisfies these assumptions.

Remarque 2.1. *We take here, for simplicity, Dirichlet boundary conditions. However, we can obtain the same results for Neumann boundary conditions, namely,*

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = \frac{\partial \alpha}{\partial \nu} = 0 \quad on \quad \Gamma, \tag{2.8}$$

where ν denotes the unit outer normal to Γ. To do so, we rewrite, owing to (1.3) *and* (1.5)*, the equations in the form*

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\left(\frac{\partial \bar{\alpha}}{\partial t} - \Delta\frac{\partial \bar{\alpha}}{\partial t}\right) + \langle u_0 + \alpha_1\rangle - \langle u\rangle, \tag{2.9}$$

$$\frac{\partial^2 \bar{\alpha}}{\partial t^2} - \Delta\frac{\partial^2 \bar{\alpha}}{\partial t^2} - \Delta\frac{\partial \bar{\alpha}}{\partial t} - \Delta\bar{\alpha} = -\frac{\partial \bar{u}}{\partial t}, \tag{2.10}$$

where $\bar{v} = v - \langle v\rangle$, $|\langle v_0\rangle| \leqslant M_1$, $|\langle \alpha_0\rangle| \leqslant M_2$, for fixed positive constants M_1 and M_2. Then, we note that

$$v \mapsto (\|(-\Delta)^{\frac{-1}{2}}\bar{v}\|^2 + \langle v\rangle^2)^{\frac{1}{2}},$$

where, here, $-\Delta$ denotes the minus Laplace operator with Neumann boundary conditions and acting on functions with null average and where it is understood that $\langle .\rangle = \frac{1}{|\Omega|}\langle ., 1\rangle_{H^{-1}(\Omega),H^1(\Omega)}$. Furthermore,

$$v \mapsto (\|\bar{v}\|^2 + \langle v\rangle^2)^{\frac{1}{2}},$$

$$v \mapsto (\|\nabla v\|^2 + \langle v\rangle^2)^{\frac{1}{2}},$$

$$v \mapsto (\|\Delta v\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

and

$$v \mapsto (\|\nabla\Delta v\|^2 + \langle v\rangle^2)^{\frac{1}{2}},$$

are norms in $H^{-1}(\Omega), L^2(\Omega), H^1(\Omega), H^2(\Omega)$ and $H^3(\Omega)$, respectively, which are equivalent to the usual ones. We further assume that

$$|f(s)| \leqslant \epsilon F(s) + c_\epsilon,\ s \in \mathbb{R}, \tag{2.11}$$

which allows to deal with term $\langle f(u)\rangle$.

3. A priori estimates

In what follows, the Poincaré, Holder and Young inegualities are extensively used, without further referring to them.

We rewrite (2.1) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}. \tag{3.1}$$

We multiply (3.1) by $\frac{\partial u}{\partial t}$ and have, integrating over Ω and by parts

$$\frac{d}{dt}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\mathrm{d}x\right) + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 = 2\left(\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\right). \tag{3.2}$$

We then multiply (2.2) by $\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}$ and obtain

$$\frac{d}{dt}\left(\|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2\right) + 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2$$

$$= -2\left(\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\right). \tag{3.3}$$

Summing (3.2) and (3.3), we find the differential egality

$$\frac{dE_1}{dt} + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2 = 0, \tag{3.4}$$

where

$$E_1 = \|\nabla u\|^2 + 2\int_\Omega F(u)\mathrm{d}x + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2, \tag{3.5}$$

satisfies

$$E_1 \geqslant c\left(\|u\|_{H^1(\Omega)}^2 + \|\alpha\|_{H^2(\Omega)}^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^2(\Omega)}^2\right) - c',\ c > 0 \tag{3.6}$$

(note indeed that $\|\frac{\partial \alpha}{\partial t}\|^2 + 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2 = \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2$).

We now multiply (3.1) by u and have, owing to (2.7), where $\|u\|^2 \leqslant c\|\nabla u\|^2$,

$$\frac{d}{dt}\|u\|_{-1}^2 + c(\|\nabla u\|^2 + \int_\Omega F(u)\mathrm{d}x) \leqslant \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2. \tag{3.7}$$

Multiplying (2.2) by $-\Delta\alpha$, we then obtain

$$\frac{d}{dt}\left(\|\Delta\alpha\|^2 - 2\left(\left(\frac{\partial \alpha}{\partial t}, \Delta\alpha\right)\right) + 2\left(\left(\Delta\frac{\partial \alpha}{\partial t}, \Delta\alpha\right)\right)\right) + \|\Delta\alpha\|^2$$
$$\leqslant \|\frac{\partial u}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2. \tag{3.8}$$

Summing (3.4), δ_1(3.7) and δ_2(3.8), where δ_1, $\delta_2 > 0$, are small enough, we find a differential ineguality of the form

$$\frac{dE_2}{dt} + c(E_2 + \|u\|_{H^2(\Omega)}^2 + \|\frac{\partial u}{\partial t}\|_{-1}^2) \leqslant c', c > 0, \tag{3.9}$$

where

$$E_2 = E_1 + \delta_1\|u\|_{-1}^2 + \delta_2\left(\|\Delta\alpha\|^2 - 2\left(\left(\frac{\partial \alpha}{\partial t}, \Delta\alpha\right)\right) + 2\left(\left(\Delta\frac{\partial \alpha}{\partial t}, \Delta\alpha\right)\right)\right), \tag{3.10}$$

satisfies

$$E_2 \geqslant c\left(\|u\|_{H^1(\Omega)}^2 + \|\alpha\|_{H^2(\Omega)}^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^2(\Omega)}^2\right) - c'. \tag{3.11}$$

We multiply (2.1) by $\dfrac{\partial u}{\partial t}$ to find

$$\frac{d}{dt}\|\Delta u\|^2 + 2\|\frac{\partial u}{\partial t}\|^2 = \left(\left(\Delta f(u), \frac{\partial u}{\partial t}\right)\right) - 2\left(\left(\Delta\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\right),$$

which yields, owing to (2.5) and the continuous embedding $H^2(\Omega) \subset C(\bar{\Omega})$,

$$\frac{d}{dt}\|\Delta u\|^2 + \|\frac{\partial u}{\partial t}\|^2 \leqslant Q(\|u\|_{H^2(\Omega)}) - 2\left(\left(\Delta\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\right). \tag{3.12}$$

Multiply also (2.2) by $-\Delta(\dfrac{\partial \alpha}{\partial t} - \Delta\dfrac{\partial \alpha}{\partial t})$ to have

$$\frac{d}{dt}\left(\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2\right) + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + 2\|\nabla\Delta\frac{\partial \alpha}{\partial t}\|^2$$
$$= 2\left(\left(\Delta\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\right). \tag{3.13}$$

Summing then (3.12) and (3.13), we obtain

$$\frac{d}{dt}\left(\|\Delta u\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2\right)$$
$$+ \|\frac{\partial u}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + 2\|\nabla\Delta\frac{\partial \alpha}{\partial t}\|^2 \leqslant Q(\|u\|_{H^2(\Omega)}). \tag{3.14}$$

In particular, setting

$$y = \|\Delta u\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + 2\|\Delta \frac{\partial \alpha}{\partial t}\|^2 + \|\nabla \Delta \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \alpha\|^2 + \|\nabla \Delta \alpha\|^2, \tag{3.15}$$

we deduce from (3.14) an inequation of the form

$$y' \leqslant Q(y). \tag{3.16}$$

Let z be the solution to the ordinary differential equation

$$z' = Q(z), z(0) = y(0). \tag{3.17}$$

If follows from the comparison principle that there exists $T_0 = T_0(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, \|\alpha_1\|_{H^3(\Omega)})$ belonging to, say, $\left(0, \frac{1}{2}\right)$ such that

$$y(t) \leqslant z(t), \forall t \in [0, T_0] \tag{3.18}$$

hence

$$\|u(t)\|_{H^2(\Omega)} + \|\alpha(t)\|_{H^3(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|_{H^3(\Omega)} \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}), t \leqslant T_0. \tag{3.19}$$

We now differentiate (3.1) with respect to time and have, noting that

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} = \Delta \frac{\partial \alpha}{\partial t} + \Delta \alpha - \frac{\partial u}{\partial t},$$

the equation

$$(-\Delta)^{-1} \frac{\partial}{\partial t} \frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t} + f'(u) \frac{\partial u}{\partial t} = \Delta \frac{\partial \alpha}{\partial t} + \Delta \alpha - \frac{\partial u}{\partial t}. \tag{3.20}$$

We multiply (3.20) by $t\frac{\partial u}{\partial t}$ and find, owing to (2.6)

$$\frac{d}{dt}\left(t\|\frac{\partial u}{\partial t}\|_{-1}^2\right) + \frac{3}{2} t \|\nabla \frac{\partial u}{\partial t}\|^2 \leqslant ct(\|\frac{\partial u}{\partial t}\|^2 + \|\nabla \alpha\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2) + \|\frac{\partial u}{\partial t}\|_{-1}^2,$$

hence, noting that $\|\frac{\partial u}{\partial t}\|^2 \leqslant c\|\frac{\partial u}{\partial t}\|_{-1}\|\nabla \frac{\partial u}{\partial t}\|$,

$$\frac{d}{dt}\left(t\|\frac{\partial u}{\partial t}\|_{-1}^2\right) + t\|\nabla \frac{\partial u}{\partial t}\|^2 \leqslant ct(\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\nabla \alpha\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2) + \|\frac{\partial u}{\partial t}\|_{-1}^2. \tag{3.21}$$

Returning to (3.9) we have in particular

$$\frac{d}{dt} E_2 + cE_2 \leqslant c'. \tag{3.22}$$

If thus follows form (3.22) and Gronwall's lemma that

$$E_2(t) \leqslant ce^{-c't}(\|u_0\|_{H^1(\Omega)}^2 + \|\alpha_0\|_{H^2(\Omega)}^2 + \|\alpha_1\|_{H^2(\Omega)}^2 + \int_\Omega F(u_0)\mathrm{d}x) + c'', c' > 0, t \geqslant 0. \tag{3.23}$$

Furthermore

$$\int_t^{t+1}(\|\frac{\partial u}{\partial t}\|_{-1}^2+\|u\|_{H^2(\Omega)}^2)\mathrm{d}\tau$$

$$\leqslant ce^{-c't}(\|u_0\|_{H^1(\Omega)}^2+\|\alpha_0\|_{H^2(\Omega)}^2+\|\alpha_1\|_{H^2(\Omega)}^2+\int_\Omega F(u_0)\mathrm{d}x)+c'',c'>0,t\geqslant 0. \tag{3.24}$$

Finally, more generally, for every $r>0$, we have

$$\int_t^{t+r}(\|\frac{\partial u}{\partial t}\|_{-1}^2+\|u\|_{H^2(\Omega)}^2)\mathrm{d}\tau$$

$$\leqslant ce^{-c't}(\|u_0\|_{H^1(\Omega)}^2+\|\alpha_0\|_{H^2(\Omega)}^2+\|\alpha_1\|_{H^2(\Omega)}^2+\int_\Omega F(u_0)\mathrm{d}x)+c''(r),c'>0,t\geqslant 0. \tag{3.25}$$

In particular, we deduce from (3.19), (3.21), (3.24) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2\leqslant\frac{1}{t}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^3(\Omega)},\|\alpha_1\|_{H^3(\Omega)}),t\in(0,T_0]\,. \tag{3.26}$$

Mutiplying then (3.20) by $\frac{\partial u}{\partial t}$ we have, proceeding as above,

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}(t)\|_{-1}^2+\|\nabla\frac{\partial u}{\partial t}\|^2\leqslant c(\|\frac{\partial u}{\partial t}\|_{-1}^2+\|\nabla\alpha\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2). \tag{3.27}$$

If thus follows form (3.24), (3.27) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2\leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^3(\Omega)},\|\alpha_1\|_{H^3(\Omega)})\|\frac{\partial u}{\partial t}(T_0)\|_{-1}^2,t\geqslant T_0, \tag{3.28}$$

hence, owing to (3.26),

$$\|\frac{\partial u}{\partial t}(t)\|_{-1}^2\leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^3(\Omega)},\|\alpha_1\|_{H^3(\Omega)}),t\geqslant T_0. \tag{3.29}$$

We now rewrite (3.1) in the form

$$-\Delta u+f(u)=h_u(t),u=0\quad on\quad\Gamma,for\quad t\geqslant T_0\quad(fixed), \tag{3.30}$$

where

$$h_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t}+\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}, \tag{3.31}$$

satisfies, owing to (3.23) and (3.29),

$$\|h_u(t)\|\leqslant e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^3(\Omega)},\|\alpha_1\|_{H^3(\Omega)}),t\geqslant T_0. \tag{3.32}$$

We multiply (3.30) by u and have, owing to (2.7)

$$\|\nabla u\|^2\leqslant c\|h_u(t)\|^2+c'. \tag{3.33}$$

Then, multiplying (3.30) by $-\Delta u$, we find, owing to (2.6),

$$\|\Delta u\|^2\leqslant c(\|h_u(t)\|^2+\|\nabla u\|^2). \tag{3.34}$$

We thus deduce from (3.32) − (3.34) that

$$\|u\|_{H^2(\Omega)} \leqslant e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}), t \geqslant T_0, \tag{3.35}$$

and, thus, owing to (2.22),

$$\|u\|_{H^2(\Omega)} \leqslant e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}), t \geqslant 0. \tag{3.36}$$

Returnig to (3.13), we have

$$\frac{d}{dt}\left(\|\nabla \frac{\partial \alpha}{\partial t}\|^2 + 2\|\Delta \frac{\partial \alpha}{\partial t}\|^2 + \|\nabla \Delta \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \alpha\|^2 + \|\nabla \Delta \alpha\|^2\right)$$

$$+ c(\|\Delta \frac{\partial \alpha}{\partial t}\|^2 + \|\nabla \Delta \frac{\partial \alpha}{\partial t}\|^2) \leqslant \|\nabla \frac{\partial u}{\partial t}\|^2. \tag{3.37}$$

Noting that it follows from (3.24), (3.27) and (3.28) that

$$\int_{T_0}^{t} \|\nabla \frac{\partial u}{\partial t}\|^2 d\tau \leqslant e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}), t \geqslant T_0. \tag{3.38}$$

We finally deduce from (3.19) and (3.36) − (3.38) that

$$\|u(t)\|_{H^2(\Omega)} + \|\alpha(t)\|_{H^3(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|_{H^3(\Omega)}$$

$$\leqslant e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}), t \geqslant 0. \tag{3.39}$$

4. Existence and uniqueness of solutions

We first have the following theorem.

Theorem 4.1. *We assume that* (2.6) *and* (2.7) *hold. Then, if* $(u_0, \alpha_0, \alpha_1) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^3(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^3(\Omega) \cap H_0^1(\Omega)\right)$ *and* $F(u_0) < \infty$, (1.1) − (1.4) *possesses at last one solution* $(u, \alpha, \frac{\partial \alpha}{\partial t})$ *such that*

$$u \in L^\infty\left(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)\right), \alpha \in L^\infty\left(\mathbb{R}_+; H^3(\Omega) \cap H_0^1(\Omega)\right),$$

$$\frac{\partial u}{\partial t} \in L^2\left(0, T; H^{-1}(\Omega)\right), \frac{\partial \alpha}{\partial t} \in L^\infty\left(\mathbb{R}_+; H^3(\Omega) \cap H_0^1(\Omega)\right).$$

The proof of this theorem is based on (3.9), (3.39) and a standard Galerkin scheme.

We then have the following theorem.

Theorem 4.2. *The system* (1.1) − (1.4) *possesses a unique solution with the above regularity.*

Proof. There only remains to prove the uniqueness. Let $\left(u^{(1)},\alpha^{(1)},\frac{\partial\alpha^{(1)}}{\partial t}\right)$ and $\left(u^{(2)},\alpha^{(2)},\frac{\partial\alpha^{(2)}}{\partial t}\right)$ be two solutions to (1.1) − (1.4) with initial data $\left(u_0^{(1)},\alpha_0^{(1)},\alpha_1^{(1)}\right)$ and $\left(u_0^{(2)},\alpha_0^{(2)},\alpha_1^{(2)}\right)$, respectively. We set

$$\left(u,\alpha,\frac{\partial\alpha}{\partial t}\right)=\left(u^{(1)},\alpha^{(1)},\frac{\partial\alpha^{(1)}}{\partial t}\right)-\left(u^{(2)},\alpha^{(2)},\frac{\partial\alpha^{(2)}}{\partial t}\right)$$

and

$$(u_0,\alpha_0,\alpha_1)=\left(u_0^{(1)},\alpha_0^{(1)},\alpha_1^{(1)}\right)-\left(u_0^{(2)},\alpha_0^{(2)},\alpha_1^{(2)}\right).$$

Then, (u,α) satisfies

$$\frac{\partial u}{\partial t}+\Delta^2u-\Delta\left(f(u^{(1)})-f(u^{(2)})\right)=-\Delta\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right),\tag{4.1}$$

$$\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t},\tag{4.2}$$

$$u|_\Gamma=\alpha|_\Gamma=0=\quad on\quad \Gamma,\tag{4.3}$$

$$u|_{t=0}=u_0,\alpha|_{t=0}=\alpha_0,\frac{\partial\alpha}{\partial t}|_{t=0}=\alpha_1.\tag{4.4}$$

Multiplying (4.1) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$, we have

$$2\|\frac{\partial u}{\partial t}\|_{-1}^2+\frac{d}{dt}\|\nabla u\|^2+2\left(\left(f(u^{(1)})-f(u^{(2)}),\frac{\partial u}{\partial t}\right)\right)$$

$$=2\left(\left(\frac{\partial u}{\partial t},\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right)\right).\tag{4.5}$$

Multiplying then (4.2) by $\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}$, we obtain

$$\frac{d}{dt}\left(\|\frac{\partial\alpha}{\partial t}\|^2+2\|\nabla\frac{\partial\alpha}{\partial t}\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2+\|\nabla\alpha\|^2+\|\Delta\alpha\|^2\right)+2\|\nabla\frac{\partial\alpha}{\partial t}\|^2+2\|\Delta\frac{\partial\alpha}{\partial t}\|^2$$

$$=-2\left(\left(\frac{\partial u}{\partial t},\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right)\right).\tag{4.6}$$

Summing (4.5) and (4.6), we find

$$\frac{d}{dt}(\|\nabla u\|^2+\|\nabla\alpha\|^2+\|\Delta\alpha\|^2+\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\|^2)+2\|\frac{\partial u}{\partial t}\|_{-1}^2+2\|\nabla\frac{\partial\alpha}{\partial t}\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2$$

$$=-2\left(\left(f(u^{(1)})-f(u^{(2)}),\frac{\partial u}{\partial t}\right)\right).$$

We have

$$2\left|\left(\left(f(u^{(1)})-f(u^{(2)}),\frac{\partial u}{\partial t}\right)\right)\right|\leqslant 2\|(-\Delta)^{\frac{1}{2}}(f(u^{(1)}-f(u^{(2)})\|\|(-\Delta)^{\frac{-1}{2}}\frac{\partial u}{\partial t}\|,$$

$$\leqslant \|(-\Delta)^{\frac{1}{2}}(f(u^{(1)} - f(u^{(2)})\|^2 + \|(-\Delta)^{\frac{-1}{2}}\frac{\partial u}{\partial t}\|^2,$$

$$\leqslant Q(\|u\|^2) + \|\frac{\partial u}{\partial t}\|_{-1}^2,$$

where, here and below

$$Q = Q(\|u_0^{(1)}\|_{H^2(\Omega)}, \|\alpha_0^{(1)}\|_{H^2(\Omega)}, \|\alpha_1^{(1)}\|_{H^2(\Omega)}, \|u_0^{(2)}\|_{H^2(\Omega)}, \|\alpha_0^{(2)}\|_{H^2(\Omega)}, \|\alpha_1^{(2)}\|_{H^2(\Omega)})$$

Therefore

$$\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\|^2) + \|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\|\nabla\frac{\partial\alpha}{\partial t}\|^2 + \|\Delta\frac{\partial\alpha}{\partial t}\|^2 \leqslant Q\|u\|^2.$$

In particular,

$$\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\|^2) \leqslant Q\|u\|^2. \tag{4.7}$$

It thus follows form (4.7) and Gronwall's lemma that

$$\|u(t)\|_{H^1(\Omega)}^2 + \|\alpha(t)\|_{H^2(\Omega)}^2 + \|\frac{\partial\alpha}{\partial t}(t)\|_{H^2(\Omega)}^2$$

$$\leqslant ce^{Qt}(\|u_0\|_{H^1(\Omega)}^2 + \|\alpha_0\|_{H^1(\Omega)}^2 + \|\alpha_1\|_{H^2(\Omega)}^2), t \geqslant 0, \tag{4.8}$$

hence the uiqueness, as well as the continuous dependence with respec to the initial data. □

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