Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 1(2): 137-143
DOI:10.3934/Math.2016.2.137
Received: 7 July 2016
Accepted: 16 July 2016
Published: 22 July 2016

Research article

Quantum Rule of Angular Momentum

Tian Ma[1] and Shouhong Wang[2,*]

[1] Department of Mathematics, Sichuan University, Chengdu, P. R. China
[2] Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

[*] **Correspondence:** Email: showang@indiana.edu; Tel: +1-812-855-8350;
Fax: +1-812-855-0046.

Abstract: In this article, we establish a quantum rule of angular momentum that only fermions with spin $J = \frac{1}{2}$ and bosons with $J = 0$ can rotate around a center with zero moment of force, and particles with $J \neq 0, \frac{1}{2}$ will move on a straight line unless there is a nonzero moment of force present.

Keywords: angular momentum; spin; fermions; bosons; Dirac equations; Klein Gordon equation

1. Main Result

Quantum physics is the study of the behavior of matter and energy at molecular, atomic, nuclear, and sub-atomic levels. Quantum physics was initiated and developed in the first half of the 20th century, following the pioneering work [7] on blackbody radiation, of [2] on photons and energy quanta, Niels Bohr on structure of atoms, and [1] on matter-wave duality. Quantum physics and the Einstein theory of general relativity have become two cornerstones of modern physics.

Quantum mechanics is then naturally developed, and its core ingredients include 1) the wave function description of the physical reality and the Bohm probability interpretation of the wave function, 2) the wave equations, dictating the dynamic behavior of the particles, 3) the Heisenberg uncertainty relation, and 4) the Pauli exclusion principle.

The main objective of this paper is to derive another core ingredient of quantum mechanics, in addition to the Heisenberg uncertainty relation and the Pauli exclusion principle:

Quantum Rule of Angular Momentum. *Only fermions with spin $J = \frac{1}{2}$ and bosons with $J = 0$ can rotate around a center with zero moment of force, and particles with $J \neq 0, \frac{1}{2}$ will move on a straight line unless there is a nonzero moment of force present.*

This quantum rule is very useful in describing how elementary particles are binding together forming subatomic particles. This rule offers a mathematical and physical foundation of the weakton model

[5] and the theory of mediator cloud structure of charged leptons and quarks, and explains why all stable fermions with mediator clouds possess spin $J = \frac{1}{2}$.

This article is part of a research program initiated in the last few years by the authors to derive experimentally verifiable laws of Nature based only on a few fundamental first principles, guided by experimental and observation evidences. The new theory we have established gives rise to solutions and explanations to a number of longstanding mysteries in modern theoretical physics. This work is synthesized in a recent book by the authors [6].

Basically, we have discovered three fundamental principles: the principle of interaction dynamics (PID) [3], the principle of representation invariance (PRI) [4], and the principle of symmetry-breaking (PSB) for unification [6].

PID takes the variation of the action under the energy-momentum conservation constraints, and is required by the dark matter and dark energy phenomena for gravity, by the quark confinements for the strong interaction, and by the Higgs field for the weak interaction.

PRI requires that the gauge theory be independent of the choices of the representation generators. These representation generators play the same role as coordinates, and in this sense, PRI is a coordinate-free invariance/covariance, reminiscent of the Einstein principle of general relativity. In other words, PRI is purely a logic requirement for the gauge theory.

PSB offers an entirely different route of unification from the Einstein unification route which uses large symmetry groups. The three sets of symmetries — the general relativistic invariance, the Lorentz and gauge invariances, as well as the Galileo invariance — are mutually independent and dictate in part the physical laws in different levels of Nature. For a system coupling different levels of physical laws, part of these symmetries must be broken.

These three new principles have important physical consequences, and, in particular, provide a new route of unification for the four interactions:

1) the general relativity and the gauge symmetries dictate the Lagrangian;
2) the coupling of the four interactions is achieved through PID and PRI in the field equations, which obey the Einstein principle of general relativity and PRI, but break spontaneously the gauge symmetry;
3) the unified field model can be easily decoupled to study an individual interaction, when the other interactions are negligible; and
4) the unified field model coupling the matter fields using PSB.

2. Conservation laws based on quantum Hamiltonian dynamics (QHD)

To prove this quantum rule, we first recall the total angular momentum $\vec{J}$ of a particle defined by

$$\vec{J} = \vec{L} + s\vec{S},$$

where $\vec{L} = \vec{r} \times \vec{p}$ is the orbital angular momentum, $\vec{p} = -i\hbar\nabla$, s is the spin, and

$$\vec{S} = (S_1, S_2, S_3), \qquad S_k = \hbar \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}.$$

Here σ_k $(1 \le k \le 3)$ are the Pauli matrices.

Let H be the Hamiltonian energy of a conservative quantum system, which can be described by the following Hamiltonian equations:

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= \hat{H}_\Phi(\Psi, \Phi), \\ \frac{\partial \Phi}{\partial t} &= -\hat{H}_\Psi(\Psi, \Phi), \end{aligned} \tag{1}$$

where

$$\hat{H}_\Phi = \frac{\delta H}{\delta \Phi}, \qquad \hat{H}_\Psi = \frac{\delta H}{\delta \Psi}.$$

Let L be an observable physical quantity with the corresponding Hermitian operator $\hat{L}$ for the conjugate fields $(\Psi, \Phi)^T$ of (1), and $\hat{L}$ is expressed as

$$\hat{L} = \begin{pmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{pmatrix}, \qquad \hat{L}_{12}^T = \hat{L}_{21}^*.$$

Then the physical quantity L of system (1) is given by

$$L = \int (\Psi^\dagger, \Phi^\dagger) \hat{L} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} dx = \int \left[\Psi^\dagger \hat{L}_{11} \Psi + \Phi^\dagger \hat{L}_{22} \Phi + 2\text{Re}(\Psi^\dagger \hat{L}_{12} \Phi) \right] dx. \tag{2}$$

It is clear that the quantity L of (2) is conserved if for the solution $(\Psi, \Phi)^T$ of (1) we have

$$\frac{dL}{dt} = 0,$$

which is equivalent to

$$\begin{aligned} \int \Big[& \hat{H}_\Phi^\dagger \hat{L}_{11} \Psi + \Psi^\dagger \hat{L}_{11} \hat{H}_\Phi - \hat{H}_\Psi^\dagger \hat{L}_{22} \Phi - \Phi^\dagger \hat{L}_{22} \hat{H}_\Psi \\ & + 2\text{Re}(\hat{H}_\Phi^\dagger \hat{L}_{12} \Phi - \Psi^\dagger \hat{L}_{12} \hat{H}_\Psi)] dx = 0. \end{aligned} \tag{3}$$

We remark here that if the QHD is described by a complex valued wave function:

$$\psi = \Psi + i\Phi,$$

and its dynamic equation is linear, then (1) can be written as

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi, \quad H = \int \psi^\dagger \hat{H} \psi dx. \tag{4}$$

In this case, the physical quantity L in (2) is in the form

$$L = \int \psi^\dagger \hat{L} \psi dx, \tag{5}$$

and the conservation law (3) of L is equivalent to

$$\hat{L}\hat{H} - \hat{H}\hat{L} = 0. \tag{6}$$

The formulas (4)-(6) are the conservation laws of the classical quantum mechanics.

Hence, the conservation law (3) is the generalization to the classical quantum mechanics, which are applicable to all conservative quantum systems, including the Klein-Gordon system and nonlinear systems.

3. Angular Momentum Rule for Fermions

Consider fermions which obey the Dirac equations as (4) with the Hamiltonian

$$\hat{H} = -i\hbar c(\alpha^k \partial_k) + mc^2\alpha^0 + V(r), \tag{7}$$

where V is the potential energy of a central field, and α^0, α^k $(1 \leq k \leq 3)$ are the Dirac matrices

$$\alpha^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \quad \text{for } 1 \leq k \leq 3, \tag{8}$$

and σ^k are the Pauli matrices.

The total angular momentum $\hat{J}$ of a particle is

$$\hat{J} = \hat{L} + s\hat{S},$$

where s is the spin, $\hat{L}$ is the orbital angular momentum

$$\begin{aligned} &\hat{L} = (\hat{L}_1, \hat{L}_2, \hat{L}_3) = \hat{r} \times \hat{p}, \quad \hat{p} = -i\hbar\nabla, \\ &\hat{L}_1 = -i\hbar(x_2\partial_3 - x_3\partial_2), \\ &\hat{L}_2 = -i\hbar(x_3\partial_1 - x_1\partial_3), \\ &\hat{L}_3 = -i\hbar(x_1\partial_2 - x_2\partial_1), \end{aligned} \tag{9}$$

and $\hat{S}$ is the spin operator

$$\hat{S} = (\hat{S}_1, \hat{S}_2, \hat{S}_3), \quad \hat{S}_k = \hbar \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \qquad \text{for } 1 \leq k \leq 3. \tag{10}$$

By (7)-(10), we see that

$$\begin{aligned} \hat{H}\hat{L}_1 - \hat{L}_1\hat{H} =& \hbar^2 c[(x_2\partial_3 - x_3\partial_2)(\alpha^2\partial_2 + \alpha^3\partial_3) - (\alpha^2\partial_2 + \alpha^3\partial_3)(x_2\partial_3 - x_3\partial_2)] \\ =& \hbar^2 c[\alpha^2\partial_3(x_2\partial_2 - \partial_2 x_2) - \alpha^3\partial_2(x_3\partial_3 - \partial_3 x_3)]. \end{aligned}$$

Notice that

$$x_2\partial_2 - \partial_2 x_2 = x_3\partial_3 - \partial_3 x_3 = -1.$$

Hence we get

$$\hat{H}\hat{L}_1 - \hat{L}_1\hat{H} = \hbar^2 c(\alpha^3\partial_2 - \alpha^2\partial_3). \tag{11}$$

Similarly we have

$$\begin{aligned} \hat{H}\hat{L}_2 - \hat{L}_2\hat{H} = \hbar^2 c(\alpha^1\partial_3 - \alpha^3\partial_1), \\ \hat{H}\hat{L}_3 - \hat{L}_3\hat{H} = \hbar^2 c(\alpha^2\partial_1 - \alpha^1\partial_2). \end{aligned} \tag{12}$$

On the other hand, we infer from (7) and (10) that

$$\hat{H}\hat{S}_j - \hat{S}_j\hat{H} = -i\hbar^2 c\gamma^5 \left[\partial_k(\sigma^k\sigma^j - \sigma^j\sigma^k)\right] = -i\hbar^2 c\gamma^5(2i\varepsilon_{kjl}\sigma^l)\partial_k = 2\hbar^2 c\varepsilon_{kjl}\alpha^l\partial_k,$$

where γ^5 is defined by

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^0 = \alpha^0, \quad \gamma^k = \alpha^0\alpha^k.$$

Hence we have

$$\begin{aligned} \hat{H}\hat{S}_1 - \hat{S}_1\hat{H} &= -2\hbar^2 c(\alpha^3\partial_2 - \alpha^2\partial_3), \\ \hat{H}\hat{S}_2 - \hat{S}_2\hat{H} &= -2\hbar^2 c(\alpha^1\partial_3 - \alpha^3\partial_1), \\ \hat{H}\hat{S}_3 - \hat{S}_3\hat{H} &= -2\hbar^2 c(\alpha^2\partial_1 - \alpha^1\partial_2). \end{aligned} \tag{13}$$

For $\hat{J} = \hat{L} + s\hat{S}$, we derive from (11)-(13) that

$$\hat{H}\hat{J} - \hat{J}\hat{H} = 0 \qquad \Longleftrightarrow \qquad \text{spin } s = \frac{1}{2}. \tag{14}$$

When fermions move on a straight line,

$$\hat{H} = c\alpha^3 p_3, \quad \hat{L} = 0.$$

In this case, by (11)-(12), for straight line motion,

$$\hat{H}\hat{J} - \hat{J}\hat{H} = 0 \qquad \text{for any } s. \tag{15}$$

Thus, by the conservation law (6), the assertion of Angular Momentum Rule for fermions follows from (14) and (15).

4. Angular Momentum Rule for Bosons

Now, consider bosons which obey the Klein-Gordon equation in the form (1). It is known that the spins J of bosons depend on the types of Klein-Gordon fields (Ψ, Φ):

$$\begin{pmatrix} \Psi \\ \Phi \end{pmatrix} = \begin{cases} \text{a scalar field} & \Rightarrow \quad J = 0, \\ \text{a 4-vector field} & \Rightarrow \quad J = 1, \\ \text{a 2nd-order tensor field} & \Rightarrow \quad J = 2, \\ \text{a real valued field} & \Rightarrow \quad \text{neutral bosons}, \\ \text{a complex valued field} & \Rightarrow \quad \text{charged bosons}. \end{cases} \tag{16}$$

For the Klein-Gordon fields $(\Psi, \Phi)^T$, the Hamiltonian for a central force field is given by

$$H = \frac{1}{2}\int \left[|\Phi|^2 + c^2|\nabla\Psi|^2 + \frac{1}{\hbar^2}(m^2c^4 + V(r))|\Psi|^2 \right] dx \tag{17}$$

The Hamiltonian energy operator $\hat{H}$ of (17) is given by

$$\hat{H} = \begin{pmatrix} \hat{H}_\Psi & 0 \\ 0 & \hat{H}_\Phi \end{pmatrix}, \qquad \hat{H}_\Phi = \Phi, \qquad \hat{H}_\Psi = \left[-c^2\Delta + \frac{1}{\hbar^2}(m^2c^4 + V) \right]\Psi. \tag{18}$$

The angular momentum operator $\hat{J}$ is

$$\hat{J} = \begin{pmatrix} \hat{L} & 0 \\ 0 & \hat{L} \end{pmatrix} + s\hbar\hat{\sigma}, \quad \hat{\sigma} = (\sigma^1, \sigma^2, \sigma^3). \tag{19}$$

where s is the spin of bosons, and $\hat{L}$ is as in (9).

For scalar bosons, spin $s = 0$ in (19) and the Hermitian operators in the conservation law (3) are

$$\hat{L}_{11} = \hat{L}_{22} = \hat{L}, \qquad \hat{L}_{12} = \hat{L}_{21} = 0, \qquad \hat{H}_\Phi, \hat{H}_\Psi \text{ as in (18)}$$

Then by

$$\int \Phi^\dagger \hat{L}\Psi dx = -\int \Psi \hat{L}^\dagger \Phi dx,$$
$$\int \hat{H}_\Psi^\dagger \hat{L}\Phi dx = -\int \Phi^\dagger \hat{L}\hat{H}_\Psi dx,$$

we derive the conservation law (3), i.e.

$$\int \left[\Phi^\dagger \hat{L}\Psi + \Psi^\dagger \hat{L}\Phi - \hat{H}_\Psi^\dagger \hat{L}\Phi - \Phi^\dagger \hat{L}\hat{H}_\Psi\right] dx = 0.$$

However, it is clear that $\hat{H}$ and $\hat{J}$ in (18) and (19) don't satisfy (3) for spin $s \neq 0$. Hence the quantum rule of angular momentum for bosons holds true.

Acknowledgments

The work was supported in part by grants from the National Science Foundation (NSF), from the Office of Naval Research (ONR), and from Chinese National Science Foundation.

References

1. L. de Broglie, *Recherches sur la théorie des quanta, Thesis Paris, 1924*.
2. A. Einstein, *Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt*, Annalen der Physik **322**, 132–148.
3. T. Ma and S. Wang, *Unified field equations coupling four forces and principle of interaction dynamics*, arXiv:1210.0448 (2012); DCDS-A, 35:3 (2015), 1103-1138.
4. T. Ma and S. Wang, *Unified field theory and principle of representation invariance*, arXiv:1212.4893 (2012); version 1 appeared in Applied Mathematics and Optimization, DOI: 10.1007/s00245-013-9226-0, 69:3 (2014), 359-392.
5. T. Ma and S. Wang, *Weakton model of elementary particles and decay mechanisms*, Indiana University Institute for Scientific Computing and Applied Mathematics Preprint Series, #1304 (May 30, 2013); Electronic Journal of Theoretical Physics, Volume 12, No. 32, 139-178.
6. T. Ma and S. Wang, *Mathematical Principles of Theoretical Physics*, Science Press, Beijing, 524pp., August, 2015.

7. M. Planck, *Ueber das Gesetz der Energieverteilung im Normalspectrum,* Annalen der Physik, **309**, 553–563.

AIMS Press