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Research article

Harmonic Maps Surfaces and Relativistic Strings

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Abstract: The harmonic map is introduced and several physical applications are presented. The classical nonlinear σ model can be looked at as the embedding of a two-dimensional surface in a three-dimensional sphere, which is itself embedded in a four-dimensional space. A system of nonlinear evolution equations are obtained by working out the zero curvature condition for the Gauss equations relevant to this geometric formulation.

Keywords: harmonic map; curvature; surface; soliton; sigma models

1. Introduction

One area in which linear and nonlinear equations appear to be in very close relationship is the embedding of Riemannian manifolds into manifolds of higher dimension. The embedded manifold is constructed by means of linear differential equations. These equations form an overdetermined set and the integrability conditions they obey in order for a solution to exist are in general nonlinear differential equations. They would be obeyed by the metric or second fundamental form of the embedded manifold, for example.

2. Discussion

The term harmonic map generally refers to a class of nonlinear field equations [6] which have a surprising number of applications. There are various applications such as the description of theories with broken symmetries, with or instead of Yang-Mills equations. They can also be quite similar to the Einstein equations for gravitation and to some of the equations which appear in string theory [5,4]. The wave or Laplace equation for a scalar field $\varphi(\mathbf{x})$

$$\frac{\partial}{\partial x^\mu}(\sqrt{g} g^{\mu\nu} \frac{\partial \varphi}{\partial x^\nu}) = 0 \tag{1}$$

characterizes harmonic functions φ from which the class of harmonic maps takes its name. The usual nonlinear geodesic equation is also a specialized subclass of the harmonic maps. The general harmonic map combines aspects of both these equations in the nonlinear partial differential equation which can be obtained from the action

$$\mathcal{A} = \frac{1}{2} \int \sqrt{g} g^{\mu\nu}(x) \frac{\partial \varphi^a}{\partial x^\mu} \frac{\partial \varphi^b}{\partial x^\nu} G_{ab}(\varphi)\, d^n x. \tag{2}$$

For example, physical theories of this class would be those where $g^{\mu\nu}(x)$ is flat Minkowski space. A nontrivial example of this class of theories is the nonlinear σ-model where $G_{ab}(\varphi)$ is the metric of a sphere and the φ^a are independent fields. In fact, Minkowski spacetime can be replaced by any d-dimensional spacetime M with a Lorentz or Euclidean signature metric G_{ab}. The action of a spin-0 particle of mass m propagating in d-dimensional spacetime is

$$\mathcal{A} = \int \sqrt{g} (\frac{1}{2} g^{\mu\nu}(x) \frac{\partial \varphi^a}{\partial x^\mu} \frac{\partial \varphi^b}{\partial x^\nu} G_{ab}(\varphi) - \frac{1}{2} m^2)\, d^n x.$$

In a quantum theory, this action would lead to the massive Klein-Gordon equation in curved spacetime which determines a wavefunction.

Harmonic maps can be used to create surfaces and of course there continues to be great interest in differential equations which can be used to induce surfaces [1-3]. Let M and M' be two pseudo-Riemannian manifolds with $\{x^\mu\}$ coordinates on M and φ^a coordinates on M'. If M is thought of as spacetime, its metric $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$ can be restricted to flat Minkowski or Euclidean space. The M' manifold is the set of possible values for some nonlinear field φ^a. Nonlinearity enters because the metric on M' can be thought of as being curved

$$dS^2 = G_{ab}(\varphi)\, d\varphi^a d\varphi^b.$$

Therefore, a mapping $\phi : M \to M'$, $x \to \phi(x)$ is represented in coordinates as $\phi^a(x^\mu)$, and will be referred to as a harmonic map if it satisifes the Euler-Lagrange equations obtained from (2). For example, let M be a flat Euclidean or Minkowski space and take M' to be the sphere $\mathcal{S}^2$ with the usual metric

$$dS^2 = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{3}$$

A mapping is a pair of fields $\theta(x^\mu)$, $\phi(x^\mu)$ which are obtained by requiring they satisfy differentiability requirements which arise from the structures of $\mathcal{S}^2$ and the spacetime M. The action in this case takes the form,

$$\mathcal{A} = \frac{1}{2} \int d^n x\, [(\vec{\nabla}\theta)^2 + \sin^2 (\vec{\nabla}\phi)^2] \tag{4}$$

and (4) leads to the following field equations

$$-\partial_\mu \partial^\mu \theta + \sin\theta \cos\theta (\vec{\nabla}\phi)^2 = 0, \tag{5}$$

$$-\partial_\mu \partial^\mu \phi - 2\cot\theta(\vec{\nabla}\theta)\cdot(\vec{\nabla}\phi) = 0. \tag{6}$$

When $(\vec{\nabla}\phi)^2$ is constant, this system reduces to the sine-Gordon equation.

In addition to harmonic functions with $\dim M' = 1$ and geodesics with $\dim M = 1$, any isometry $M \to M'$ or covering of Riemannian manifolds $M \to M'$ is a harmonic map. Minimal hypersurfaces are coordinate conditions in constructing solutions of Einstein's equations. In fact, any minimal immersion $M \to M'$ of Riemannian manifolds is a harmonic map.

Harmonic maps can help in understanding some of the nonlinearities that occur in the Einstein equations of general relativity as the Yang-Mills equations have done. In two space-time dimensions, the classical nonlinear σ model may be studied as the embedding of a two-dimensional surface in a three-dimensional sphere which is itself embedded in four-dimensional Euclidean space.

The nonlinear σ model in two-dimensional space-time which will be studied here consists of four scalar fields $\varphi^i(x_1, x_2)$, $i = 1, \cdots, 4$, which undergo self-interaction defined by the constraint

$$\varphi^i \varphi^i = 1. \tag{7}$$

The Lagrangian density for this system is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi^i \partial^\mu \varphi^i + \frac{1}{2}\lambda(\varphi^i\varphi^i - 1), \tag{8}$$

and λ in (8) is a Lagrange multiplier with $i = 1, 2$. The equations of motion which result from (8) are

$$\partial_\mu \partial^\mu \varphi^i - \lambda \varphi^i = 0, \tag{9}$$

$$\varphi^i \varphi^i = 1. \tag{10}$$

The fields φ^i in these equations can be interpreted as the components of a vector in a four-dimensional space which is Euclidean. Constraint (10) implies that this vector must reside on the surface of a three-dimensional sphere. A solution φ^i of (9) describes a two-dimensional surface embedded in this sphere. The problem of solving (9) and (10) then reduces to the problem of embedding a surface in a three-dimensional sphere which in turn is itself embedded in a four-dimensional Euclidean space. The metric on the four-dimensional Euclidean space has the form,

$$ds^2 = d\varphi^i \otimes d\varphi^i. \tag{11}$$

This induces a metric on the two-dimensional surface $\varphi^i(\sigma, \tau)$ given by

$$ds^2 = \frac{\partial\varphi^i}{\partial\sigma}\frac{\partial\varphi^i}{\partial\sigma} d\sigma \otimes d\sigma + 2\frac{\partial\varphi^i}{\partial\sigma}\frac{\partial\varphi^i}{\partial\tau} d\sigma \otimes d\tau + \frac{\partial\varphi^i}{\partial\tau}\frac{\partial\varphi^i}{\partial\tau} d\tau \otimes d\tau. \tag{12}$$

In this context, it is always possible to choose the coordinates σ, τ so that the following system holds:

$$\frac{\partial\varphi^i}{\partial\sigma}\frac{\partial\varphi^i}{\partial\sigma} + \frac{\partial\varphi^i}{\partial\tau}\frac{\partial\varphi^i}{\partial\tau} = 1, \qquad \frac{\partial\varphi^i}{\partial\sigma}\frac{\partial\varphi^i}{\partial\tau} = 0. \tag{13}$$

Consequently, the metric (12) of the surface can be expressed in terms of a single scalar field $\vartheta(\sigma,\tau)$ as follows

$$ds^2 = \cos^2\vartheta\, d\sigma^2 + \sin^2\vartheta\, d\tau^2. \tag{14}$$

To complete the description of a surface embedded in a higher-dimensional space, the second fundamental form is required.

The extrinsic curvature is given by a symmetric tensor $\Omega_{\mu\nu}$ which has the following four components

$$\Omega_{11} = \frac{\partial^2\vec{\varphi}}{\partial\sigma^2}\cdot\vec{X}_3, \quad \Omega_{12} = \Omega_{21} = \frac{\partial^2\vec{\varphi}}{\partial\sigma\partial\tau}\cdot\vec{X}_3, \quad \Omega_{22} = \frac{\partial^2\vec{\varphi}}{\partial\tau^2}\cdot\vec{X}_3. \tag{15}$$

As $\vec{\varphi}_\tau$, $\vec{\varphi}_\sigma$ span the tangent plane to the three-sphere $\vec{X} = \vec{n}$ is defined to be a unit vector which is orthogonal to these vectors. Let $\vec{X}_1$, $\vec{X}_2$ be unit vectors parallel to $\vec{\varphi}_\sigma$ and $\vec{\varphi}_\tau$, respectively. To generate an orthonormal tetrad in the surrounding Euclidean space, it suffices to include the element $\vec{X}_4 = \vec{\varphi}$ as the final element in the set.

The components of the metric tensor $g_{\mu\nu}$ can be obtained from (14),

$$g_{11} = \cos^2\vartheta, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = \sin^2\vartheta.$$

Expanding out equations (9) in terms of the (σ,τ)-variables, $\vec{\varphi}$ must satisfy

$$\frac{\partial^2\vec{\varphi}}{\partial\sigma^2} - \frac{\partial^2\vec{\varphi}}{\partial\tau^2} = \lambda\vec{\varphi}. \tag{16}$$

By writing the scalar product of (16) with $\vec{n}_3$ using (15) and the identification $\vec{\varphi} = \vec{n}_4$, the following important constraint is obtained

$$\Omega_{11} - \Omega_{22} = \lambda\vec{\varphi}\cdot\vec{n}_3 = 0. \tag{17}$$

Therefore, equation (17) implies that the diagonal components of Ω are equal, $\Omega_{11} = \Omega_{22}$. The Gauss-Weingarten equations assume the following form,

$$\frac{\partial\bar{N}}{\partial\sigma} = A\bar{N}, \qquad \frac{\partial\bar{N}}{\partial\tau} = B\bar{N}. \tag{18}$$

Once the Gauss-Weingarten equations have been obtained, they can be used to construct a surface. The integrability conditions for (18) are the Gauss-Codazzi equations. The quantity $\bar{N}$ is a four-component object which consists of the four vectors $\vec{n}_i$,

$$\bar{N} = \begin{pmatrix} \vec{n}_1 \\ \vec{n}_2 \\ \vec{n}_3 \\ \vec{n}_4 \end{pmatrix} \tag{19}$$

The matrices A and B which appear in (18) are given explicitly in the following form,

$$A = \begin{pmatrix} 0 & \frac{\partial\vartheta}{\partial\tau} & \frac{\Omega_{11}}{\cos\vartheta} & -\cos\vartheta \\ -\frac{\partial\vartheta}{\partial\tau} & 0 & \frac{\Omega_{12}}{\sin\vartheta} & 0 \\ -\frac{\Omega_{11}}{\cos\vartheta} & -\frac{\Omega_{12}}{\sin\vartheta} & 0 & 0 \\ \cos\vartheta & 0 & 0 & 0 \end{pmatrix} \tag{20}$$

$$B = \begin{pmatrix} 0 & \frac{\partial \vartheta}{\partial \sigma} & \frac{\Omega_{12}}{\cos \vartheta} & 0 \\ -\frac{\partial \vartheta}{\partial \sigma} & 0 & \frac{\Omega_{11}}{\sin \vartheta} & -\sin \vartheta \\ -\frac{\Omega_{12}}{\cos \vartheta} & -\frac{\Omega_{11}}{\sin \vartheta} & 0 & 0 \\ 0 & \sin \vartheta & 0 & 0 \end{pmatrix} \tag{21}$$

Substituting A and B into the pair of equations (18) and working out the components of each one, the following five equations result,

$$\frac{\partial^2 \vartheta}{\partial \tau^2} - \frac{\partial^2 \vartheta}{\partial \sigma^2} + \frac{1}{\sin \vartheta \cos \vartheta}(\Omega_{12}^2 - \Omega_{11}^2) - \sin \vartheta \cos \vartheta = 0, \tag{22}$$

$$-\frac{\partial}{\partial \tau}(\frac{\Omega_{11}}{\cos \vartheta}) + \frac{\partial}{\partial \sigma}(\frac{\Omega_{12}}{\cos \vartheta}) + \frac{\Omega_{12}}{\sin \vartheta}\frac{\partial \vartheta}{\partial \sigma} - \frac{\Omega_{11}}{\sin \vartheta}\frac{\partial \vartheta}{\partial \tau} = 0, \tag{23}$$

$$\frac{\partial}{\partial \tau}(\cos \vartheta) + \sin \vartheta \, \frac{\partial \vartheta}{\partial \tau} = 0, \tag{24}$$

$$-\frac{\partial}{\partial \tau}(\frac{\Omega_{12}}{\sin \vartheta}) + \frac{\partial}{\partial \sigma}\frac{\Omega_{11}}{\sin \vartheta} - \frac{\Omega_{11}}{\cos \vartheta}\frac{\partial \vartheta}{\partial \sigma} + \frac{\Omega_{12}}{\cos \vartheta}\frac{\partial \vartheta}{\partial \tau} = 0, \tag{25}$$

$$-\frac{\partial}{\partial \sigma}\sin \vartheta + \cos \vartheta \frac{\partial \vartheta}{\partial \sigma} = 0. \tag{26}$$

Equation 22 can be written in the form,

$$\sin \vartheta \cos \vartheta (\frac{\partial^2 \vartheta}{\partial \tau^2} - \frac{\partial^2 \vartheta}{\partial \sigma^2}) - \sin^2 \vartheta \, \cos^2 \vartheta + \Omega_{12}^2 - \Omega_{11}^2 = 0. \tag{27}$$

The two quantities Ω_{11} and Ω_{12} satisfy the equation

$$\sin \vartheta \frac{\partial}{\partial v}(\frac{\Omega}{\cos \vartheta}) + \Omega \frac{\partial \vartheta}{\partial v} = \frac{\partial}{\partial v}(\frac{\sin \vartheta}{\cos \vartheta}\Omega)$$

for $v = \sigma, \tau$ and $\Omega = \Omega_{11}, \Omega_{12}$, respectively, (23) can be put in the form,

$$\frac{\partial}{\partial \tau}(\tan \vartheta \, \Omega_{11}) = \frac{\partial}{\partial \sigma}(\tan \vartheta \, \Omega_{12}). \tag{28}$$

From (28), it follows there exists a function or field called $\beta(\sigma, \tau)$ such that Ω_{11} and Ω_{12} can be expressed as

$$\Omega_{11} = \cot \vartheta \, \frac{\partial \beta}{\partial \sigma}, \qquad \Omega_{12} = \cot \vartheta \, \frac{\partial \beta}{\partial \tau}. \tag{29}$$

This choice puts (27) into the form of a compatibility condition for β, and the remaining two equations (25)-(26) then take the form,

$$\begin{gathered} \sin\vartheta\cos\vartheta(\tfrac{\partial^2\vartheta}{\partial\tau^2}-\tfrac{\partial^2\vartheta}{\partial\sigma^2})-\sin^2\vartheta\cos^2\vartheta+\cot^2\vartheta((\tfrac{\partial\beta}{\partial\tau})^2-(\tfrac{\partial\beta}{\partial\sigma})^2)=0, \\ \tfrac{\partial}{\partial\tau}(\cot^2\vartheta\,\tfrac{\partial\beta}{\partial\tau})=\tfrac{\partial}{\partial\sigma}(\cot^2\vartheta\tfrac{\partial\beta}{\partial\sigma}). \end{gathered} \tag{30}$$

The matrices A and B defined in equations (20) and (21) are elements of the Lie algebra $O(4) = O(3) \oplus O(3)$ and they are uniquely determined by two three-dimensional rotations

$$A = C + D, \qquad B = E + F. \tag{31}$$

In fact, C and E can be put in the following forms

$$C = \begin{pmatrix} 0 & \frac{\partial\vartheta}{\partial\tau} & -\frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\sigma} \\ -\frac{\partial\vartheta}{\partial\tau} & 0 & -\cos\vartheta-\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\tau} \\ \frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\vartheta} & \cos\vartheta+\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\tau} & 0 \end{pmatrix} \tag{32}$$

$$E = \begin{pmatrix} 0 & \frac{\partial\vartheta}{\partial\sigma} & \sin\vartheta-\frac{1}{\sin\theta}\frac{\partial\beta}{\partial\tau} \\ -\frac{\partial\vartheta}{\partial\sigma} & 0 & -\frac{\cos\vartheta}{\sin\vartheta}\frac{\partial\beta}{\partial\sigma} \\ -\sin\vartheta+\frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\tau} & \frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\sigma} & 0 \end{pmatrix} \tag{33}$$

The matrices D and F can be obtained from the matrices C and E by means of the discrete transformation

$$\vartheta \to \pi - \vartheta, \qquad \cos\vartheta \to -\cos\vartheta, \qquad \sin\vartheta \to \sin\vartheta, \qquad \beta \to -\beta. \tag{34}$$

The matrices D and F have the following structure

$$D = \begin{pmatrix} 0 & -\frac{\partial\vartheta}{\partial\tau} & \frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\tau} \\ \frac{\partial\vartheta}{\partial\tau} & 0 & \cos\vartheta-\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\tau} \\ \frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\sigma} & -\cos\vartheta+\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\tau} & 0 \end{pmatrix} \tag{35}$$

$$F = \begin{pmatrix} 0 & -\frac{\partial\vartheta}{\partial\tau} & \sin\vartheta+\frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\tau} \\ \frac{\partial\vartheta}{\partial\sigma} & 0 & -\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\sigma} \\ -\sin\vartheta-\frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\tau} & \frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\sigma} & 0 \end{pmatrix} \tag{36}$$

The transformation leaves the metric and extrinsic curvature of the surface unaltered. It is possible to introduce a set of unit vectors $\vec{Y}_i$, $\vec{Z}_i$, $i = 1, 2, 3$ in three-dimensional space so that the system (18) takes the following form

$$\frac{\partial\bar{Y}}{\partial\sigma} = C\bar{Y}, \qquad \frac{\partial\bar{Z}}{\partial\sigma} = D\bar{Z}, \tag{37}$$

$$\frac{\partial\bar{Y}}{\partial\tau} = E\bar{Y}, \qquad \frac{\partial\bar{Z}}{\partial\tau} = F\bar{Z}. \tag{38}$$

Differentiating both (37) and (38) with respect to τ and σ respectively, the zero-curvature condition for $\bar{Y}$ implies the following relation satisifed by C and E,

$$C_\tau - E_\sigma + CE - EC = 0. \tag{39}$$

Similarly, the $\bar{Z}$ field implies the following relation satisifed by D and F,

$$D_\tau - F_\sigma + DF - FD = 0. \tag{40}$$

It should be stated that D and F are in a one-to-one correspondence with C and E, so it suffices to work out just one of these equations. Substituting the matrices (33) and (34) into (39), the diagonal elements of the zero curvature condition are found to sum to zero, and we are left with the following nontrivial results. After simplifying the first column and second row, the following equation is obtained

$$\vartheta_{\tau\tau} - \vartheta_{\sigma\sigma} - \sin\vartheta\cos\vartheta \frac{\cos\vartheta}{\sin^3\vartheta}((\frac{\partial\beta}{\partial\tau})^2 - (\frac{\partial\beta}{\partial\sigma})^2) = 0. \tag{41}$$

From the first column and third row we have

$$\frac{\partial}{\partial\tau}(\frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\sigma}) + \frac{\partial}{\partial\sigma}(\sin\vartheta - \frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\tau}) - \frac{\partial\vartheta}{\partial\sigma}(\cos\vartheta + \frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\tau}) + \frac{\partial\vartheta}{\partial\tau}\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\sigma} = 0. \tag{42}$$

This can be put in the form of an identity

$$-\cos\vartheta\frac{\partial\vartheta}{\partial\tau}\frac{\partial\beta}{\partial\sigma} + \sin\vartheta\frac{\partial^2\beta}{\partial\tau\partial\sigma} + \cos\vartheta\frac{\partial\vartheta}{\partial\sigma}\frac{\partial\beta}{\partial\tau} - \sin\vartheta\frac{\partial^2\beta}{\partial\tau\partial\sigma} = 0.$$

Finally, from the second column and the third row, the last equation is found to be

$$\frac{\partial}{\partial\tau}(\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\tau}) - \frac{\partial}{\partial\sigma}(\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\beta}{\partial\sigma}) + \frac{1}{\sin\vartheta}\frac{\partial\beta}{\partial\sigma}\frac{\partial\vartheta}{\partial\sigma} - \frac{1}{\sin\vartheta}\frac{\partial\vartheta}{\partial\tau}\frac{\partial\beta}{\partial\tau} = 0. \tag{43}$$

Applying the product rule, the following relation holds

$$\frac{\partial}{\partial\tau}(\frac{1}{\cos\vartheta}\cot^2\vartheta\frac{\partial\beta}{\partial\tau}) = \frac{1}{\sin\vartheta}\frac{\partial\vartheta}{\partial\tau}\frac{\partial\beta}{\partial\tau} + \frac{1}{\cos\vartheta}\frac{\partial}{\partial\tau}(\cot^2\vartheta\frac{\partial\beta}{\partial\tau}). \tag{44}$$

Using (44), (43) can be put in the following form after some simplification

$$\frac{\partial}{\partial\tau}(\cot^2\vartheta\frac{\partial\beta}{\partial\tau}) = \frac{\partial}{\partial\sigma}(\cot^2\vartheta\frac{\partial\beta}{\partial\sigma}). \tag{45}$$

These constitute the system of equations which result as a consequence of applying zero curvature condition (39) from the $\bar{Y}$ field. Therefore, the following Theorem has been proved and it is summarized below.

Theorem. Compatibility condition (39) resulting from (37) for the 3×3 matrix problem defined by the matrices (32) and (33) is equivalent to the following system of coupled partial differential equations for ϑ and β,

$$\frac{\partial^2\vartheta}{\partial\tau^2} - \frac{\partial^2\vartheta}{\partial\sigma^2} - \sin\vartheta\cos\vartheta + \frac{\cos\vartheta}{\sin^2\vartheta}((\frac{\partial\beta}{\partial\tau})^2 - (\frac{\partial\beta}{\partial\sigma})^2) = 0, \tag{46}$$

$$\frac{\partial}{\partial\tau}(\cot^2\vartheta\frac{\partial\beta}{\partial\tau}) = \frac{\partial}{\partial\sigma}(\cot^2\vartheta\frac{\partial\beta}{\partial\sigma}). \tag{47}$$

Moreover, the results in these equations are completely consistent with the equations in (30) which were obtained from Gauss-Weingarten equations (18). □

3. Conclusion

This is not the first time these equations have appeared. Equations (46) and (47) have also been obtained by Pohlmeyer [7] by means of a study of the nonlinear σ model in field theory. This approach however is more geometric than the one in Pohlmeyer [7]. It should also be stated that this model has led to a system of two coupled, Lorentz-invariant, nonlinear equations in two independent variables which will possess solitary wave solutions. From the theorem, it is seen that one of the fields is massless and moves in a background geometry that has a dynamical evolution of its own specified by a second field which has a sine-Gordon type self-interaction.

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