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Research article

Categorification of VB-Lie algebroids and VB-Courant algebroids

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Abstract: In this paper, first we introduce the notion of a VB-Lie 2-algebroid, which can be viewed as the categorification of a VB-Lie algebroid. The tangent prolongation of a Lie 2-algebroid is a VB-Lie 2-algebroid naturally. We show that after choosing a splitting, there is a one-to-one correspondence between VB-Lie 2-algebroids and flat superconnections of a Lie 2-algebroid on a 3-term complex of vector bundles. Then we introduce the notion of a VB-CLWX 2-algebroid, which can be viewed as the categorification of a VB-Courant algebroid. We show that there is a one-to-one correspondence between split Lie 3-algebroids and split VB-CLWX 2-algebroids. Finally, we introduce the notion of an E-CLWX 2-algebroid and show that associated to a VB-CLWX 2-algebroid, there is an E-CLWX 2-algebroid structure on the graded fat bundle naturally. By this result, we give a construction of a new Lie 3-algebra from a given Lie 3-algebra, which provides interesting examples of Lie 3-algebras including the higher analogue of the string Lie 2-algebra.

Keywords: Lie 3-algebroid; VB-Lie algebroid; VB-Courant algebroid; superconnection; VB-Lie 2-algebroid; VB-CLWX 2-algebroid; higher analogue of the string Lie 2-algebra
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1. Introduction

In this paper, we study the categorification of VB-Lie algebroids and VB-Courant algebroids, and establish the relations between these higher structures and super representations of Lie 2-algebroids, tangent prolongations of Lie 2-algebroids, N-manifolds of degree 3, tangent prolongations of CLWX 2-algebroids and higher analogues of the string Lie 2-algebra.

1.1. Lie n-algebroids, Courant algebroids and CLWX 2-algebroids

An **NQ-manifold** is an N-manifold $\mathcal{M}$ together with a degree 1 vector field Q satisfying $[Q, Q] = 0$. It is well known that a degree 1 NQ manifold corresponds to a Lie algebroid. Thus, people usually think that

An NQ-manifold of degree n corresponds to a Lie n-algebroid.

Some work in this direction appeared in [54]. Strictly speaking, a Lie n-algebroid gives arise to an NQ-manifold only after a degree 1 shift, just as a Lie algebroid A corresponds to a degree 1 NQ manifold $A[1]$. To make the shifting manifest, and to present a Lie n-algebroid in a way more used to differential geometers, that is, to use the language of vector bundles, the authors introduced the notion of a split Lie n-algebroid in [52] to study the integration of a Courant algebroid. The equivalence between the category of split NQ manifolds and the category of split Lie n-Lie algebroids was proved in [5]. The language of split Lie n-algebroids has slowly become a useful tool for differential geometers to study problems related to NQ-manifolds ([14, 24, 25]). Since Lie 2-algebras are the categorification of Lie algebras ([4]), we will view Lie 2-algebroids as the categorification of Lie algebroids.

To study the double of a Lie bialgebroid ([42]), Liu, Weinstein and Xu introduced the notion of a Courant algebroid in [35]. See [44] for an alternative definition. There are many important applications of Courant algebroids, e.g. in generalized complex geometry ([8, 17, 22]), Poisson geometry ([33]), moment maps ([9]), Poisson-Lie T-duality ([47, 48]) and topological field theory ([46]). In [34], the authors introduced the notion of a CLWX 2-algebroid (named after Courant-Liu-Weinstein-Xu), which can be viewed as the categorification of a Courant algebroid. Furthermore, CLWX 2-algebroids are in one-to-one correspondence with QP-manifolds (symplectic NQ-manifolds) of degree 3, and have applications in the fields theory. See [23] for more details. The underlying algebraic structure of a CLWX 2-algebroid is a Leibniz 2-algebra, or a Lie 3-algebra. There is also a close relationship between CLWX 2-algebroids and the first Pontryagin classes of quadratic Lie 2-algebroids, which are represented by closed 5-forms. More precisely, as the higher analogue of the results given in [6, 13], it was proved in [49] that the first Pontryagin class of a quadratic Lie algebroid is the obstruction of the existence of a CLWX-extension.

1.2. VB-*Lie algebroids and* VB-*Courant algebroids*

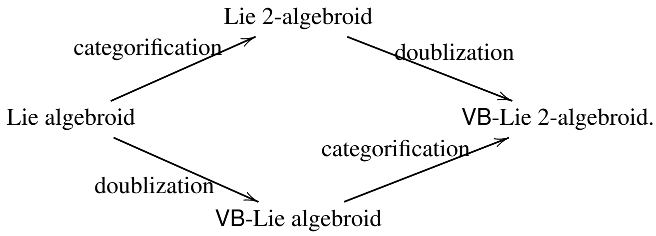
Double structures in geometry can be traced back to the work of Ehresmann on connection theory, and have been found many applications in Poisson geometry. See [40] for more details. We use the word "doublization" to indicate putting geometric structures on double vector bundles in the sequel. In [19], Gracia-Saz and Mehta introduced the notion of a VB-Lie algebroid, which is equivalent to Mackenzie's $\mathcal{LA}$-vector bundle ([38]). A VB-Lie algebroid is a Lie algebroid object in the category of vector bundles and one important property is that it is closely related to superconnection (also called representation up to homotopy [1, 2]) of a Lie algebroid on a 2-term complex of vector bundles. Recently, the relation between VB-algebroid morphisms and representations up to homotopy were studied in [15].

In his PhD thesis [32], Li-Bland introduced the notion of a VB-Courant algebroid which is the doublization of a Courant algebroid [35], and established abstract correspondence between NQ-manifolds of degree 2 and VB-Courant algebroids. Then in [24], Jotz Lean provided a more concrete description of the equivalence between the category of split Lie 2-algebroids and the category of decomposed VB-Courant algebroids.

Double structures, such as double principle (vector) bundles ([12, 16, 26, 30]), double Lie algebroids ([18, 37, 38, 39, 41, 55]), double Lie groupoids ([43]), VB-Lie algebroids ([7, 19]) and VB-Lie groupoids ([7, 20]) became more and more important recently and are widely studied. In particular, the Lie theory relating VB-Lie algebroids and VB-Lie groupoids, i.e. their relation via differentiation and integration, is established in [7].

1.3. Summary of the results and outline of the paper

In this paper, we combine the aforementioned higher structures and double structures. First we introduce the notion of a VB-Lie 2-algebroid, which can be viewed as the categorification of a VB-Lie algebroid, or doublization of a Lie 2-algebroid:



We show that the tangent prolongation of a Lie 2-algebroid is a VB-Lie 2-algebroid and the graded fat bundle associated to a VB-Lie 2-algebroid is Lie 2-algebroid. Consequently, the graded jet bundle of a Lie 2-algebroid is also a Lie 2-algebroid. In [19], the authors showed that a VB-Lie algebroid is equivalent to a flat superconnection (representation up to homotopy ([1])) of a Lie algebroid on a 2-term complex of vector bundles after choosing a splitting. Now for a VB-Lie 2-algebroid, we establish a higher analogous result, namely, we show that after choosing a splitting, it is equivalent to a flat superconnection of a Lie 2-algebroid on a 3-term complex of vector bundles.

Then we introduce the notion of a VB-CLWX 2-algebroid, which can be viewed as both the doublization of a CLWX 2-algebroid and the categorification of a VB-Courant algebroid. More importantly, we show that after choosing a splitting, there is a one-to-one correspondence between VB-CLWX 2-algebroids and split Lie 3-algebroids (NQ-manifolds of degree 3). The tangent prolongation of a CLWX 2-algebroid is a VB-CLWX 2-algebroid naturally. We go on defining E-CLWX 2-algebroid, which can be viewed as the categorification of an E-Courant algebroid introduced in [11]. As a higher analogue of the result that associated to a VB-Courant algebroid, there is an E-Courant algebroid [24, 31], we show that on the graded fat bundle associated to a VB-CLWX 2-algebroid, there is an E-CLWX 2-algebroid structure naturally. Similar to the case of a CLWX 2-algebroid, an E-CLWX 2-algebroid also gives rise to a Lie 3-algebra naturally. Thus through the following procedure:

$$\begin{array}{c}\text{Lie}\\ \text{3-algebra}\end{array} \longmapsto \begin{array}{c}\mathsf{VB\text{-}CLWX}\\ \text{2-algebroid}\end{array} \longmapsto \begin{array}{c}E\text{-}\mathsf{CLWX}\\ \text{2-algebroid}\end{array} \longmapsto \begin{array}{c}\text{Lie}\\ \text{3-algebra,}\end{array}$$

we can construct a Lie 3-algebra from a Lie 3-algebra. We obtain new interesting examples, including the higher analogue of the string Lie 2-algebra.

The paper is organized as follows. In Section 2, we recall double vector bundles, VB-Lie algebroids and VB-Courant algebroids. In Section 3, we introduce the notion of a VB-Lie 2-algebroid, and show that both the graded side bundle and the graded fat bundle are Lie 2-algebroids. The tangent prolongation of a Lie 2-algebroid is a VB-Lie 2-algebroid naturally. In Section 4, first we construct a strict Lie 3-algebroid $\mathrm{End}(\mathcal{E}) = (\mathrm{End}^{-2}(\mathcal{E}), \mathrm{End}^{-1}(\mathcal{E}), \mathfrak{D}(\mathcal{E}), \mathfrak{p}, \mathrm{d}, [\cdot,\cdot]_C)$ from a 3-term complex of vector bundles $\mathcal{E} : E_{-2} \xrightarrow{\pi} E_{-1} \xrightarrow{\pi} E_0$ and then we define a flat superconnection of a Lie 2-algebroid $\mathcal{A} = (A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ on this 3-term complex of vector bundles to be a morphism from $\mathcal{A}$ to $\mathrm{End}(\mathcal{E})$. We show that after choosing a splitting, VB-Lie 2-algebroids one-to-one correspond to flat superconnections of a Lie 2-algebroid on a 3-term complex of vector bundles. In Section 5, we introduce the notion of a VB-CLWX 2-algebroid and show that after choosing a splitting, there is a one-to-one correspondence between VB-CLWX 2-algebroids and Lie 3-algebroids. In Section 6, we introduce the notion of an E-CLWX 2-algebroid and show that the graded fat bundle associated to a VB-CLWX 2-algebroid is an E-CLWX 2-algebroid naturally. In particular, the graded jet bundle of a CLWX 2-algebroid, which is the graded fat bundle of the tangent prolongation of this CLWX 2-algebroid, is a T^*M-CLWX 2-algebroid. We can also obtain a Lie 3-algebra from an E-CLWX 2-algebroid. In Section 7, we construct a Lie 3-algebra from a given Lie 3-algebra using the theories established in Section 5 and Section 6, and give interesting examples. In particular, we show that associated to a quadratic Lie 2-algebra, we can obtain a Lie 3-algebra, which can be viewed as the higher analogue of the string Lie 2-algebra.

2. Preliminaries

See [40, Definition 9.1.1] for the precise definition of a double vector bundle. We denote a double vector bundle

$$\begin{array}{ccccc} D & \xrightarrow{\pi_B} & B & & \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle q_B} & & \\ A & \xrightarrow[q_A]{} & M & \longleftarrow & C \end{array}$$

with core C by $(D; A, B; M)$. We use D^B and D^A to denote vector bundles $D \longrightarrow B$ and $D \longrightarrow A$ respectively. For a vector bundle A, both the tangent bundle TA and the cotangent bundle T^*A are double vector bundles:

$$\begin{array}{ccc} TA & \longrightarrow & TM \\ \downarrow & & \downarrow \\ A & \longrightarrow & M, \end{array} \qquad \begin{array}{ccc} T^*A & \longrightarrow & A^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M. \end{array}$$

A morphism of double vector bundles

$$(\varphi; f_A, f_B; f_M) : (D; A, B; M) \to (D'; A', B'; M')$$

consists of maps φ: $D \to D'$, $f_A : A \to A'$, $f_B : B \to B'$, $f_M : M \to M'$, such that each of (φ, f_B), (φ, f_A), (f_A, f_M) and (f_B, f_M) is a morphism of the relevant vector bundles.

The space of sections $\Gamma_B(D)$ of the vector bundle D^B is generated as a $C^\infty(B)$-module by core sections $\Gamma^c_B(D)$ and linear sections $\Gamma^l_B(D)$. See [41] for more details. For a section $c : M \to C$, the corresponding **core section** $c^\dagger : B \to D$ is defined as

$$c^\dagger(b_m) = \tilde{0}_{b_m} +_A \overline{c(m)}, \quad \forall\, m \in M,\ b_m \in B_m,$$

where $\bar{\cdot}$ means the inclusion $C \hookrightarrow D$. A section $\xi : B \to D$ is called **linear** if it is a bundle morphism from $B \to M$ to $D \to A$ over a section $X \in \Gamma(A)$. We will view $B^* \otimes C$ both as $\mathrm{Hom}(B, C)$ and $\mathrm{Hom}(C^*, B^*)$ depending on what it acts. Given $\psi \in \Gamma(B^* \otimes C)$, there is a linear section $\tilde{\psi} : B \to D$ over the zero section $0^A : M \to A$ given by

$$\widetilde{\psi}(b_m) = \tilde{0}_{b_m} +_A \overline{\psi(b_m)}.$$

Note that $\Gamma^l_B(D)$ is locally free as a $C^\infty(M)$-module. Therefore, $\Gamma^l_B(D)$ is equal to $\Gamma(\hat{A})$ for some vector bundle $\hat{A} \to M$. The vector bundle $\hat{A}$ is called the **fat bundle** of the double vector bundle $(D; A, B; M)$. Moreover, we have the following short exact sequence of vector bundles over M

$$0 \to B^* \otimes C \longrightarrow \hat{A} \xrightarrow{\mathrm{pr}} A \to 0. \tag{2.1}$$

Definition 2.1. ([19, Definition 3.4]) A VB**-Lie algebroid** is a double vector bundle $(D; A, B; M)$ equipped with a Lie algebroid structure $(D^B, a, [\cdot,\cdot]_D)$ such that the anchor $a : D \longrightarrow TB$ is linear, i.e. $a : (D; A, B; M) \longrightarrow (TB; TM, B; M)$ is a morphism of double vector bundles, and the Lie bracket $[\cdot,\cdot]_D$ is linear:

$$[\Gamma^l_B(D), \Gamma^l_B(D)]_D \subset \Gamma^l_B(D),\ [\Gamma^l_B(D), \Gamma^c_B(D)]_D \subset \Gamma^c_B(D),\ [\Gamma^c_B(D), \Gamma^c_B(D)]_D = 0.$$

The vector bundle $A \longrightarrow M$ is then also a Lie algebroid, with the anchor $\mathfrak{a}$ and the bracket $[\cdot,\cdot]_A$ defined as follows: if ξ_1, ξ_2 are linear over $X_1, X_2 \in \Gamma(A)$, then the bracket $[\xi_1, \xi_2]_D$ is linear over $[X_1, X_2]_A$.

Definition 2.2. ([32, Definition 3.1.1]) A VB**-Courant algebroid** is a metric double vector bundle $(D; A, B; M)$ such that $(D^B, S, [\![\cdot,\cdot]\!], \rho)$ is a Courant algebroid and the following conditions are satisfied:

(i) The anchor map $\rho : D \to TB$ is linear;

(ii) The Courant bracket is linear. That is

$$[\![\Gamma^l_B(D), \Gamma^l_B(D)]\!] \subseteq \Gamma^l_B(D), \quad [\![\Gamma^l_B(D), \Gamma^c_B(D)]\!] \subseteq \Gamma^c_B(D), \quad [\![\Gamma^c_B(D), \Gamma^c_B(D)]\!] = 0.$$

Theorem 2.3. ([32, Proposition 3.2.1]) *There is a one-to-one correspondence between Lie 2-algebroids and* VB*-Courant algebroids.*

3. VB-Lie 2-algebroids

In this section, we introduce the notion of a VB-Lie 2-algebroid, which can be viewed as the categorification of a VB-Lie algebroid introduced in [19]. First we recall the notion of a Lie n-algebroid. See [28, 29] for more information of L_∞-algebras.

Definition 3.1. ([52, Definition 2.1]) A split Lie n-algebroid is a non-positively graded vector bundle $\mathcal{A} = A_0 \oplus A_{-1} \oplus \cdots \oplus A_{-n+1}$ over a manifold M equipped with a bundle map $a : A_0 \longrightarrow TM$ (called the anchor), and $n+1$ many brackets $l_i : \Gamma(\wedge^i \mathcal{A}) \longrightarrow \Gamma(\mathcal{A})$ with degree $2-i$ for $1 \le i \le n+1$, such that

1. $\Gamma(\mathcal{A})$ is an n-term L_∞-algebra:

$$\sum_{i+j=k+1} (-1)^{i(j-1)} \sum_{\sigma \in Sh^{-1}_{i,k-i}} \mathrm{sgn}(\sigma)\mathrm{Ksgn}(\sigma)$$
$$l_j(l_i(X_{\sigma(1)}, \cdots, X_{\sigma(i)}), X_{\sigma(i+1)}, \cdots, X_{\sigma(k)}) = 0,$$

   where the summation is taken over all $(i, k-i)$-unshuffles $Sh^{-1}_{i,k-i}$ with $i \ge 1$ and "Ksgn(σ)" is the Koszul sign for a permutation $\sigma \in S_k$, i.e.

$$X_1 \wedge \cdots \wedge X_k = \mathrm{Ksgn}(\sigma) X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(k)}.$$

2. l_2 satisfies the Leibniz rule with respect to the anchor a:

$$l_2(X^0, fX) = f l_2(X^0, X) + a(X^0)(f)X, \quad \forall\, X^0 \in \Gamma(A_0),\ f \in C^\infty(M),\ X \in \Gamma(\mathcal{A}).$$

3. For $i \neq 2$, l_i's are $C^\infty(M)$-linear.

Denote a split Lie n-algebroid by $(A_{-n+1}, \cdots, A_0, a, l_1, \cdots, l_{n+1})$, or simply by $\mathcal{A}$. We will only use a split Lie 2-algebroid $(A_{-1}, A_0, a, l_1, l_2, l_3)$ and a split Lie 3-algebroid $(A_{-2}, A_{-1}, A_0, a, l_1, l_2, l_3, l_4)$. For a split Lie n-algebroid, we have a generalized Chevalley-Eilenberg complex $(\Gamma(\mathsf{Symm}(\mathcal{A}[1])^*), \delta)$. See [5, 52] for more details. Then $\mathcal{A}[1]$ is an NQ-manifold of degree n. A split Lie n-algebroid morphism $\mathcal{A} \to \mathcal{A}'$ can be defined to be a graded vector bundle morphism $f : \mathsf{Symm}(\mathcal{A}[1]) \to \mathsf{Symm}(\mathcal{A}'[1])$ such that the induced pull-back map $f^* : C(\mathcal{A}'[1]) \to C(\mathcal{A}[1])$ between functions is a morphism of NQ manifolds. However it is rather complicated to write down a morphism between split Lie n-algebroids in terms of vector bundles, anchors and brackets, please see [5, Section 4.1] for such details. We only give explicit formulas of a morphism from a split Lie 2-algebroid to a strict split Lie 3-algebroid ($l_3 = 0, l_4 = 0$) and this is what we will use in this paper to define flat superconnections.

Definition 3.2. Let $\mathcal{A} = (A_{-1}, A_0, a, l_1, l_2, l_3)$ be a split Lie 2-algebroid and $\mathcal{A}' = (A'_{-2}, A'_{-1}, A'_0, a', l'_1, l'_2)$ a strict split Lie 3-algebroid. A morphism F from $\mathcal{A}$ to $\mathcal{A}'$ consists of:

- a bundle map $F^0 : A_0 \longrightarrow A'_0$,
- a bundle map $F^1 : A_{-1} \longrightarrow A'_{-1}$,
- a bundle map $F^2_0 : \wedge^2 A_0 \longrightarrow A'_{-1}$,
- a bundle map $F^2_1 : A_0 \wedge A_{-1} \longrightarrow A'_{-2}$,
- a bundle map $F^3 : \wedge^3 A_0 \longrightarrow A'_{-2}$,

such that for all $X^0, Y^0, Z^0, X^0_i \in \Gamma(A_0)$, $i = 1, 2, 3, 4$, $X^1, Y^1 \in \Gamma(A_{-1})$, we have

$$
\begin{aligned}
& a' \circ F^0 = a, \\
& l'_1 \circ F_1 = F_0 \circ l_1, \\
& F^0 l_2(X^0, Y^0) - l'_2(F^0(X^0), F^0(Y^0)) = l'_1 F^2_0(X^0, Y^0), \\
& F^1 l_2(X^0, Y^1) - l'_2(F^0(X^0), F^1(Y^1)) = F^2_0(X^0, l_1(Y^1)) - l'_1 F^2_1(X^0, Y^1), \\
& l'_2(F^1(X^1), F^1(Y^1)) = F^2_1(l_1(X^1), Y^1) - F^2_1(X^1, l_1(Y^1)), \\
& l'_2(F^0(X^0), F^2(Y^0, Z^0)) - F^2_0(l_2(X^0, Y^0), Z^0) + c.p. = F^1(l_3(X^0, Y^0, Z^0)) \\
& \qquad + l'_1 F^3(X^0, Y^0, Z^0), \\
& l'_2(F^0(X^0), F^2_1(Y^0, Z^1)) + l'_2(F^0(Y^0), F^2_1(Z^1, X^0)) + l'_2(F^1(Z^1), F^2_0(X^0, Y^0)) \\
& = F^2_1(l_2(X^0, Y^0), Z^1) + c.p. + F^3(X^0, Y^0, l_1(Z^1)),
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{i=1}^{4} (-1)^{i+1} \Big(F^2_1(X^0_i, l_3(X^0_1, \cdots, \widehat{X^0_i}, \cdots X^0_4)) + l'_2(F^0(X^0_i), F^3(X^0_1, \cdots, \widehat{X^0_i}, \cdots X^0_4)) \Big) \\
& + \sum_{i<j} (-1)^{i+j} \Big(F^3(l_2(X^0_i, X^0_j), X^0_k, X^0_l) + c.p. - \frac{1}{2} l'_2(F^2_0(X^0_i, X^0_j), F^2_0(X^0_k, X^0_l)) \Big) = 0,
\end{aligned}
$$

where $k < l$ and $\{k, l\} \cap \{i, j\} = \emptyset$.

Let $(A_{-1}, A_0, a, l_1, l_2, l_3)$ be a split Lie 2-algebroid. Then for all $X^0, Y^0 \in \Gamma(A_0)$ and $X^1 \in \Gamma(A_{-1})$, Lie derivatives $L^0_{X^0} : \Gamma(A^*_{-i}) \longrightarrow \Gamma(A^*_{-i})$, $i = 0, 1$, $L^1_{X^1} : \Gamma(A^*_{-1}) \longrightarrow \Gamma(A^*_0)$ and $L^3_{X^0,Y^0} : \Gamma(A^*_{-1}) \longrightarrow \Gamma(A^*_0)$ are defined by

$$\left\{ \begin{array}{rcl} \langle L^0_{X^0}\alpha^0, Y^0\rangle &=& \rho(X^0)\langle Y^0, \alpha^0\rangle - \langle \alpha^0, l_2(X^0, Y^0)\rangle, \\ \langle L^0_{X^0}\alpha^1, Y^1\rangle &=& \rho(X^0)\langle Y^1, \alpha^1\rangle - \langle \alpha^1, l_2(X^0, Y^1)\rangle, \\ \langle L^1_{X^1}\alpha^1, Y^0\rangle &=& -\langle \alpha^1, l_2(X^1, Y^0)\rangle, \\ \langle L^3_{X^0,Y^0}\alpha^1, Z^0\rangle &=& -\langle \alpha^1, l_3(X^0, Y^0, Z^0)\rangle, \end{array} \right. \tag{3.1}$$

for all $\alpha^0 \in \Gamma(A^*_0)$, $\alpha^1 \in \Gamma(A^*_{-1})$, $Y^1 \in \Gamma(A_{-1})$, $Z^0 \in \Gamma(A_0)$. If $(\mathcal{A}^*[1], \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ is also a split Lie 2-algebroid, we denote by $\mathcal{L}^0, \mathcal{L}^1, \mathcal{L}^3, \delta_*$ the corresponding operations.

A graded double vector bundle consists of a double vector bundle of degree -1 and a double vector bundle of degree 0:

$$\begin{array}{ccccc} D_{-1} & \xrightarrow{\pi^{B_{-1}}} & B_{-1} & & \\ {\scriptstyle \pi^{A_{-1}}}\downarrow & & \downarrow{\scriptstyle q^{B_{-1}}} & & \\ A_{-1} & \xrightarrow[q^{A_{-1}}]{} & M_{-1} & \longleftarrow & C_{-1}, \end{array} \qquad \begin{array}{ccccc} D_{0} & \xrightarrow{\pi^{B_{0}}} & B_{0} & & \\ {\scriptstyle \pi^{A_{0}}}\downarrow & & \downarrow{\scriptstyle q^{B_{0}}} & & \\ A_{0} & \xrightarrow[q^{A_{0}}]{} & M_{0} & \longleftarrow & C_{0}. \end{array}$$

We denote a graded double vector bundle by $\begin{pmatrix} D_{-1}; & A_{-1}, B_{-1}; & M_{-1} \\ D_0; & A_0, B_0; & M_0 \end{pmatrix}$. Morphisms between graded double vector bundles can be defined in an obvious way. We will denote by $\mathcal{D}$ and $\mathcal{A}$ the graded vector bundles $D^B_0 \oplus D^B_{-1}$ and $A_0 \oplus A_{-1}$ respectively. Now we are ready to introduce the main object in this section.

Definition 3.3. A VB-Lie 2-algebroid is a graded double vector bundle

$$\begin{pmatrix} D_{-1}; & A_{-1}, B; & M \\ D_0; & A_0, B; & M \end{pmatrix}$$

equipped with a Lie 2-algebroid structure $(D^B_{-1}, D^B_0, a, l_1, l_2, l_3)$ on $\mathcal{D}$ such that

(i) The anchor $a : D_0 \longrightarrow TB$ is linear, i.e. we have a bundle map $\mathfrak{a} : A_0 \longrightarrow TM$ such that $(a; \mathfrak{a}, \mathrm{id_B}; \mathrm{id_M})$ is a double vector bundle morphism (see Diagram (i));

(ii) l_1 is linear, i.e. we have a bundle map $\mathfrak{l}_1 : A_{-1} \longrightarrow A_0$ such that $(l_1; \mathfrak{l}_1, \mathrm{id_B}; \mathrm{id_M})$ is a double vector bundle morphism (see Diagram (ii));

(iii) l_2 is linear, i.e.

$$\begin{array}{ll} l_2(\Gamma^l_B(D_0), \Gamma^l_B(D_0)) \subset \Gamma^l_B(D_0), & l_2(\Gamma^l_B(D_0), \Gamma^c_B(D_0)) \subset \Gamma^c_B(D_0), \\ l_2(\Gamma^l_B(D_0), \Gamma^l_B(D_{-1})) \subset \Gamma^l_B(D_{-1}), & l_2(\Gamma^l_B(D_0), \Gamma^c_B(D_{-1})) \subset \Gamma^c_B(D_{-1}), \\ l_2(\Gamma^c_B(D_0), \Gamma^l_B(D_{-1})) \subset \Gamma^c_B(D_{-1}), & l_2(\Gamma^c_B(D_0), \Gamma^c_B(D_{-1})) = 0; \\ l_2(\Gamma^c_B(D_0), \Gamma^c_B(D_0)) = 0. & \end{array}$$

(iv) l_3 is linear, i.e.

$$\begin{array}{l} l_3(\Gamma^l_B(D_0), \Gamma^l_B(D_0), \Gamma^l_B(D_0)) \subset \Gamma^l_B(D_{-1}), \\ l_3(\Gamma^l_B(D_0), \Gamma^l_B(D_0), \Gamma^c_B(D_0)) \subset \Gamma^c_B(D_{-1}), \\ l_3(\Gamma^c_B(D_0), \Gamma^c_B(D_0), \cdot) = 0. \end{array}$$

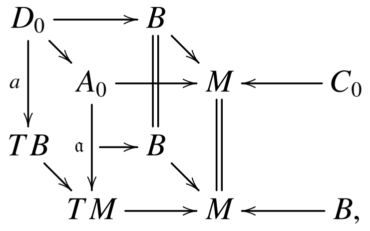


Diagram (i)

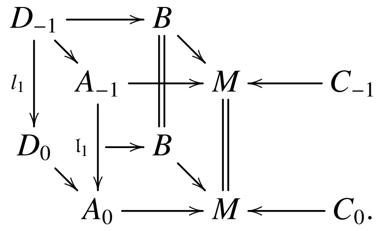


Diagram (ii)

Since Lie 2-algebroids are the categorification of Lie algebroids, VB-Lie 2-algebroids can be viewed as the categorification of VB-Lie algebroids.

Recall that if $(D; A, B; M)$ is a VB-Lie algebroid, then A is a Lie algebroid. The following result is its higher analogue.

Theorem 3.4. *Let* $\begin{pmatrix} D_{-1}; & A_{-1}, B; & M \\ D_0; & A_0, B; & M \end{pmatrix}$ *be a* VB*-Lie 2-algebroid. Then*

$$(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$$

is a split Lie 2-algebroid, where $\mathfrak{l}_2$ is defined by the property that if $\xi_1^0,\ \xi_2^0,\ \xi^0 \in \Gamma_B^l(D_0)$ are linear sections over $X_1^0,\ X_2^0,\ X^0 \in \Gamma(A_0)$, and $\xi^1 \in \Gamma_B^l(D_{-1})$ is a linear section over $X^1 \in \Gamma(A_{-1})$, then $l_2(\xi_1^0, \xi_2^0) \in \Gamma_B^l(D_0)$ is a linear section over $\mathfrak{l}_2(X_1^0, X_2^0) \in \Gamma(A_0)$ and $l_2(\xi^0, \xi^1) \in \Gamma_B^l(D_{-1})$ is a linear section over $\mathfrak{l}_2(X^0, X^1) \in \Gamma(A_{-1})$. Similarly, $\mathfrak{l}_3$ is defined by the property that if $\xi_1^0,\ \xi_2^0,\ \xi_3^0 \in \Gamma_B^l(D_0)$ are linear sections over $X_1^0,\ X_2^0,\ X_3^0 \in \Gamma(A_0)$, then $l_3(\xi_1^0, \xi_2^0, \xi_3^0) \in \Gamma_B^l(D_{-1})$ is a linear section over $\mathfrak{l}_3(X_1^0, X_2^0, X_3^0) \in \Gamma(A_{-1})$.

Proof. Since l_2 is linear, for any $\xi^i \in \Gamma_B^l(D_{-i})$ satisfying $\pi^{A_{-i}}(\xi^i) = 0$, we have

$$\pi^{A_{-(i+j)}}(l_2(\xi^i, \eta^j)) = 0, \quad \forall\ \eta^j \in \Gamma_B^l(D_{-j}).$$

This implies that $\mathfrak{l}_2$ is well-defined. Similarly, $\mathfrak{l}_3$ is also well-defined.

By the fact that $l_1 : D_{-1} \longrightarrow D_0$ is a double vector bundle morphism over $\mathfrak{l}_1 : A_{-1} \longrightarrow A_0$, we can deduce that $(\Gamma(A_{-1}), \Gamma(A_0), \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ is a Lie 2-algebra. We only give a proof of the property

$$\mathfrak{l}_1(\mathfrak{l}_2(X_0, X_1)) = \mathfrak{l}_2(X_0, \mathfrak{l}_1(X_1)), \quad \forall X^0 \in \Gamma(A_0),\ X^1 \in \Gamma(A_{-1}). \tag{3.2}$$

The other conditions in the definition of a Lie 2-algebra can be proved similarly. In fact, let $\xi^0 \in \Gamma_B^l(D_0)$, $\xi^1 \in \Gamma_B^l(D_{-1})$ be linear sections over X^0, X^1 respectively, then by the equality $l_1(l_2(\xi^0, \xi^1)) = l_2(\xi^0, l_1(\xi^1))$, we have

$$\pi^{A_0} l_1(l_2(\xi^0, \xi^1)) = \pi^{A_0} l_2(\xi^0, l_1(\xi^1)).$$

Since $l_1 : D_{-1} \longrightarrow D_0$ is a double vector bundle morphism over $\mathfrak{l}_1 : A_{-1} \longrightarrow A_0$, the left hand side is equal to

$$\pi^{A_0} l_1(l_2(\xi^0, \xi^1)) = \mathfrak{l}_1 \pi^{A_{-1}} l_2(\xi^0, \xi^1) = \mathfrak{l}_1 \mathfrak{l}_2(X^0, X^1),$$

and the right hand side is equal to

$$\pi^{A_0} l_2(\xi^0, l_1(\xi^1)) = \mathfrak{l}_2(\pi^{A_0}(\xi^0), \pi^{A_0}(l_1(\xi^1))) = \mathfrak{l}_2(X_0, \mathfrak{l}_1(X^1)).$$

Thus, we deduce that (3.2) holds.

Finally, for all $X^0 \in \Gamma(A_0)$, $Y^i \in \Gamma(A_{-i})$ and $f \in C^\infty(M)$, let $\xi^0 \in \Gamma^l_B(D_0)$ and $\eta^i \in \Gamma^l_B(D_{-i}),\ i = 0, 1$ be linear sections over X^0 and Y^i. Then $q_B^*(f)\eta^i$ is a linear section over fY^i. By the fact that a is a double vector bundle morphism over $\mathfrak{a}$, we have

$$
\begin{aligned}
\mathfrak{l}_2(X^0, fY^i) &= \pi^{A_{-i}} l_2(\xi^0, q_B^*(f)\eta^i) = \pi^{A_{-i}}(q_B^*(f) l_2(\xi^0, \eta^i) + a(\xi^0)(q_B^*(f))\eta^i) \\
&= f\mathfrak{l}_2(X^0, Y^i) + \mathfrak{a}(X^0)(f)Y^i.
\end{aligned}
$$

Therefore, $(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ is a Lie 2-algebroid.

Remark 1. *By the above theorem, we can view a* VB*-Lie* 2*-algebroid as a Lie* 2*-algebroid object in the category of double vector bundles.*

Consider the associated graded fat bundle $\hat{A}_{-1} \oplus \hat{A}_0$, obviously we have

Proposition 1. *Let* $\begin{pmatrix} D_{-1}; & A_{-1}, B; & M \\ D_0; & A_0, B; & M \end{pmatrix}$ *be a* VB*-Lie* 2*-algebroid. Then* $(\hat{A}_{-1}, \hat{A}_0, \hat{a}, \hat{l}_1, \hat{l}_2, \hat{l}_3)$ *is a split Lie* 2*-algebroid, where* $\hat{a} = \mathfrak{a} \circ \mathrm{pr}$ *and* $\hat{l}_1,\ \hat{l}_2,\ \hat{l}_3$ *are the restriction of* $l_1,\ l_2,\ l_3$ *on linear sections respectively.*

Consequently, we have the following exact sequences of split Lie 2*-algebroids:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B^* \otimes C_{-1} & \longrightarrow & \hat{A}_{-1} & \xrightarrow{\mathrm{pr}} & A_{-1} & \longrightarrow & 0 \\
 & & \big\downarrow & & \hat{l}_1\big\downarrow & & \mathfrak{l}_1\big\downarrow & & \\
0 & \longrightarrow & B^* \otimes C_0 & \longrightarrow & \hat{A}_0 & \xrightarrow{\mathrm{pr}} & A_0 & \longrightarrow & 0
\end{array}
\tag{3.3}
$$

It is helpful to give the split Lie 2-algebroid structure on $B^* \otimes C_{-1} \oplus B^* \otimes C_0$. Since l_1 is linear, it induces a bundle map $l_1^C : C_{-1} \longrightarrow C_0$. The restriction of $\hat{l}_1$ on $B^* \otimes C_{-1}$ is given by

$$
\hat{l}_1(\phi^1) = l_1^C \circ \phi^1, \quad \forall \phi^1 \in \Gamma(B^* \otimes C_{-1}) = \Gamma(\mathrm{Hom}(B, C_{-1})). \tag{3.4}
$$

Since the anchor $a : D_0 \longrightarrow TB$ is a double vector bundle morphism, it induces a bundle map $\varrho : C_0 \longrightarrow B$ via

$$
\langle \varrho(c^0), \xi \rangle = -a(c^0)(\xi), \quad \forall c^0 \in \Gamma(C_0),\ \xi \in \Gamma(B^*). \tag{3.5}
$$

Then by the Leibniz rule, we deduce that the restriction of $\hat{l}_2$ on $\Gamma(B^* \otimes C_{-1} \oplus B^* \otimes C_0)$ is given by

$$
\hat{l}_2(\phi^0, \psi^0) = \phi^0 \circ \varrho \circ \psi^0 - \psi^0 \circ \varrho \circ \phi^0, \tag{3.6}
$$

$$
\hat{l}_2(\phi^0, \psi^1) = -\hat{l}_2(\psi^1, \phi^0) = -\psi^1 \circ \varrho \circ \phi^0, \tag{3.7}
$$

for all $\phi^0, \psi^0 \in \Gamma(B^* \otimes C_0) = \Gamma(\mathrm{Hom}(B, C_0))$, $\psi^1 \in \Gamma(B^* \otimes C_{-1}) = \Gamma(\mathrm{Hom}(B, C_{-1}))$. Since l_3 is linear, the restriction of l_3 on $B^* \otimes C_{-1} \oplus B^* \otimes C_0$ vanishes. Obviously, the anchor is trivial. Thus, the split Lie 2-algebroid structure on $B^* \otimes C_{-1} \oplus B^* \otimes C_0$ is exactly given by (3.4), (3.6) and (3.7). Therefore, $B^* \otimes C_{-1} \oplus B^* \otimes C_0$ is a graded bundle of strict Lie 2-algebras.

An important example of VB-Lie algebroids is the tangent prolongation of a Lie algebroid. Now we explore the tangent prolongation of a Lie 2-algebroid. Recall that for a Lie algebroid $A \longrightarrow M$, TA is a Lie algebroid over TM. A section $\sigma : M \longrightarrow A$ gives rise to a linear section $\sigma_T \triangleq d\sigma : TM \longrightarrow TA$

and a core section $\sigma_C : TM \longrightarrow TA$ by contraction. Any section of TA over TM is generated by such sections. A function $f \in C^\infty(M)$ induces two types of functions on TM by

$$f_C = q^* f, \quad f_T = df,$$

where $q : TM \longrightarrow M$ is the projection. We have the following relations about the module structure:

$$(f\sigma)_C = f_C\sigma_C, \quad (f\sigma)_T = f_T\sigma_C + f_C\sigma_T. \tag{3.8}$$

In particular, for $A = TM$, we have

$$X_T(f_T) = X(f)_T, \quad X_T(f_C) = X(f)_C, \quad X_C(f_T) = X(f)_C, \quad X_C(f_C) = 0, \tag{3.9}$$

for all $X \in \mathfrak{X}(M)$. See [32, Example 2.5.4] and [40] for more details.

Now for split Lie 2-algebroids, we have

Proposition 2. *Let $\mathcal{A} = (A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ be a split Lie 2-algebroid. Then*

$$(TA_{-1}, TA_0, a, l_1, l_2, l_3)$$

is a split Lie 2-algebroid over TM, where $a : TA_0 \longrightarrow TTM$ is given by

$$a(\sigma^0_T) = \mathfrak{a}(\sigma^0)_T, \quad a(\sigma^0_C) = \mathfrak{a}(\sigma^0)_C, \tag{3.10}$$

$l_1 : \Gamma_{TM}(TA_{-1}) \longrightarrow \Gamma_{TM}(TA_0)$ is given by

$$l_1(\sigma^1_T) = \mathfrak{l}_1(\sigma^1)_T, \quad l_1(\sigma^1_C) = \mathfrak{l}_1(\sigma^1)_C, \tag{3.11}$$

$l_2 : \Gamma_{TM}(TA_{-i}) \times \Gamma_{TM}(TA_{-j}) \longrightarrow \Gamma_{TM}(TA_{-(i+j)})$ is given by

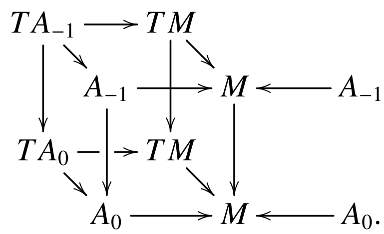
$$\begin{aligned}
&l_2(\sigma^0_T, \tau^0_T) = \mathfrak{l}_2(\sigma^0, \tau^0)_T,\ l_2(\sigma^0_T, \tau^0_C) = \mathfrak{l}_2(\sigma^0, \tau^0)_C,\ l_2(\sigma^0_C, \tau^0_C) = 0,\\
&l_2(\sigma^0_T, \tau^1_T) = \mathfrak{l}_2(\sigma^0, \tau^1)_T,\ l_2(\sigma^0_T, \tau^1_C) = \mathfrak{l}_2(\sigma^0, \tau^1)_C,\ l_2(\sigma^0_C, \tau^1_T) = \mathfrak{l}_2(\sigma^0, \tau^1)_C,\\
&l_2(\sigma^0_C, \tau^1_C) = 0,
\end{aligned}$$

and $l_3 : \wedge^3\Gamma_{TM}(TA_0) \longrightarrow \Gamma_{TM}(TA_{-1})$ is given by

$$l_3(\sigma^0_T, \tau^0_T, \varsigma^0_T) = \mathfrak{l}_3(\sigma^0, \tau^0, \varsigma^0)_T, \quad l_3(\sigma^0_T, \tau^0_T, \varsigma^0_C) = \mathfrak{l}_3(\sigma^0, \tau^0, \varsigma^0)_C, \tag{3.12}$$

and $l_3(\sigma^0_T, \tau^0_C, \varsigma^0_C) = 0$, for all $\sigma^0, \tau^0, \varsigma^0 \in \Gamma(A_0)$ and $\sigma^1, \tau^1 \in \Gamma(A_{-1})$.

Moreover, we have the following VB*-Lie 2-algebroid:*



Proof. By the fact that $\mathcal{A} = (A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ is a split Lie 2-algebroid, it is straightforward to deduce that $(TA_{-1}, TA_0, a, l_1, l_2, l_3)$ is a split Lie 2-algebroid over TM. Moreover, $a,\ l_1,\ l_2,\ l_3$ are all linear, which implies that it is a VB-Lie 2-algebroid. ■

The associated fat bundles of double vector bundles $(TA_{-1}; A_{-1}, TM; M)$ and $(TA_0; A_0, TM; M)$ are the jet bundles $\mathfrak{J}A_{-1}$ and $\mathfrak{J}A_0$ respectively. By Proposition 2 and Proposition 1, we obtain the following result, which is the higher analogue of the fact that the jet bundle of a Lie algebroid is a Lie algebroid.

Corollary 1. *Let $(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ be a split Lie 2-algebroid. Then we obtain that $(\mathfrak{J}A_{-1}, \mathfrak{J}A_0, \hat{a}, \hat{l}_1, \hat{l}_2, \hat{l}_3)$ is a split Lie 2-algebroid, where $\hat{a},\ \hat{l}_1,\ \hat{l}_2,\ \hat{l}_3$ is given by*

$$\begin{array}{rcl} \hat{a}(\sigma^0_T) & = & \mathfrak{a}(\sigma^0), \\ \hat{l}_2(\sigma^0_T, \tau^0_T) & = & \mathfrak{l}_2(\sigma^0, \tau^0)_T, \\ \hat{l}_2(\sigma^0_T, \tau^1_T) & = & \mathfrak{l}_2(\sigma^0, \tau^1)_T, \\ \hat{l}_3(\sigma^0_T, \tau^0_T, \zeta^0_T) & = & \mathfrak{l}_2(\sigma^0, \tau^0, \zeta^0)_T, \end{array}$$

for all $\sigma^0,\ \tau^0,\ \zeta^0 \in \Gamma(A_0)$ and $\tau^1 \in \Gamma(A_{-1})$.

4. Superconnections of a split Lie 2-algebroid on a 3-term complex of vector bundles

In the section, we introduce the notion of a superconnection of a split Lie 2-algebroid on a 3-term complex of vector bundles, which generalizes the notion of a superconnection of a Lie algebroid on a 2-term complex of vector bundles studied in [19]. We show that a VB-Lie 2-algebroid structure on a split graded double vector bundle is equivalent to a flat superconnection of a split Lie 2-algebroid on a 3-term complex of vector bundles.

Denote a 3-term complex of vector bundles $E_{-2} \xrightarrow{\pi} E_{-1} \xrightarrow{\pi} E_0$ by $\mathcal{E}$. Sections of the covariant differential operator bundle $\mathfrak{D}(\mathcal{E})$ are of the form $\mathfrak{d} = (\mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2)$, where $\mathfrak{d}_i : \Gamma(E_{-i}) \longrightarrow \Gamma(E_{-i})$ are $\mathbb{R}$-linear maps such that there exists $X \in \mathfrak{X}(M)$ satisfying

$$\mathfrak{d}_i(fe^i) = f\mathfrak{d}_i(e^i) + X(f)e^i, \quad \forall f \in C^\infty(M),\ e^i \in \Gamma(E_{-i}).$$

Equivalently, $\mathfrak{D}(\mathcal{E}) = \mathfrak{D}(E_0) \times_{TM} \mathfrak{D}(E_{-1}) \times_{TM} \mathfrak{D}(E_{-2})$. Define $\mathfrak{p} : \mathfrak{D}(\mathcal{E}) \longrightarrow TM$ by

$$\mathfrak{p}(\mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2) = X. \tag{4.1}$$

Then the covariant differential operator bundle $\mathfrak{D}(\mathcal{E})$ fits the following exact sequence:

$$0 \longrightarrow \mathrm{End}(E_0) \oplus \mathrm{End}(E_{-1}) \oplus \mathrm{End}(E_{-2}) \longrightarrow \mathfrak{D}(\mathcal{E}) \longrightarrow TM \longrightarrow 0. \tag{4.2}$$

Denote by $\mathrm{End}^{-1}(\mathcal{E}) = \mathrm{Hom}(E_0, E_{-1}) \oplus \mathrm{Hom}(E_{-1}, E_{-2})$. Denote by $\mathrm{End}^{-2}(\mathcal{E}) = \mathrm{Hom}(E_0, E_{-2})$. Define $\mathrm{d} : \mathrm{End}^{-2}(\mathcal{E}) \longrightarrow \mathrm{End}^{-1}(\mathcal{E})$ by

$$\mathrm{d}(\theta^2) = \pi \circ \theta^2 - \theta^2 \circ \pi, \quad \forall \theta^2 \in \Gamma(\mathrm{Hom}(E_0, E_{-2})), \tag{4.3}$$

and define $\mathrm{d} : \mathrm{End}^{-1}(\mathcal{E}) \longrightarrow \mathfrak{D}(\mathcal{E})$ by

$$\mathrm{d}(\theta^1) = \pi \circ \theta^1 + \theta^1 \circ \pi, \quad \forall \theta^1 \in \Gamma(\mathrm{Hom}(E_0, E_{-1}) \oplus \mathrm{Hom}(E_{-1}, E_{-2})). \tag{4.4}$$

Then we define a degree 0 graded symmetric bracket operation $[\cdot,\cdot]_C$ on the section space of the graded bundle $\mathrm{End}^{-2}(\mathcal{E}) \oplus \mathrm{End}^{-1}(\mathcal{E}) \oplus \mathfrak{D}(\mathcal{E})$ by

$$[\mathfrak{d},\mathfrak{t}]_C = \mathfrak{d}\circ\mathfrak{t} - \mathfrak{t}\circ\mathfrak{d}, \quad \forall \mathfrak{d},\mathfrak{t} \in \Gamma(\mathfrak{D}(\mathcal{E})), \tag{4.5}$$

$$[\mathfrak{d},\theta^i]_C = \mathfrak{d}\circ\theta^i - \theta^i\circ\mathfrak{d}, \quad \forall \mathfrak{d} \in \Gamma(\mathfrak{D}(\mathcal{E})),\ \theta^i \in \Gamma(\mathrm{End}^{-i}(\mathcal{E})), \tag{4.6}$$

$$[\theta^1,\vartheta^1]_C = \theta^1\circ\vartheta^1 + \vartheta^1\circ\theta^1, \quad \forall \theta^1,\vartheta^1 \in \Gamma(\mathrm{End}^{-1}(\mathcal{E})). \tag{4.7}$$

Denote by $\mathfrak{D}_\pi(\mathcal{E}) \subset \mathfrak{D}(\mathcal{E})$ the subbundle of $\mathfrak{D}(\mathcal{E})$ whose section $\mathfrak{d} \in \Gamma(\mathfrak{D}_\pi(\mathcal{E}))$ satisfying $\pi\circ\mathfrak{d} = \mathfrak{d}\circ\pi$, or in term of components,

$$\mathfrak{d}_0\circ\pi = \pi\circ\mathfrak{d}_1, \quad \mathfrak{d}_1\circ\pi = \pi\circ\mathfrak{d}_2.$$

It is obvious that $\Gamma(\mathfrak{D}_\pi(\mathcal{E}))$ is closed under the bracket operation $[\cdot,\cdot]_C$ and

$$\mathrm{d}(\mathrm{End}^{-1}(\mathcal{E})) \subset \mathfrak{D}_\pi(\mathcal{E}).$$

Then it is straightforward to verify that

Theorem 4.1. *Let $E_{-2} \xrightarrow{\pi} E_{-1} \xrightarrow{\pi} E_0$ be a 3-term complex of vector bundles over M. Then $(\mathrm{End}^{-2}(\mathcal{E}), \mathrm{End}^{-1}(\mathcal{E}), \mathfrak{D}_\pi(\mathcal{E}), \mathfrak{p}, \mathrm{d}, [\cdot,\cdot]_C)$ is a strict split Lie 3-algebroid.*

With above preparations, we give the definition of a superconnection of a split Lie 2-algebroid on a 3-term complex of vector bundles as follows.

Definition 4.2. A **superconnection** of a split Lie 2-algebroid $(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ on a 3-term complex of vector bundles $E_{-2} \xrightarrow{\pi} E_{-1} \xrightarrow{\pi} E_0$ consists of:

- a bundle morphism $F^0 : A_0 \longrightarrow \mathfrak{D}_\pi(\mathcal{E})$,
- a bundle morphism $F^1 : A_{-1} \longrightarrow \mathrm{End}^{-1}(\mathcal{E})$,
- a bundle morphism $F^2_0 : \wedge^2 A_0 \longrightarrow \mathrm{End}^{-1}(\mathcal{E})$,
- a bundle morphism $F^2_1 : A_0 \wedge A_{-1} \longrightarrow \mathrm{End}^{-2}(\mathcal{E})$,
- a bundle morphism $F^3 : \wedge^3 A_0 \longrightarrow \mathrm{End}^{-2}(\mathcal{E})$.

A superconnection is called **flat** if $(F^0, F^1, F^2_0, F^2_1, F^3)$ is a Lie n-algebroid morphism from the split Lie 2-algebroid $(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ to the strict split Lie 3-algebroid $(\mathrm{End}^{-2}(\mathcal{E}), \mathrm{End}^{-1}(\mathcal{E}), \mathfrak{D}_\pi(\mathcal{E}), \mathfrak{p}, \mathrm{d}, [\cdot,\cdot]_C)$.

Remark 2. *If the split Lie 2-algebroid reduces to a Lie algebroid A and the 3-term complex reduces to a 2-term complex $E_{-1} \xrightarrow{\pi} E_0$, a superconnection will only consists of*

- *a bundle morphism $F^0 = (F^0_0, F^0_1) : A \longrightarrow \mathfrak{D}_\pi(\mathcal{E})$,*
- *a bundle morphism $F^2_0 : \wedge^2 A_0 \longrightarrow \mathrm{Hom}(E_0, E_{-1})$.*

Thus, we recover the notion of a superconnection (also called representation up to homotopy if it is flat) of a Lie algebroid on a 2-term complex of vector bundles. See [1, 19] for more details.

Now we come back to VB-Lie 2-algebroids. Let $(D^B_{-1}, D^B_0, a, l_1, l_2, l_3)$ be a VB-Lie 2-algebroid structure on the graded double vector bundle $\begin{pmatrix} D_{-1}; & A_{-1}, B; & M \\ D_0; & A_0, B; & M \end{pmatrix}$. Recall from Theorem 3.4 and Proposition 1 that both $(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ and $(\hat{A}_{-1}, \hat{A}_0, \hat{a}, \hat{l}_1, \hat{l}_2, \hat{l}_3)$ are split Lie 2-algebroids.

Choose a horizontal lift $s = (s_0, s_1) : A_0 \oplus A_{-1} \longrightarrow \hat{A}_0 \oplus \hat{A}_{-1}$ of the short exact sequence of split Lie 2-algebroids (3.3). Define $\nabla^B : A_0 \longrightarrow \mathfrak{D}(B)$ by

$$\langle \nabla^B_{X^0} b, \xi \rangle = \mathfrak{a}(X^0)\langle \xi, b\rangle - \langle b, \hat{a}(s_0(X^0))(\xi)\rangle, \quad \forall X^0 \in \Gamma(A_0),\ b \in \Gamma(B),\ \xi \in \Gamma(B^*).$$

Since for all $\phi^0 \in \Gamma(B^* \otimes C_0)$, we have $\hat{a}(\phi^0) = 0$, it follows that ∇^B is well-defined.

We define $\nabla^0 : A_0 \longrightarrow \mathfrak{D}(C_0)$ and $\nabla^1 : A_0 \longrightarrow \mathfrak{D}(C_{-1})$ by

$$\nabla^0_{X^0} c^0 = l_2(s_0(X^0), c^0), \quad \nabla^1_{X^0} c^1 = l_2(s_0(X^0), c^1), \tag{4.8}$$

for all $X^0 \in \Gamma(A_0),\ c^0 \in \Gamma(C_0),\ c^1 \in \Gamma(C_{-1})$.

Define $\Upsilon^1 : A_{-1} \longrightarrow \mathrm{Hom}(B, C_0)$ and $\Upsilon^2 : A_{-1} \longrightarrow \mathrm{Hom}(C_0, C_{-1})$ by

$$\Upsilon^1_{X^1} = s_0(\mathfrak{l}_1(X^1)) - \hat{l}_1(s_1(X^1)), \quad \Upsilon^2_{X^1} c^0 = l_2(s_1(X^1), c^0), \tag{4.9}$$

for all $X^1 \in \Gamma(A_{-1}),\ c^0 \in \Gamma(C_0)$. Since l_2 is linear, ∇^0, ∇^1 and Υ are well-defined.

Define $R^0 : \wedge^2\Gamma(A_0) \longrightarrow \Gamma(\mathrm{Hom}(B, C_0))$, $\Lambda : \wedge^2\Gamma(A_0) \longrightarrow \Gamma(\mathrm{Hom}(C_0, C_{-1}))$ and $R^1 : \Gamma(A_0) \wedge \Gamma(A_{-1}) \longrightarrow \Gamma(\mathrm{Hom}(B, C_{-1}))$ by

$$\begin{aligned} R^0(X^0, Y^0) &= s_0\mathfrak{l}_2(X^0, Y^0) - \hat{l}_2(s_0(X^0), s_0(Y^0)), & (4.10)\\ \Lambda(X^0, Y^0)(c^0) &= -l_3(s_0(X^0), s_0(Y^0), c^0), & (4.11)\\ R^1(X^0, Y^1) &= s_1\mathfrak{l}_2(X^0, Y^1) - \hat{l}_2(s_0(X^0), s_1(Y^1)), & (4.12) \end{aligned}$$

for all $X^0,\ Y^0 \in \Gamma(A_0)$ and $Y^1 \in \Gamma(A_{-1})$

Finally, define $\Xi : \wedge^3\Gamma(A_0) \longrightarrow \mathrm{Hom}(B, C_{-1})$ by

$$\Xi(X^0, Y^0, Z^0)) = s_1\mathfrak{l}_3(X^0, Y^0, Z^0) - \hat{l}_3(s_0(X^0), s_0(Y^0), s_0(Z^0)). \tag{4.13}$$

By the equality $l_1 l_2(s_0(X^0), c^1) = l_2(s_0(X^0), l_1^C(c^1))$, we obtain

$$l_1^C \circ \nabla^1_{X^0} = \nabla^0_{X^0} \circ l_1^C. \tag{4.14}$$

By the fact that $a : D_0 \longrightarrow TB$ preserves the bracket operation, we obtain

$$\begin{aligned} \langle \nabla^B_{X^0}\varrho(c^0), \xi\rangle &= \mathfrak{a}(X^0)\langle \varrho(c^0), \xi\rangle - \langle \varrho(c^0), a(s_0(X^0))(\xi)\rangle \\ &= -[a(s_0(X^0)), a(c^0)]_{TB}(\xi) = -a(l_2(s_0(X^0), c^0))(\xi) \\ &= \langle \varrho\nabla^0_{X^0} c^0, \xi\rangle, \end{aligned}$$

which implies that

$$\nabla^B_{X^0} \circ \varrho = \varrho \circ \nabla^0_{X^0}. \tag{4.15}$$

By (4.14) and (4.15), we deduce that $(\nabla^B_{X^0}, \nabla^0_{X^0}, \nabla^1_{X^0}) \in \mathfrak{D}(\mathcal{E})$, where $\mathcal{E}$ is the 3-term complex of vector bundles $C_{-1} \xrightarrow{l_1^C} C_0 \xrightarrow{\varrho} B$. Then we obtain a superconnection $(F^0, F^1, F^2_0, F^2_1, F^3)$ of the Lie 2-algebroid $(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ on the 3-term complex of vector bundles $C_{-1} \xrightarrow{l_1^C} C_0 \xrightarrow{\varrho} B$, where

$$F^0 = (\nabla^B, \nabla^0, \nabla^1), \quad F^1 = (\Upsilon^1, \Upsilon^2), \quad F^2_0 = (R^0, \Lambda), \quad F^2_1 = R^1, \quad F^3 = \Xi.$$

Theorem 4.3. *There is a one-to-one correspondence between* VB*-Lie* 2*-algebroids* $\begin{pmatrix} D_{-1}; & A_{-1}, B; & M \\ D_0; & A_0, B; & M \end{pmatrix}$ *and flat superconnections* $(F^0, F^1, F^2_0, F^2_1, F^3)$ *of the split Lie* 2*-algebroid* $(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ *on the* 3*-term complex of vector bundles* $C_{-1} \xrightarrow{l_1^C} C_0 \xrightarrow{\varrho} B$ *by choosing a horizontal lift* $s = (s_0, s_1) : A_0 \oplus A_{-1} \longrightarrow \hat{A}_0 \oplus \hat{A}_{-1}$.

Proof. First it is obvious that

$$\mathfrak{p} \circ F^0 = \mathfrak{a}. \tag{4.16}$$

Using equalities $\mathfrak{a} \circ \mathfrak{l}_1 = 0$ and $a \circ l_1 = 0$, we have

$$\langle \nabla^B_{\mathfrak{l}_1 X^1} b, \xi \rangle = \mathfrak{a}(\mathfrak{l}_1(X^1)) \langle b, \xi \rangle - \langle b, a(s_0(\mathfrak{l}_1(X^1)))(\xi) \rangle = -\langle b, a(\Upsilon^1_{X^1})(\xi) \rangle,$$

which implies that

$$\nabla^B_{\mathfrak{l}_1 X^1} = \varrho \circ \Upsilon^1_{X^1}. \tag{4.17}$$

For ∇^0, we can obtain

$$\begin{aligned} \nabla^0_{\mathfrak{l}_1(X^1)} &= l_2(s_0 \mathfrak{l}_1(X^1), \cdot)|_{C_0} = l_2(l_1(s_1(X^1)) + \Upsilon^1_{X^1}, \cdot)|_{C_0} \\ &= l_1^C \circ \Upsilon^2_{X^1} + \Upsilon^1_{X^1} \circ \varrho. \end{aligned} \tag{4.18}$$

For ∇^1, we have

$$\nabla^1_{\mathfrak{l}_1(X^1)} = l_2(s_0 \mathfrak{l}_1(X^1), \cdot)|_{C_1} = l_2(l_1(s_1(X^1)) + \Upsilon^1_{X^1}, \cdot)|_{C_1} = \Upsilon^2_{X^1} \circ l_1^C. \tag{4.19}$$

By (4.17), (4.18) and (4.19), we deduce that

$$F^0 \circ \mathfrak{l}_1 = \mathrm{d} \circ F^1. \tag{4.20}$$

By straightforward computation, we have

$$\begin{aligned} &\langle \nabla^B_{\mathfrak{l}_2(X^0, Y^0)} b - \nabla^B_{X^0} \nabla^B_{Y^0} b + \nabla^B_{Y^0} \nabla^B_{X^0} b, \xi \rangle \\ = &\ \langle b, a(\hat{l}_2(s_0(X^0), s_0(Y_0)) - s_0 \mathfrak{l}_2(X^0, Y^0))(\xi) \rangle \\ = &\ \langle b, -a(R^0(X^0, Y^0))(\xi) \rangle, \end{aligned}$$

which implies that

$$\nabla^B_{\mathfrak{l}_2(X^0, Y^0)} - \nabla^B_{X^0} \nabla^B_{Y^0} + \nabla^B_{Y^0} \nabla^B_{X^0} = \varrho \circ R^0(X^0, Y^0). \tag{4.21}$$

Similarly, we have

$$\begin{aligned} &\nabla^0_{\mathfrak{l}_2(X^0, Y^0)} c^0 - \nabla^0_{X^0} \nabla^0_{Y^0} c^0 + \nabla^0_{Y^0} \nabla^0_{X^0} c^0 \\ = &\ l_2(s_0 \mathfrak{l}_2(X^0, Y^0), c^0) - l_2(s_0(X^0), l_2(s_0(Y_0), c^0)) + l_2(s_0(Y^0), l_2(s_0(X_0), c^0)) \\ = &\ -l_1 l_3(s_0(X^0), s_0(Y_0), c^0) + l_2(R^0(X^0, Y^0), c^0), \end{aligned}$$

which implies that

$$\nabla^0_{\mathfrak{l}_2(X^0, Y^0)} - \nabla^0_{X^0} \nabla^0_{Y^0} + \nabla^0_{Y^0} \nabla^0_{X^0} = l_1^C \circ \Lambda(X^0, Y^0) + R^0(X^0, Y^0) \circ \varrho, \tag{4.22}$$

and

$$
\begin{aligned}
& \nabla^1_{\mathfrak{l}_2(X^0,Y^0)}c^1 - \nabla^1_{X^0}\nabla^1_{Y^0}c^1 + \nabla^1_{Y^0}\nabla^1_{X^0}c^1 \\
= \; & l_2(s_0\mathfrak{l}_2(X^0,Y^0),c^1) - l_2(s_0(X^0), l_2(s_0(Y_0),c^1)) + l_2(s_0(Y^0), l_2(s_0(X_0),c^1)) \\
= \; & -l_3(s_0(X^0), s_0(Y^0), l_1(c^1)) + l_2(R^0(X^0,Y^0),c^1),
\end{aligned}
$$

which implies that

$$\nabla^1_{\mathfrak{l}_2(X^0,Y^0)} - \nabla^1_{X^0}\nabla^1_{Y^0} + \nabla^1_{Y^0}\nabla^1_{X^0} = \Lambda(X^0,Y^0)\circ l_1^C. \tag{4.23}$$

By (4.21), (4.22) and (4.23), we obtain

$$F^0(\mathfrak{l}_2(X^0,Y^0)) - [F^0(X^0),F^0(Y^0)]_C = \mathrm{d}F_0^2(X^0,Y^0). \tag{4.24}$$

By the equality

$$l_2(s_0(X^0), l_2(s_1(Y^1),c^0)) + c.p. = \hat{l}_3(s_0(X^0), l_1(s_1(Y^1)), c^0),$$

we obtain

$$[F^0(X^0), \Upsilon^2_{Y^1}]_C - \Upsilon^2_{\mathfrak{l}_2(X^0,Y^1)} = -\Lambda(X^0,\mathfrak{l}_1(Y^1)) - R^1(X^0,Y^1)\circ\varrho. \tag{4.25}$$

Furthermore, we have

$$
\begin{aligned}
\Upsilon^1_{\mathfrak{l}_2(X^0,Y^1)} &= s_0\mathfrak{l}_1(\mathfrak{l}_2(X^0,Y^1)) - \hat{l}_1 s_1(\mathfrak{l}_2(X^0,Y^1)) \\
&= s_0\mathfrak{l}_2(X^0,\mathfrak{l}_1(Y^1)) - \hat{l}_1\hat{l}_2(s_0(X^0),s_1(Y^1)) - \hat{l}_1 R^1(X^0,Y^1) \\
&= s_0\mathfrak{l}_2(X^0,\mathfrak{l}_1(Y^1)) - \hat{l}_2(s_0(X^0),\hat{l}_1 s_1(Y^1)) - l_1^C\circ R^1(X^0,Y^1) \\
&= s_0\mathfrak{l}_2(X^0,\mathfrak{l}_1(Y^1)) - \hat{l}_2(s_0(X^0), s_0\mathfrak{l}_1(Y^1) - \Upsilon^1_{Y^1}) - l_1^C\circ R^1(X^0,Y^1) \\
&= [F^0(X^0),\Upsilon^1_{Y^1}]_C + R^0(X^0,\mathfrak{l}_1(Y^1)) - l_1^C\circ R^1(X^0,Y^1).
\end{aligned} \tag{4.26}
$$

By (4.25) and (4.26), we deduce that

$$F^1(\mathfrak{l}_2(X^0,Y^1)) - [F^0(X^0),F^1(Y^1)]_C = F_0^2(X^0,\mathfrak{l}_1(Y^1)) - \mathrm{d}F_1^2(X^0,Y^1). \tag{4.27}$$

By straightforward computation, we have

$$
\begin{aligned}
& R^1(\mathfrak{l}_1(X^1),Y^1) - R^1(X^1,\mathfrak{l}_1(Y^1)) \\
= \; & s_1\mathfrak{l}_2(\mathfrak{l}_1(X^1),Y^1) - \hat{l}_2(s_0\mathfrak{l}_1(X^1),s_1(Y^1)) \\
& -s_1\mathfrak{l}_2(X^1,\mathfrak{l}_1(Y^1)) + \hat{l}_2(s_1(X^1),s_0\mathfrak{l}_1(Y^1)) \\
= \; & \hat{l}_2(s_1(X^1),\hat{l}_1 s_1(Y^1)) + \hat{l}_2(s_1(X^1),\Upsilon^1_{Y^1}) - \hat{l}_2(s_0\mathfrak{l}_1(X^1),s_1(Y^1)) \\
= \; & -\hat{l}_2(\Upsilon^1_{X^1},s_1(Y^1)) + \hat{l}_2(s_1(X^1),\Upsilon^1_{Y^1}) \\
= \; & [\Upsilon^1_{X^1}+\Upsilon^2_{X^1}, \Upsilon^1_{Y^1}+\Upsilon^2_{Y^1}]_C.
\end{aligned} \tag{4.28}
$$

By the equality

$$\hat{l}_2(s_0(X^0),\hat{l}_2(s_0(Y^0),s_0(Z^0))) + c.p. = \hat{l}_1\hat{l}_3(s_0(X^0),s_0(Y^0),s_0(Z^0)),$$

we deduce that

$$\begin{aligned}&[F^0(X^0), R^0(Y^0, Z^0)]_C + R^0(X^0, \mathfrak{l}_2(Y^0, Z^0)) + c.p.\\ = \ & \Upsilon^1_{\mathfrak{l}_3(X^0,Y^0,Z^0)} + l_1^C \circ \Xi(X^0, Y^0, Z^0).\end{aligned} \tag{4.29}$$

By the equality

$$l_2(s_0(X^0), l_3(s_0(Y^0), s_0(Z^0), c^0)) - l_3(l_2(s_0(X^0), s_0(Y^0)), s_0(Z^0), c^0) + c.p. = 0,$$

we deduce that

$$\begin{aligned}&-[F^0(X^0), \Lambda(Y^0, Z^0)]_C + \Lambda(\mathfrak{l}_2(X^0, Y^0), Z^0) + c.p.\\ &+\Upsilon^2_{\mathfrak{l}_3(X^0,Y^0,Z^0)} - \Xi(X^0, Y^0, Z^0) \circ \varrho = 0.\end{aligned} \tag{4.30}$$

By (4.29) and (4.30), we obtain

$$\begin{aligned}&[F^0(X^0), F_0^2(Y^0, Z^0)]_C + F_0^2(X^0, \mathfrak{l}_2(Y^0, Z^0)) + c.p.\\ = \ & F^1(\mathfrak{l}_3(X^0, Y^0, Z^0)) + \mathrm{d}F^3(X^0, Y^0, Z^0).\end{aligned} \tag{4.31}$$

Then by the equality

$$\hat{l}_2(s_0(X^0), \hat{l}_2(s_0(Y^0), s_1(Z^1))) + c.p. = \hat{l}_3(s_0(X^0), s_0(Y^0), \hat{l}_1(s_1(Z^1))),$$

we deduce that

$$\begin{aligned}&[F^0(X^0), R^1(Y^0, Z^1)]_C + [F^0(Y^0), R^1(Z^1, X^0)]_C + [\Upsilon^2_{Z^1}, R^0(X^0, Y^0)]_C\\ &+R^1(X^0, \mathfrak{l}_2(Y^0, Z^1)) + R^1(Y^0, \mathfrak{l}_2(Z^1, X^0)) + R^1(Z^1, \mathfrak{l}_2(X^0, Y^0))\\ = \ & \Xi(X^0, Y^0, \mathfrak{l}_1(Z^1)) - [\Lambda(X^0, Y^0), \Upsilon^1_{Z^1}]_C.\end{aligned} \tag{4.32}$$

Finally, by the equality

$$\begin{aligned}&\sum_{i=1}^{4}(-1)^{i+1}\hat{l}_2(s_0(X_i^0), \hat{l}_3(s_0(X_1^0), \cdots, \widehat{s_0(X_i^0)}, \cdots, s_0(X_4^0)))\\ &+\sum_{i<j,k<l}(-1)^{i+j}\hat{l}_3(\hat{l}_2(s_0(X_i^0), s_0(X_j^0)), s_0(X_k^0), s_0(X_l^0)) = 0,\end{aligned}$$

we deduce that

$$\begin{aligned}&\sum_{i=1}^{4}(-1)^{i+1}\Big([F^0(X_i^0), \Xi(X_1^0, \cdots, \widehat{X_i^0}, \cdots, X_4^0)]_C\\ &+R^1(X_i^0, \mathfrak{l}_3(X_1^0, \cdots, \widehat{X_i^0}, \cdots, X_4^0))\Big)\\ &+\sum_{i<j}(-1)^{i+j}\Big(\Xi(\mathfrak{l}_2(X_i^0, X_j^0), X_1^0, \cdots, \widehat{X_i^0}, \cdots, \widehat{X_j^0}, \cdots, X_4^0)\\ &-[R^0(X_i^0, X_j^0), \Lambda(X_1^0, \cdots, \widehat{X_i^0}, \cdots, \widehat{X_j^0}, \cdots, X_4^0)]_C\Big) = 0.\end{aligned} \tag{4.33}$$

By (4.16), (4.20), (4.24), (4.27), (4.28), (4.31)-(4.33), we deduce that $(F^0, F^1, F^2_0, F^2_1, F^3)$ is a morphism from the split Lie 2-algebroid $(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ to the strict split Lie 3-algebroid

$$(\mathrm{End}^{-2}(\mathcal{E}), \mathrm{End}^{-1}(\mathcal{E}), \mathfrak{D}_\pi(\mathcal{E}), \mathfrak{p}, \mathrm{d}, [\cdot,\cdot]_C).$$

Conversely, let $(A_{-1}, A_0, \mathfrak{a}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3)$ be a split Lie 2-algebroid and $(F^0, F^1, F^2_0, F^2_1, F^3)$ a flat superconnection on the 3-term complex $C_{-1} \xrightarrow{l_1^C} C_0 \xrightarrow{\varrho} B$. Then we can obtain a VB-Lie 2-algebroid structure on the split graded double vector bundle $\begin{pmatrix} A_{-1} \oplus B \oplus C_{-1}; & A_{-1}, B; & M \\ A_0 \oplus B \oplus C_0; & A_0, B; & M \end{pmatrix}$. We leave the details to readers. The proof is finished.

5. VB-CLWX 2-algebroids

In this section, first we recall the notion of a CLWX 2-algebroid. Then we explore what is a metric graded double vector bundle, and introduce the notion of a VB-CLWX 2-algebroid, which can be viewed as the categorification of a VB-Courant algebroid introduced in [32].

As a model for "Leibniz algebras that satisfy Jacobi identity up to all higher homotopies", the notion of a strongly homotopy Leibniz algebra, or a Lod_∞-algebra was given in [36] by Livernet, which was further studied by Ammar and Poncin in [3]. In [50], the authors introduced the notion of a Leibniz 2-algebra, which is the categorification of a Leibniz algebra, and proved that the category of Leibniz 2-algebras and the category of 2-term Lod_∞-algebras are equivalent. Due to this reason, a 2-term Lod_∞-algebra will be called a Leibniz 2-algebra directly in the sequel.

Definition 5.1. ([34]) A CLWX **2-algebroid** is a graded vector bundle $\mathcal{E} = E_{-1} \oplus E_0$ over M equipped with a non-degenerate graded symmetric bilinear form S on $\mathcal{E}$, a bilinear operation $\diamond : \Gamma(E_{-i}) \times \Gamma(E_{-j}) \longrightarrow \Gamma(E_{-(i+j)})$, $0 \le i + j \le 1$, which is skewsymmetric on $\Gamma(E_0) \times \Gamma(E_0)$, an E_{-1}-valued 3-form Ω on E_0, two bundle maps $\partial : E_{-1} \longrightarrow E_0$ and $\rho : E_0 \longrightarrow TM$, such that E_{-1} and E_0 are isotropic and the following axioms are satisfied:

(i) $(\Gamma(E_{-1}), \Gamma(E_0), \partial, \diamond, \Omega)$ is a Leibniz 2-algebra;

(ii) for all $e \in \Gamma(\mathcal{E})$, $e \diamond e = \frac{1}{2}\mathcal{D}S(e, e)$, where $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E_{-1})$ is defined by

$$S(\mathcal{D}f, e^0) = \rho(e^0)(f), \quad \forall f \in C^\infty(M),\ e^0 \in \Gamma(E_0); \tag{5.1}$$

(iii) for all $e^1_1, e^1_2 \in \Gamma(E_{-1})$, $S(\partial(e^1_1), e^1_2) = S(e^1_1, \partial(e^1_2))$;

(iv) for all $e_1, e_2, e_3 \in \Gamma(\mathcal{E})$, $\rho(e_1)S(e_2, e_3) = S(e_1 \diamond e_2, e_3) + S(e_2, e_1 \diamond e_3)$;

(v) for all $e^0_1, e^0_2, e^0_3, e^0_4 \in \Gamma(E_0)$, $S(\Omega(e^0_1, e^0_2, e^0_3), e^0_4) = -S(e^0_3, \Omega(e^0_1, e^0_2, e^0_4))$.

Denote a CLWX 2-algebroid by $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$, or simply by $\mathcal{E}$. Since the section space of a CLWX 2-algebroid is a Leibniz 2-algebra, the section space of a Courant algebroid is a Leibniz algebra and Leibniz 2-algebras are the categorification of Leibniz algebras, we can view CLWX 2-algebroids as the categorification of Courant algebroids.

As a higher analogue of Roytenberg's result about symplectic NQ manifolds of degree 2 and Courant algebroids ([45]), we have

Theorem 5.2. ([34]) *Let* $(T^*[3]A^*[2], \Theta)$ *be a symplectic NQ manifold of degree* 3*, where* A *is an ordinary vector bundle and* Θ *is a degree* 4 *function on* $T^*[3]A^*[2]$ *satisfying* $\{\Theta, \Theta\} = 0$*. Here* $\{\cdot, \cdot\}$ *is the canonical Poisson bracket on* $T^*[3]A^*[2]$*. Then* $(A^*[1], A, \partial, \rho, S, \diamond, \Omega)$ *is a* CLWX 2*-algebroid, where the bilinear form* S *is given by*

$$S(X + \alpha, Y + \beta) = \langle X, \beta\rangle + \langle Y, \alpha\rangle, \quad \forall\, X, Y \in \Gamma(A), \alpha, \beta \in \Gamma(A^*),$$

and ∂*,* ρ*,* $\diamond$ *and* Ω *are given by derived brackets. More precisely, we have*

$$\begin{array}{rcll} \partial\alpha &=& \{\alpha, \Theta\}, & \forall\, \alpha \in \Gamma(A^*), \\ \rho(X)(f) &=& \{f, \{X, \Theta\}\}, & \forall\, X \in \Gamma(A), f \in C^\infty(M), \\ X \diamond Y &=& \{Y, \{X, \Theta\}\}, & \forall\, X, Y \in \Gamma(A), \\ X \diamond \alpha &=& \{\alpha, \{X, \Theta\}\}, & \forall\, X \in \Gamma(A), \alpha \in \Gamma(A^*), \\ \alpha \diamond X &=& -\{X, \{\alpha, \Theta\}\}, & \forall\, X \in \Gamma(A), \alpha \in \Gamma(A^*), \\ \Omega(X, Y, Z) &=& \{Z, \{Y, \{X, \Theta\}\}\}, & \forall\, X, Y, Z \in \Gamma(A). \end{array}$$

See [27, 53] for more information of derived brackets. Note that various kinds of geometric structures were obtained in the study of QP manifolds of degree 3, e.g. Grutzmann's H-twisted Lie algebroids [21] and Ikeda-Uchino's Lie algebroids up to homotopy [23].

Definition 5.3. A **metric graded double vector bundle** is a graded double vector bundle $\left(\begin{array}{ccc} D_{-1}; & A_{-1}, B; & M \\ D_0; & A_0, B; & M \end{array}\right)$ equipped with a degree 1 nondegenerate graded symmetric bilinear form S on the graded bundle $D^B_{-1} \oplus D^B_0$ such that it induces an isomorphism between graded double vector bundles

$$\begin{array}{ccccc} D_{-1} & \longrightarrow & B & & \\ \searrow & & \searrow & & \\ & A_{-1} & \longrightarrow & M & \longleftarrow C_{-1} \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} D_0^{\star B}[1] & \longrightarrow & B & & \\ \searrow & & \searrow & & \\ & C_0^* & \longrightarrow & M & \longleftarrow A_0^* \end{array}$$

$$\begin{array}{ccccc} D_0 & \longrightarrow & B & & \\ \searrow & & \searrow & & \\ & A_0 & \longrightarrow & M & \longleftarrow C_0 \end{array} \qquad \qquad \begin{array}{ccccc} D_{-1}^{\star B}[1] & \longrightarrow & B & & \\ \searrow & & \searrow & & \\ & C_{-1}^* & \longrightarrow & M & \longleftarrow A_{-1}^*, \end{array}$$

where $\star B$ means dual over B.

Given a metric graded double vector bundle, we have

$$C_0 \cong A^*_{-1}, \quad C_{-1} \cong A^*_0.$$

In the sequel, we will always identify C_0 with A^*_{-1}, C_{-1} with A^*_0. Thus, a metric graded double vector bundle is of the following form:

$$\begin{array}{ccccc} D_{-1} & \longrightarrow & B & & \\ \searrow & & \searrow & & \\ & A_{-1} & \longrightarrow & M & \longleftarrow A_0^* \end{array}$$

$$\begin{array}{ccccc} D_0 & \longrightarrow & B & & \\ \searrow & & \searrow & & \\ & A_0 & \longrightarrow & M & \longleftarrow A_{-1}^*. \end{array}$$

Now we are ready to put a CLWX 2-algebroid structure on a graded double vector bundle.

Definition 5.4. A VB-CLWX **2-algebroid** is a metric graded double vector bundle

$$\left(\left(\begin{array}{ccc} D_{-1}; & A_{-1}, B; & M \\ D_0; & A_0, B; & M \end{array}\right), S\right),$$

equipped with a CLWX 2-algebroid structure $(D^B_{-1}, D^B_0, \partial, \rho, S, \diamond, \Omega)$ such that

(i) ∂ is linear, i.e. there exists a unique bundle map $\overline{\partial} : A_{-1} \longrightarrow A_0$ such that $\partial : D_{-1} \longrightarrow D_0$ is a double vector bundle morphism over $\overline{\partial} : A_{-1} \longrightarrow A_0$ (see Diagram (iii));

(ii) the anchor ρ is a linear, i.e. there exists a unique bundle map $\overline{\rho} : A_0 \longrightarrow TM$ such that $\rho : D_0 \longrightarrow TB$ is a double vector bundle morphism over $\overline{\rho} : A_0 \longrightarrow TM$ (see Diagram (iv));

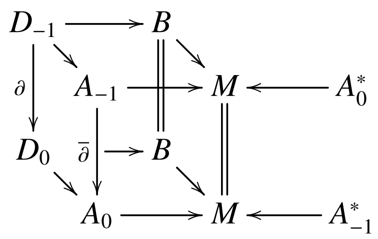


Diagram (iii)

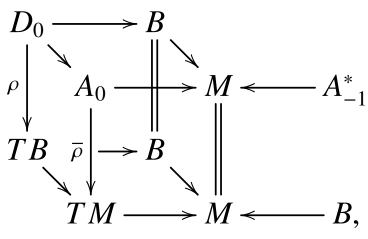


Diagram (iv)

(iii) the operation $\diamond$ is linear;

(iv) Ω is linear.

Since a CLWX 2-algebroid can be viewed as the categorification of a Courant algebroid, we can view a VB-CLWX 2-algebroid as the categorification of a VB-Courant algebroid.

Example 1. Let $(A_{-1}, A_0, a, l_1, l_2, l_3)$ be a Lie 2-algebroid. Let $E_0 = A_0 \oplus A^*_{-1}$, $E_{-1} = A_{-1} \oplus A^*_0$ and $\mathcal{E} = E_0 \oplus E_{-1}$. Then $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$ is a CLWX 2-algebroid, where $\partial : E_{-1} \longrightarrow E_0$ is given by

$$\partial(X^1 + \alpha^0) = l_1(X^1) + l^*_1(\alpha^0), \quad \forall X^1 \in \Gamma(A_{-1}),\ \alpha^0 \in \Gamma(A^*_0),$$

$\rho : E_0 \longrightarrow TM$ is given by

$$\rho(X^0 + \alpha^1) = a(X^0), \quad \forall X^0 \in \Gamma(A_0),\ \alpha^1 \in \Gamma(A^*_{-1}),$$

the symmetric bilinear form $S = (\cdot, \cdot)_+$ is given by

$$(X^0 + \alpha^1 + X^1 + \alpha^0, Y^0 + \beta^1 + Y^1 + \beta^0)_+ = \langle X^0, \beta^0\rangle + \langle Y^0, \alpha^0\rangle + \langle X^1, \beta^1\rangle + \langle Y^1, \alpha^1\rangle,$$

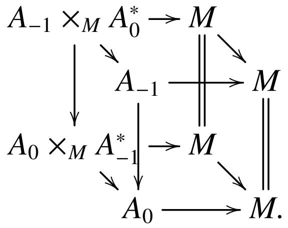
the operation $\diamond$ is given by

$$\left\{\begin{array}{rcl} (X^0 + \alpha^1) \diamond (Y^0 + \beta^1) & = & l_2(X^0, Y^0) + L^0_{X^0}\beta^1 - L^0_{Y^0}\alpha^1, \\ (X^0 + \alpha^1) \diamond (X^1 + \alpha^0) & = & l_2(X^0, X^1) + L^0_{X^0}\alpha^0 + \iota_{X^1}\delta(\alpha^1), \\ (X^1 + \alpha^0) \diamond (X^0 + \alpha^1) & = & l_2(X^1, X^0) + L^1_{X^1}\alpha^1 - \iota_{X^0}\delta(\alpha^0), \end{array}\right. \tag{5.2}$$

and the E_{-1}-valued 3-form Ω is defined by

$$\Omega(X^0 + \alpha^1, Y^0 + \beta^1, Z^0 + \zeta^1) = l_3(X^0, Y^0, Z^0) + L^3_{X^0, Y^0}\zeta^1 + L^3_{Z^0, X^0}\beta^1 + L^3_{Y^0, Z^0}\alpha^1,$$

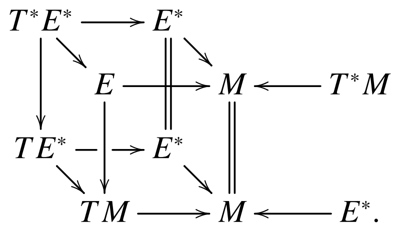
where L^0, L^1, L^3 are given by (3.1). It is straightforward to see that this CLWX 2-algebroid gives rise to a VB-CLWX 2-algebroid:



Example 2. For any manifold M, $(T^*[1]M, TM, \partial = 0, \rho = \mathrm{id}, S, \diamond, \Omega = 0)$ is a CLWX 2-algebroid, where S is the natural symmetric pairing between TM and T^*M, and $\diamond$ is the standard Dorfman bracket given by

$$(X + \alpha) \diamond (Y + \beta) = [X, Y] + L_X\beta - \iota_Y d\alpha, \quad \forall\, X, Y \in \mathfrak{X}(M),\ \alpha, \beta \in \Omega^1(M). \tag{5.3}$$

See [34, Remark 3.4] for more details. In particular, for any vector bundle E, $(T^*E^*, TE^*, \partial = 0, \rho = \mathrm{id}, S, \diamond, \Omega = 0)$ is a CLWX 2-algebroid, which gives rise to a VB-CLWX 2-algebroid:



We have a higher analogue of Theorem 2.3:

Theorem 5.5. *There is a one-to-one correspondence between split Lie* 3*-algebroids and split* VB-CLWX 2*-algebroids.*

Proof. Let $\mathcal{A} = (A_{-2}, A_{-1}, A_0, a, l_1, l_2, l_3, l_4)$ be a split Lie 3-algebroid. Then $T^*[3]\mathcal{A}[1]$ is a symplectic NQ manifold of degree 3. Note that

$$T^*[3]\mathcal{A}[1] = T^*[3](A_0 \times_M A_{-1}^* \times_M A_{-2}^*)[1],$$

where $A_0 \times_M A_{-1}^* \times_M A_{-2}^*$ is viewed as a vector bundle over the base A_{-2}^* and $A_{-1} \times_M A_0^* \times_M A_{-2}^*$ is its dual bundle. Denote by $(x^i, \mu_j, \xi^k, \theta_l, p_i, \mu^j, \xi_k, \theta^l)$ a canonical (Darboux) coordinate on $T^*[3](A_0 \times_M A_{-1}^* \times_M A_{-2}^*)[1]$, where x^i is a smooth coordinate on M, $\mu_j \in \Gamma(A_{-2})$ is a fibre coordinate on A_{-2}^*, $\xi^k \in \Gamma(A_0^*)$ is a fibre coordinate on A_0, $\theta_l \in \Gamma(A_{-1})$ is a fibre coordinate on A_{-1}^* and $(p_i, \mu^j, \xi_k, \theta^l)$ are the momentum coordinates for $(x^i, \mu_j, \xi^k, \theta_l)$. About their degrees, we have

$$\begin{pmatrix} x^i & \mu_j & \xi^k & \theta_l & p_i & \mu^j & \xi_k & \theta^l \\ 0 & 0 & 1 & 1 & 3 & 3 & 2 & 2 \end{pmatrix}$$

The symplectic structure is given by

$$\omega = dx^i dp_i + d\mu_j d\mu^j + d\xi^k d\xi_k + d\theta_l d\theta^l,$$

which is degree 3. The Lie 3-algebroid structure gives rise to a degree 4 function Θ satisfying $\{\Theta, \Theta\} = 0$. By Theorem 5.2, we obtain a CLWX 2-algebroid $(D_{-1}, D_0, \partial, \rho, S, \diamond, \Omega)$, where $D_{-1} = A_{-1} \times_M A_0^* \times_M A_{-2}^*$

and $D_0 = A_0 \times_M A^*_{-1} \times_M A^*_{-2}$ are vector bundles over A^*_{-2}. Obviously, they give the graded double vector bundle

$$\left(\begin{array}{lll} A_{-1} \times_M A^*_0 \times_M A^*_{-2}; & A_{-1}, A^*_{-2}; & M \\ A_0 \times_M A^*_{-1} \times_M A^*_{-2}; & A_0, A^*_{-2}; & M \end{array}\right).$$

The section space $\Gamma_{A^*_{-2}}(D_0)$ are generated by $\Gamma(A^*_{-1})$ (the space of core sections) and $\Gamma(A_{-2} \otimes A^*_{-1}) \oplus \Gamma(A_0)$ (the space of linear sections) as $C^\infty(A^*_{-2})$-module. Similarly, The section space $\Gamma_{A^*_{-2}}(D_{-1})$ are generated by $\Gamma(A^*_0)$ and $\Gamma(A_{-2} \otimes A^*_0) \oplus \Gamma(A_{-1})$ as $C^\infty(A^*_{-2})$-module. Thus, in the sequel we only consider core sections and linear sections.

The graded symmetric bilinear form S is given by

$$\begin{aligned} S(e^0, e^1) &= S(X^0 + \psi^1 + \alpha^1, X^1 + \psi^0 + \alpha^0) \\ &= \langle \alpha_1, X^1 \rangle + \langle \alpha^0, X_0 \rangle + \psi^1(X^1) + \psi^0(X^0), \end{aligned}$$

for all $e^0 = X^0 + \psi^1 + \alpha^1 \in \Gamma_{A^*_{-2}}(D_0)$ and $e^1 = X^1 + \psi^0 + \alpha^0 \in \Gamma_{A^*_{-2}}(D_{-1})$, where $X^i \in \Gamma(A_{-i})$, $\psi^i \in \Gamma(A_{-2} \otimes A^*_{-i})$ and $\alpha^i \in \Gamma(A^*_{-i})$. Then it is obvious that

$$\left(\left(\begin{array}{lll} A_{-1} \times_M A^*_0 \times_M A^*_{-2}; & A_{-1}, A^*_{-2}; & M \\ A_0 \times_M A^*_{-1} \times_M A^*_{-2}; & A_0, A^*_{-2}; & M \end{array}\right), S\right)$$

is a metric graded double vector bundle.

The bundle map $\partial : D_{-1} \longrightarrow D_0$ is given by

$$\partial(X^1 + \psi^0 + \alpha^0) = l_1(X^1) + l_2(X^1, \cdot)|_{A_{-1}} + \psi^0 \circ l_1 + l_1^*(\alpha^0).$$

Thus, $\partial : D_{-1} \longrightarrow D_0$ is a double vector bundle morphism over $l_1 : A_{-1} \longrightarrow A_0$.

Note that functions on A^*_{-2} are generated by fibrewise constant functions $C^\infty(M)$ and fibrewise linear functions $\Gamma(A_{-2})$. For all $f \in C^\infty(M)$ and $X^2 \in \Gamma(A_{-2})$, the anchor $\rho : D_0 \longrightarrow TA^*_{-2}$ is given by

$$\rho(X^0 + \psi^1 + \alpha^1)(f + X^2) = a(X^0)(f) + \langle \alpha^1, l_1(X^2) \rangle + l_2(X^0, X^2) + \psi^1(l_1(X^2)).$$

Therefore, for a linear section $X^0 + \psi^1 \in \Gamma^l_{A^*_{-2}}(D_0)$, the image $\rho(X^0 + \psi^1)$ is a linear vector field and for a core section $\alpha^1 \in \Gamma(A^*_{-1})$, the image $\rho(\alpha^1)$ is a constant vector field. Thus, ρ is linear.

The bracket operation $\diamond$ is given by

$$\begin{aligned} & (X^0 + \psi^1 + \alpha^1) \diamond (Y^0 + \phi^1 + \beta^1) \\ = \;& l_2(X^0, Y^0) + l_3(X^0, Y^0, \cdot)|_{A_{-1}} + l_2(X^0, \phi^1(\cdot)) - \phi^1 \circ l_2(X^0, \cdot)|_{A_{-1}} + L^0_{X_0}\beta^1 \\ & + \psi^1 \circ l_2(Y^0, \cdot)|_{A_{-1}} - l_2(Y^0, \psi^1(\cdot)) + \psi^1 \circ l_1 \circ \phi^1 - \phi^1 \circ l_1 \circ \psi^1 - \beta^1 \circ l_1 \circ \psi^1 \\ & - L^0_{Y_0}\alpha^1 + \alpha^1 \circ l_1 \circ \phi^1, \\ & (X^0 + \psi^1 + \alpha^1) \diamond (Y^1 + \phi^0 + \beta^0) \\ = \;& l_2(X^0, Y^1) + l_3(X^0, \cdot, Y^1)|_{A_0} + l_2(X^0, \phi^0(\cdot)) - \phi^0 \circ l_2(X^0, \cdot)|_{A_0} + L^0_{X^0}\beta^0 \\ & - \psi^1 l_2(\cdot, Y^1)|_{A_0} + \delta(\psi^1(Y^1)) + \psi^1 \circ l_1 \circ \phi^0 + \iota_{Y_1}\delta\alpha^1 + \alpha^1 \circ l_1 \circ \phi^0, \\ & (Y^1 + \phi^0 + \beta^0) \diamond (X^0 + \psi^1 + \alpha^1) \\ = \;& l_2(Y^1, X^0) - l_3(X^0, \cdot, Y^1)|_{A_0} - l_2(X^0, \phi^0(\cdot)) + \phi^0 \circ l_2(X^0, \cdot)|_{A_0} + \delta(\phi^0(X^0)) \end{aligned}$$

$$-\iota_{X^0}\delta\beta^0 + \psi^1 l_2(\cdot, Y^1)|_{A_0} - \psi^1 \circ l_1 \circ \phi^0 + L^1_{Y_1}\alpha^1 - \alpha^1 \circ l_1 \circ \phi^0.$$

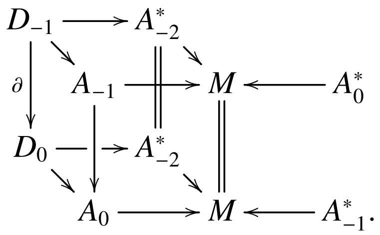
Then it is straightforward to see that the operation $\diamond$ is linear.

Finally, Ω is given by

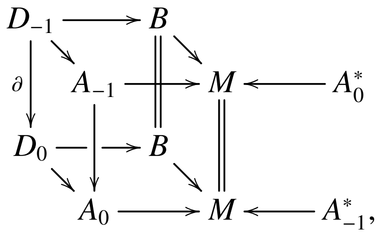
$$\begin{aligned} & \Omega(X^0 + \psi^1 + \alpha^1, Y^0 + \phi^1 + \beta^1, Z^0 + \varphi^1 + \gamma^1) \\ = \; & l_3(X^0, Y^0, Z^0) + l_4(X^0, Y^0, Z^0, \cdot) \\ & -\varphi^1 \circ l_3(X^0, Y^0, \cdot)|_{A_0} - \phi^1 \circ l_3(Z^0, X^0, \cdot)|_{A_0} - \psi^1 \circ l_3(Y^0, Z^0, \cdot)|_{A_0} \\ & + L^3_{X^0,Y^0}\gamma^1 + L^3_{Y^0,Z^0}\alpha^1 + L^3_{Z^0,X^0}\beta^1, \end{aligned}$$

which implies that Ω is also linear.

Thus, a split Lie 3-algebroid gives rise to a split VB-CLWX 2-algebroid:



Conversely, given a split VB-CLWX 2-algebroid:



where $D_{-1} = A_{-1} \times_M A_0^* \times_M B$ and $D_0 = A_0 \times_M A_{-1}^* \times_M B$, then we can deduce that the corresponding symplectic NQ-manifold of degree 3 is $T^*[3]\mathcal{A}[1]$, where $\mathcal{A} = A_0 \oplus A_{-1} \oplus B$ is a graded vector bundle in which B is of degree -2, and the Q-structure gives rise to a Lie 3-algebroid structure on $\mathcal{A}$. We omit details.

Remark 3. *Since every double vector bundle is splitable, every* VB-CLWX *2-algebroid is isomorphic to a split one. Meanwhile, by choosing a splitting, we obtain a split Lie 3-algebroid from an NQ-manifold of degree 3 (Lie 3-algebroid). Thus, we can enhance the above result to be a one-to-one correspondence between Lie 3-algebroids and* VB-CLWX *2-algebroids. We omit such details.*

Recall that the tangent prolongation of a Courant algebroid is a VB-Courant algebroid ([32, Proposition 3.4.1]). Now we show that the tangent prolongation of a CLWX 2-algebroid is a VB-CLWX 2-algebroid. The notations used below is the same as the ones used in Section 3.

Proposition 3. *Let* $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$ *be a* CLWX *2-algebroid. Then we obtain that* $(TE_{-1}, TE_0, \widetilde{\partial}, \widetilde{\rho}, \widetilde{S}, \widetilde{\diamond}, \widetilde{\Omega})$ *is a* CLWX *2-algebroid over* TM*, where the bundle map* $\widetilde{\partial} : TE_{-1} \longrightarrow TE_0$ *is given by*

$$\widetilde{\partial}(\sigma^1_T) = \partial(\sigma^1)_T, \quad \widetilde{\partial}(\sigma^1_C) = \partial(\sigma^1)_C,$$

the bundle map $\widetilde{\rho}: TE_0 \longrightarrow TTM$ *is given by*

$$\widetilde{\rho}(\sigma^0_T) = \rho(\sigma^0)_T, \quad \widetilde{\rho}(\sigma^0_C) = \rho(\sigma^0)_C,$$

the degree 1 *bilinear form* $\widetilde{S}$ *is given by*

$$\begin{array}{rcl} \widetilde{S}(\sigma^0_T, \tau^1_T) & = & S(\sigma^0, \tau^1)_T, \ \widetilde{S}(\sigma^0_T, \tau^1_C) = S(\sigma^0, \tau^1)_C, \\ \widetilde{S}(\sigma^0_C, \tau^1_T) & = & S(\sigma^0, \tau^1)_C, \ \widetilde{S}(\sigma^0_C, \tau^1_C) = 0, \end{array}$$

the bilinear operation $\widetilde{\diamond}$ *is given by*

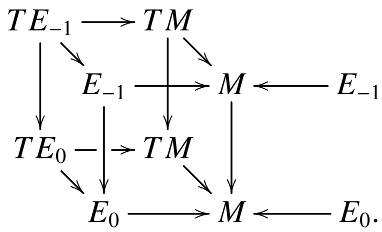
$$\begin{array}{rclrclrcl} \sigma^0_T \widetilde{\diamond} \tau^0_T & = & (\sigma^0 \diamond \tau^0)_T, & \sigma^0_T \widetilde{\diamond} \tau^0_C & = & -\tau^0_C \widetilde{\diamond} \sigma^0_T = (\sigma^0 \diamond \tau^0)_C, & \sigma^0_C \widetilde{\diamond} \tau^0_C & = & 0, \\ \sigma^0_T \widetilde{\diamond} \tau^1_T & = & (\sigma^0 \diamond \tau^1)_T, & \sigma^0_T \widetilde{\diamond} \tau^1_C & = & \sigma^0_C \widetilde{\diamond} \tau^1_T = (\sigma^0 \diamond \tau^1)_C, & \sigma^0_C \widetilde{\diamond} \tau^1_C & = & 0, \\ \tau^1_T \widetilde{\diamond} \sigma^0_T & = & (\tau^1 \diamond \sigma^0)_T, & \tau^1_C \widetilde{\diamond} \sigma^0_T & = & \tau^1_T \widetilde{\diamond} \sigma^0_C = (\tau^1 \diamond \sigma^0)_C, & \tau^1_C \widetilde{\diamond} \sigma^0_C & = & 0, \end{array}$$

and $\widetilde{\Omega}: \wedge^3 TE_0 \longrightarrow TE_{-1}$ *is given by*

$$\widetilde{\Omega}(\sigma^0_T, \tau^0_T, \varsigma^0_T) = \Omega(\sigma^0, \tau^0, \varsigma^0)_T, \quad \widetilde{\Omega}(\sigma^0_T, \tau^0_T, \varsigma^0_C) = \Omega(\sigma^0, \tau^0, \varsigma^0)_C, \quad \widetilde{\Omega}(\sigma^0_T, \tau^0_C, \varsigma^0_C) = 0,$$

for all $\sigma^0, \tau^0, \varsigma^0 \in \Gamma(E_0)$ *and* $\sigma^1, \tau^1 \in \Gamma(E_{-1})$.

Moreover, we have the following VB-CLWX 2*-algebroid:*



Proof. Since $(E_{-1}, E_0, \partial, \rho, S, \diamond, \Omega)$ is a CLWX 2-algebroid, it is straightforward to deduce that $(TE_{-1}, TE_0, \widetilde{\partial}, \widetilde{\rho}, \widetilde{S}, \widetilde{\diamond}, \widetilde{\Omega})$ is a CLWX 2-algebroid over TM. Moveover, it is obvious that $\widetilde{\partial}, \widetilde{\rho}, \widetilde{S}, \widetilde{\diamond}, \widetilde{\Omega}$ are all linear, which implies that we have a VB-CLWX 2-algebroid.

6. E-CLWX 2-algebroid

In this section, we introduce the notion of an E-CLWX 2-algebroid as the categorification of an E-Courant algebroid introduced in [11]. We show that associated to a VB-CLWX 2-algebroid, there is an E-CLWX 2-algebroid structure on the corresponding graded fat bundle.

There is an E-valued pairing $\langle \cdot, \cdot \rangle_E$ between the jet bundle $\mathfrak{J}E$ and the first order covariant differential operator bundle $\mathfrak{D}E$ defined by

$$\langle \mu, \mathfrak{d} \rangle_E \triangleq \mathfrak{d}(u), \quad \forall\ \mathfrak{d} \in (\mathfrak{D}E)_m,\ \mu \in (\mathfrak{J}E)_m,\ u \in \Gamma(E) \text{ statisfying } \mu = [u]_m.$$

Definition 6.1. Let E be a vector bundle. An E-CLWX 2-algebroid is a 6-tuple $(\mathcal{K}, \partial, \rho, \mathcal{S}, \diamond, \Omega)$, where $\mathcal{K} = K_{-1} \oplus K_0$ is a graded vector bundle over M and

- $\partial : K_{-1} \longrightarrow K_0$ is a bundle map;
- $\mathcal{S} : \mathcal{K} \otimes \mathcal{K} \rightarrow E$ is a surjective graded symmetric nondegenerate E-valued pairing of degree 1, which induces an embedding: $\mathcal{K} \hookrightarrow \mathrm{Hom}(\mathcal{K}, E)$;
- $\rho : K_0 \rightarrow \mathfrak{D}E$ is a bundle map, called the anchor, such that $\rho^\star(\mathfrak{J}E) \subset K_{-1}$, i.e.

$$\mathcal{S}(\rho^\star(\mu),\ e^0) = \left\langle \mu, \rho(e^0) \right\rangle_E,\ \forall\, \mu \in \Gamma(\mathfrak{J}E),\ e^0 \in \Gamma(K_0);$$

- $\diamond : \Gamma(K_{-i}) \times \Gamma(K_{-j}) \longrightarrow \Gamma(K_{-(i+j)}),\ 0 \le i + j \le 1$ is an $\mathbb{R}$-bilinear operation;
- $\Omega : \wedge^3 K_0 \longrightarrow K_{-1}$ is a bundle map,

such that the following properties hold:

(E1) $(\Gamma(\mathcal{K}), \partial, \diamond, \Omega)$ is a Leibniz 2-algebra;

(E2) for all $e \in \Gamma(\mathcal{K})$, $e \diamond e = \frac{1}{2}\mathcal{D}\mathcal{S}(e, e)$, where $\mathcal{D} : \Gamma(E) \longrightarrow \Gamma(K_{-1})$ is defined by

$$\mathcal{S}(\mathcal{D}u, e^0) = \rho(e^0)(u), \quad \forall u \in \Gamma(E),\ e^0 \in \Gamma(K_0); \tag{6.1}$$

(E3) for all $e^1_1, e^1_2 \in \Gamma(K_{-1})$, $\mathcal{S}(\partial(e^1_1), e^1_2) = \mathcal{S}(e^1_1, \partial(e^1_2))$;

(E4) for all $e_1, e_2, e_3 \in \Gamma(\mathcal{K})$, $\rho(e_1)\mathcal{S}(e_2, e_3) = \mathcal{S}(e_1 \diamond e_2, e_3) + \mathcal{S}(e_2, e_1 \diamond e_3)$;

(E5) for all $e^0_1, e^0_2, e^0_3, e^0_4 \in \Gamma(K_0)$, $\mathcal{S}(\Omega(e^0_1, e^0_2, e^0_3), e^0_4) = -\mathcal{S}(e^0_3, \Omega(e^0_1, e^0_2, e^0_4))$;

(E6) for all $e^0_1, e^0_2 \in \Gamma(K_0)$, $\rho(e^0_1 \diamond e^0_2) = [\rho(e^0_1), \rho(e^0_2)]_{\mathfrak{D}}$, where $[\cdot, \cdot]_{\mathfrak{D}}$ is the commutator bracket on $\Gamma(\mathfrak{D}E)$.

A CLWX 2-algebroid can give rise to a Lie 3-algebra ([34, Theorem 3.11]). Similarly, an E-CLWX 2-algebroid can also give rise to a Lie 3-algebra. Consider the graded vector space $\mathfrak{e} = \mathfrak{e}_{-2} \oplus \mathfrak{e}_{-1} \oplus \mathfrak{e}_0$, where $\mathfrak{e}_{-2} = \Gamma(E)$, $\mathfrak{e}_{-1} = \Gamma(K_{-1})$ and $\mathfrak{e}_0 = \Gamma(K_0)$. We introduce a skew-symmetric bracket on $\Gamma(\mathcal{K})$,

$$[\![e_1, e_2]\!] = \frac{1}{2}(e_1 \diamond e_2 - e_2 \diamond e_1), \quad \forall\, e_1, e_2 \in \Gamma(\mathcal{K}), \tag{6.2}$$

which is the skew-symmetrization of $\diamond$.

Theorem 6.2. *An E-CLWX 2-algebroid $(\mathcal{K}, \partial, \rho, \mathcal{S}, \diamond, \Omega)$ gives rise to a Lie 3-algebra $(\mathfrak{e}, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4)$, where $\mathfrak{l}_i$ are given by*

$$\begin{array}{rcll}
\mathfrak{l}_1(u) &=& \mathcal{D}(u), & \forall\, u \in \Gamma(E),\\
\mathfrak{l}_1(e^1) &=& \partial(e^1), & \forall\, e^1 \in \Gamma(K_{-1}),\\
\mathfrak{l}_2(e^0_1, e^0_2) &=& [\![e^0_1, e^0_2]\!], & \forall\, e^0_1, e^0_2 \in \Gamma(K_0),\\
\mathfrak{l}_2(e^0, e^1) &=& [\![e^0, e^1]\!], & \forall\, e^0 \in \Gamma(K_0), e^1 \in \Gamma(K_{-1}),\\
\mathfrak{l}_2(e^0, f) &=& \frac{1}{2}\mathcal{S}(e^0, \mathcal{D}f), & \forall\, e^0 \in \Gamma(K_0), f \in \Gamma(E),\\
\mathfrak{l}_2(e^1_1, e^1_2) &=& 0, & \forall\, e^1_1, e^1_2 \in \Gamma(K_{-1}),\\
\mathfrak{l}_3(e^0_1, e^0_2, e^0_3) &=& \Omega(e^0_1, e^0_2, e^0_3), & \forall\, e^0_1, e^0_2, e^0_3 \in \Gamma(K_0),\\
\mathfrak{l}_3(e^0_1, e^0_2, e^1) &=& -T(e^0_1, e^0_2, e^1), & \forall\, e^0_1, e^0_2 \in \Gamma(K_0), e^1 \in \Gamma(K_{-1}),\\
\mathfrak{l}_4(e^0_1, e^0_2, e^0_3, e^0_4) &=& \overline{\Omega}(e^0_1, e^0_2, e^0_3, e^0_4), & \forall\, e^0_1, e^0_2, e^0_3, e^0_4 \in \Gamma(K_0),
\end{array}$$

where the totally skew-symmetric $T:\Gamma(K_0)\times\Gamma(K_0)\times\Gamma(K_{-1})\longrightarrow\Gamma(E)$ *is given by*

$$T(e_1^0,e_2^0,e^1)=\frac{1}{6}(\mathcal{S}(e_1^0,[\![e_2^0,e^1]\!])+\mathcal{S}(e^1,[\![e_1^0,e_2^0]\!])+\mathcal{S}(e_2^0,[\![e^1,e_1^0]\!])), \tag{6.3}$$

and $\overline{\Omega}:\wedge^4\Gamma(K_0)\longrightarrow\Gamma(E)$ *is given by*

$$\overline{\Omega}(e_1^0,e_2^0,e_3^0,e_4^0)=\mathcal{S}(\Omega(e_1^0,e_2^0,e_3^0),e_4^0).$$

Proof. The proof is totally parallel to the proof of [34, Theorem 3.11], we omit the details.

Let $(D_{-1}^B,D_0^B,\partial,\rho,S,\diamond,\Omega)$ be a VB-CLWX 2-algebroid on the graded double vector bundle $\begin{pmatrix} D_{-1}; & A_{-1},B; & M\\ D_0; & A_0,B; & M\end{pmatrix}$. Then we have the associated graded fat bundles $\hat{A}_{-1}\oplus\hat{A}_0$, which fit the exact sequences:

$$0\to B^*\otimes A_0^*\longrightarrow\hat{A}_{-1}\longrightarrow A_{-1}\to 0,$$
$$0\to B^*\otimes A_{-1}^*\longrightarrow\hat{A}_{0}\longrightarrow A_{0}\to 0.$$

Since the bundle map ∂ is linear, it induces a bundle map $\hat{\partial}:\hat{A}_{-1}\longrightarrow\hat{A}_0$. Since the anchor ρ is linear, it induces a bundle map $\hat{\rho}:\hat{A}_0\longrightarrow\mathfrak{D}B^*$, where sections of $\mathfrak{D}B^*$ are viewed as linear vector fields on B. Furthermore, the restriction of S on linear sections will give rise to linear functions on B. Thus, we obtain a B^*-valued degree 1 graded symmetric bilinear form $\hat{S}$ on the graded fat bundle $\hat{A}_{-1}\oplus\hat{A}_0$. Since the operation $\diamond$ is linear, it induces an operation $\hat{\diamond}:\hat{A}_{-i}\times\hat{A}_{-j}\longrightarrow\hat{A}_{-(i+j)}$, $0\le i+j\le 1$. Finally, since Ω is linear, it induces an $\hat{\Omega}:\Gamma(\wedge^3\hat{A}_0)\longrightarrow\hat{A}_{-1}$. Then we obtain:

Theorem 6.3. *A* VB-CLWX *2-algebroid gives rise to a* B^*-CLWX *2-algebroid structure on the corresponding graded fat bundle. More precisely, let* $(D_{-1}^B,D_0^B,\partial,\rho,S,\diamond,\Omega)$ *be a* VB-CLWX *2-algebroid on the graded double vector bundle* $\begin{pmatrix} D_{-1}; & A_{-1},B; & M\\ D_0; & A_0,B; & M\end{pmatrix}$ *with the associated graded fat bundle* $\hat{A}_{-1}\oplus\hat{A}_0$. *Then* $(\hat{A}_{-1},\hat{A}_0,\hat{\partial},\hat{\rho},\hat{S},\hat{\diamond},\hat{\Omega})$ *is a* B^*-CLWX *2-algebroid.*

Proof. Since all the structures defined on the graded fat bundle $\hat{A}_{-1}\oplus\hat{A}_0$ are the restriction of the structures in the VB-CLWX 2-algebroid, it is straightforward to see that all the axioms in Definition 6.1 hold.

Example 3. Consider the VB-CLWX 2-algebroid given in Example 2, the corresponding E-CLWX 2-algebroid is $((\mathfrak{J}E)[1],\mathfrak{D}E,\partial=0,\rho=\mathrm{id},\mathcal{S}=(\cdot,\cdot)_E,\diamond,\Omega=0)$, where the graded symmetric nondegenerate E-valued pairing $(\cdot,\cdot)_E$ is given by

$$(\mathfrak{d}+\mu,\mathfrak{t}+\nu)_E=\langle\mu,\mathfrak{t}\rangle_E+\langle\nu,\mathfrak{d}\rangle_E,\quad\forall\ \mathfrak{d}+\mu,\ \mathfrak{t}+\nu\in\mathfrak{D}E\oplus\mathfrak{J}E,$$

and $\diamond$ is given by

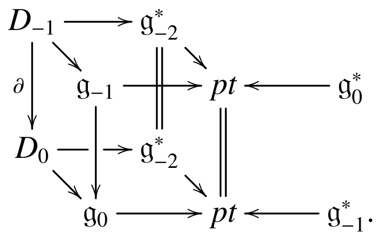
$$(\mathfrak{d}+\mu)\diamond(\mathfrak{r}+\nu)=[\mathfrak{d},\mathfrak{r}]_{\mathfrak{D}}+\mathfrak{L}_{\mathfrak{d}}\nu-\mathfrak{L}_{\mathfrak{r}}\mu+\mathbb{d}\langle\mu,\mathfrak{r}\rangle_E.$$

See [10] for more details.

Example 4. Consider the VB-CLWX 2-algebroid given in Proposition 3. The graded fat bundle is $\mathfrak{J}E_{-1}\oplus\mathfrak{J}E_0$. It follows that the graded jet bundle associated to a CLWX 2-algebroid is a T^*M-CLWX 2-algebroid. This is the higher analogue of the result that the jet bundle of a Courant algebroid is T^*M-Courant algebroid given in [11]. See also [24] for more details.

7. Constructions of Lie 3-algebras

As applications of E-CLWX 2-algebroids introduced in the last section, we construct Lie 3-algebras from Lie 3-algebras in this section. Let $(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}, \mathfrak{g}_0, l_1, l_2, l_3, l_4)$ be a Lie 3-algebra. By Theorem 5.5, the corresponding VB-CLWX 2-algebroid is given by



where $D_{-1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0^* \oplus \mathfrak{g}_{-2}^*$ and $D_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}^* \oplus \mathfrak{g}_{-2}^*$.

By Theorem 6.3, we obtain:

Proposition 4. *Let* $(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}, \mathfrak{g}_0, l_1, l_2, l_3, l_4)$ *be a Lie* 3*-algebra. Then there is an* E-CLWX 2-*algebroid* $(\mathrm{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-2}) \oplus \mathfrak{g}_{-1}, \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-2}) \oplus \mathfrak{g}_0, \partial, \rho, \mathcal{S}, \diamond, \Omega)$*, where for all* $x^i, y^i, z^i \in \mathfrak{g}_{-i}$, $\phi^i, \psi^i, \varphi^i \in \mathrm{Hom}(\mathfrak{g}_{-i}, \mathfrak{g}_{-2})$, $\partial : \mathrm{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-2}) \oplus \mathfrak{g}_{-1} \longrightarrow \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-2}) \oplus \mathfrak{g}_0$ *is given by*

$$\partial(\phi^0 + x^1) = \phi^0 \circ l_1 + l_2(x^1, \cdot)|_{\mathfrak{g}_{-1}} + l_1(x^1), \tag{7.1}$$

$\rho : \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-2}) \oplus \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(\mathfrak{g}_{-2})$ *is given by*

$$\rho(\phi^1 + x^0) = \phi^1 \circ l_1 + l_2(x^0, \cdot)|_{\mathfrak{g}_{-2}}, \tag{7.2}$$

the $\mathfrak{g}_{-2}$*-valued pairing* $\mathcal{S}$ *is given by*

$$\mathcal{S}(\phi^1 + x^0, \psi^0 + y^1) = \phi^1(y^1) + \psi^0(x^0), \tag{7.3}$$

the operation $\diamond$ *is given by*

$$\begin{cases} (x^0 + \psi^1) \diamond (y^0 + \phi^1) = l_2(x^0, y^0) + l_3(x^0, y^0, \cdot)|_{\mathfrak{g}_{-1}} + l_2(x^0, \phi^1(\cdot)) - \phi^1 \circ l_1 \circ \psi^1 \\ \quad -\phi^1 \circ l_2(x^0, \cdot)|_{\mathfrak{g}_{-1}} + \psi^1 \circ l_2(y^0, \cdot)|_{\mathfrak{g}_{-1}} - l_2(y^0, \psi^1(\cdot)) + \psi^1 \circ l_1 \circ \phi^1, \\ (x^0 + \psi^1) \diamond (y^1 + \phi^0) = l_2(x^0, y^1) + l_3(x^0, \cdot, y^1)|_{\mathfrak{g}_0} + l_2(x^0, \phi^0(\cdot)) \\ \quad -\phi^0 \circ l_2(x^0, \cdot)|_{\mathfrak{g}_0} - \psi^1 l_2(\cdot, y^1)|_{\mathfrak{g}_0} + \delta(\psi^1(y^1)) + \psi^1 \circ l_1 \circ \phi^0, \\ (y^1 + \phi^0) \diamond (x^0 + \psi^1) = l_2(y^1, x^0) - l_3(x^0, \cdot, y^1)|_{\mathfrak{g}_0} - l_2(x^0, \phi^0(\cdot)) \\ \quad +\phi^0 \circ l_2(x^0, \cdot)|_{\mathfrak{g}_0} + \delta(\phi^0(x^0)) + \psi^1 l_2(\cdot, y^1)|_{\mathfrak{g}_0} - \psi^1 \circ l_1 \circ \phi^0, \end{cases} \tag{7.4}$$

and Ω *is given by*

$$\begin{aligned} \Omega(\phi^1 + x^0, \psi^1 + y^0 + \varphi^1 + z^0) &= l_3(x^0, y^0, z^0) + l_4(x^0, y^0, z^0, \cdot) \\ &\quad -\varphi^1 \circ l_3(x^0, y^0, \cdot)|_{\mathfrak{g}_0} - \phi^1 \circ l_3(z^0, x^0, \cdot)|_{\mathfrak{g}_0} - \psi^1 \circ l_3(y^0, z^0, \cdot)|_{\mathfrak{g}_0}. \end{aligned} \tag{7.5}$$

By (7.2), it is straightforward to deduce that the corresponding $\mathcal{D} : \mathfrak{g}_{-2} \longrightarrow \mathrm{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-2}) \oplus \mathfrak{g}_{-1}$ is given by

$$\mathcal{D}(x^2) = l_2(\cdot, x^2) + l_1(x^2) \tag{7.6}$$

Then by Theorem 6.2, we obtain:

Proposition 5. *Let* $(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}, \mathfrak{g}_0, l_1, l_2, l_3, l_4)$ *be a Lie* 3*-algebra. Then there is a Lie* 3*-algebra* $(\overline{\mathfrak{g}}_{-2}, \overline{\mathfrak{g}}_{-1}, \overline{\mathfrak{g}}_0, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4)$*, where* $\overline{\mathfrak{g}}_{-2} = \mathfrak{g}_{-2}$*,* $\overline{\mathfrak{g}}_{-1} = \mathrm{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-2}) \oplus \mathfrak{g}_{-1}$*,* $\overline{\mathfrak{g}}_0 = \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-2}) \oplus \mathfrak{g}_0$*, and* $\mathfrak{l}_i$ *are given by*

$$\begin{array}{rcll}
\mathfrak{l}_1(x^2) &=& \mathcal{D}(x^2), & \forall\, x^2 \in \mathfrak{g}_{-2},\\
\mathfrak{l}_1(\phi^0 + x^1) &=& \phi^0 \circ l_1 + l_2(x^1, \cdot)|_{\mathfrak{g}_{-1}} + l_1(x^1), & \forall\, \phi^0 + x^1 \in \overline{\mathfrak{g}}_{-1},\\
\mathfrak{l}_2(e^0_1, e^0_2) &=& e^0_1 \diamond e^0_2, & \forall\, e^0_1, e^0_2 \in \overline{\mathfrak{g}}_0,\\
\mathfrak{l}_2(e^0, e^1) &=& \frac{1}{2}(e^0 \diamond e^1 - e^1 \diamond e^0), & \forall\, e^0 \in \overline{\mathfrak{g}}_0, e^1 \in \overline{\mathfrak{g}}_{-1},\\
\mathfrak{l}_2(e^0, x^2) &=& \frac{1}{2}\mathcal{S}(e^0, \mathcal{D}x^2), & \forall\, e^0 \in \overline{\mathfrak{g}}_0, x^2 \in \mathfrak{g}_{-2},\\
\mathfrak{l}_2(e^1_1, e^1_2) &=& 0, & \forall\, e^1_1, e^1_2 \in \overline{\mathfrak{g}}_{-1},\\
\mathfrak{l}_3(e^0_1, e^0_2, e^0_3) &=& \Omega(e^0_1, e^0_2, e^0_3), & \forall\, e^0_1, e^0_2, e^0_3 \in \overline{\mathfrak{g}}_0,\\
\mathfrak{l}_3(e^0_1, e^0_2, e^1) &=& -T(e^0_1, e^0_2, e^1), & \forall\, e^0_1, e^0_2 \in \overline{\mathfrak{g}}_0, e^1 \in \overline{\mathfrak{g}}_{-1},\\
\mathfrak{l}_4(e^0_1, e^0_2, e^0_3, e^0_4) &=& \overline{\Omega}(e^0_1, e^0_2, e^0_3, e^0_4), & \forall\, e^0_1, e^0_2, e^0_3, e^0_4 \in \overline{\mathfrak{g}}_0,
\end{array}$$

where the operation $\mathcal{D}$*,* $\diamond$*,* Ω *are given by* (7.6)*,* (7.4)*,* (7.5) *respectively,* $T : \overline{\mathfrak{g}}_0 \times \overline{\mathfrak{g}}_0 \times \overline{\mathfrak{g}}_{-1} \longrightarrow \mathfrak{g}_{-2}$ *is given by*

$$T(e^0_1, e^0_2, e^1) = \frac{1}{6}(\mathcal{S}(e^0_1, \mathfrak{l}_2(e^0_2, e^1)) + \mathcal{S}(e^1, \mathfrak{l}_2(e^0_1, e^0_2)) + \mathcal{S}(e^0_2, \mathfrak{l}_2(e^1, e^0_1))),$$

and $\overline{\Omega} : \wedge^4\overline{\mathfrak{g}}_0 \longrightarrow \mathfrak{g}_{-2}$ *is given by*

$$\overline{\Omega}(e^0_1, e^0_2, e^0_3, e^0_4) = \mathcal{S}(\Omega(e^0_1, e^0_2, e^0_3), e^0_4).$$

By Proposition 5, we can give interesting examples of Lie 3-algebras.

Example 5. We view a 3-term complex of vector spaces $V_{-2} \xrightarrow{l_1} V_{-1} \xrightarrow{l_1} V_0$ as an abelian Lie 3-algebra. By Proposition 5, we obtain the Lie 3-algebra

$$(V_{-2}, \mathrm{Hom}(V_0, V_{-2}) \oplus V_{-1}, \mathrm{Hom}(V_{-1}, V_{-2}) \oplus V_0, \mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4 = 0),$$

where $\mathfrak{l}_i, i = 1, 2, 3$ are given by

$$\begin{array}{rcl}
\mathfrak{l}_1(x^2) &=& l_1(x^2),\\
\mathfrak{l}_1(\phi^0 + y^1) &=& \phi^0 \circ l_1 + l_1(y^1),\\
\mathfrak{l}_2(\psi^1 + x^0, \phi^1 + y^0) &=& \psi^1 \circ l_1 \circ \phi^1 - \phi^1 \circ l_1 \circ \psi^1,\\
\mathfrak{l}_2(\psi^1 + x^0, \phi^0 + y^1) &=& \frac{1}{2} l_1(\psi^1(y^1) - \phi^0(x^0)) + \psi^1 \circ l_1 \circ \phi^0,\\
\mathfrak{l}_2(\psi^1 + x^0, x^2) &=& \frac{1}{2}\psi^1(l_1(x^2)),\\
\mathfrak{l}_2(\psi^0 + x^1, \phi^0 + y^1) &=& 0,\\
\mathfrak{l}_3(\psi^1 + x^0, \phi^1 + y^0, \varphi^1 + z^0) &=& 0,\\
\mathfrak{l}_3(\psi^1 + x^0, \phi^1 + y^0, \varphi^0 + z^1) &=& -\frac{1}{4}\Big(\psi^1 \circ l_1 \circ \phi^1(z^1) - \phi^1 \circ l_1 \circ \psi^1(z^1)\\
&& -\psi^1 \circ l_1 \circ \varphi^0(y^0) + \phi^1 \circ l_1 \circ \varphi^0(x^0)\Big),
\end{array}$$

for all $x^2 \in V_{-2}$, $\psi^0+x^1, \phi^0+y^1, \varphi^0+z^1 \in \mathrm{Hom}(V_0, V_{-2})\oplus V_{-1}$, $\psi^1+x^0, \phi^1+y^0, \varphi^1+z^0 \in \mathrm{Hom}(V_{-1}, V_{-2})\oplus V_0$.

Example 6. (Higher analogue of the Lie 2-algebra of string type)
A Lie 2-algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0, \widetilde{l_1}, \widetilde{l_2}, \widetilde{l_3})$ gives rise to a Lie 3-algebra $(\mathbb{R}, \mathfrak{g}_{-1}, \mathfrak{g}_0, l_1, l_2, l_3, l_4 = 0)$ naturally, where l_i, $i = 1, 2, 3$ is given by

$$l_1(r) = 0, \quad l_1(x^1) = \widetilde{l_1}(x^1),$$

$$l_2(x^0,y^0) = \widetilde{l_2}(x^0,y^0),\ l_2(x^0,y^1)=\widetilde{l_2}(x^0,y^1),\ l_2(x^0,r)=0,\ l_2(x^1,y^1)=0,$$
$$l_3(x^0,y^0,z^0) = \widetilde{l_3}(x^0,y^0,z^0),\quad l_3(x^0,y^0,z^1)=0,$$

for all $x^0,y^0,z^0 \in \mathfrak{g}_0$, $x^1,y^1,z^1 \in \mathfrak{g}_{-1}$, and $r,s\in\mathbb{R}$. By Proposition 5, we obtain the Lie 3-algebra $(\mathbb{R}, \mathfrak{g}_{-1}\oplus\mathfrak{g}_0^*, \mathfrak{g}_0\oplus\mathfrak{g}_{-1}^*, \mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_4)$, where $\mathfrak{l}_i$, $i=1,2,3,4$ are given by

$$\begin{aligned}
\mathfrak{l}_1(r) &= 0,\\
\mathfrak{l}_1(x^1+\alpha^0) &= l_1(x^1)+l_1^*(\alpha^0),\\
\mathfrak{l}_2(x^0+\alpha^1,y^0+\beta^1) &= l_2(x^0,y^0)+\mathrm{ad0}^*_{x^0}\beta^1-\mathrm{ad0}^*_{y^0}\alpha^1,\\
\mathfrak{l}_2(x^0+\alpha^1,y^1+\beta^0) &= l_2(x^0,y^1)+\mathrm{ad0}^*_{x^0}\beta^0-\mathrm{ad1}^*_{y^1}\alpha^1,\\
\mathfrak{l}_2(x^1+\alpha^0,y^1+\beta^0) &= 0,\\
\mathfrak{l}_2(x^0+\alpha^1,r) &= 0,\\
\mathfrak{l}_3(x^0+\alpha^1,y^0+\beta^1,z^0+\zeta^1) &= l_3(x^0,y^0,z^0)+\mathrm{ad3}^*_{x^0,y^0}\zeta^1+\mathrm{ad3}^*_{y^0,z^0}\alpha^1\\
&\quad+\mathrm{ad3}^*_{z^0,x^0}\beta^1,\\
\mathfrak{l}_3(x^0+\alpha^1,y^0+\beta^1,z^1+\zeta^0) &= \frac{1}{2}(\langle\alpha^1,l_2(y^0,z^1)\rangle+\langle\beta^1,l_2(z^1,x^0)\rangle\\
&\quad+\langle\zeta^0,l_2(x^0,y^0)\rangle),\\
\mathfrak{l}_4(x^0+\alpha^1,y^0+\beta^1,z^0+\zeta^1,u^0+\gamma^1) &= \langle\gamma^1,l_3(x^0,y^0,z^0)\rangle-\langle\zeta^1,l_3(x^0,y^0,u^0)\rangle\\
&\quad-\langle\alpha^1,l_3(y^0,z^0,u^0)\rangle-\langle\beta^1,l_3(z^0,x^0,u^0)\rangle
\end{aligned}$$

for all $x^0,y^0,z^0,u^0\in\mathfrak{g}_0$, $x^1,y^1,z^1\in\mathfrak{g}_{-1}$, $\alpha^1,\beta^1,\zeta^1,\gamma^1\in\mathfrak{g}_{-1}^*$, $\alpha^0,\beta^0\in\mathfrak{g}_0^*$, where $\mathrm{ad0}^*_{x^0}:\mathfrak{g}_{-i}^*\longrightarrow\mathfrak{g}_{-i}^*$, $\mathrm{ad1}^*_{x^1}:\mathfrak{g}_{-1}^*\longrightarrow\mathfrak{g}_0^*$ and $\mathrm{ad3}^*_{x^0,y^0}:\mathfrak{g}_{-1}^*\longrightarrow\mathfrak{g}_0^*$ are defined respectively by

$$\begin{aligned}
\langle\mathrm{ad0}^*_{x^0}\alpha^1,x^1\rangle &= -\langle\alpha^1,l_2(x^0,x^1)\rangle, & \langle\mathrm{ad0}^*_{x^0}\alpha^0,y^0\rangle &= -\langle\alpha^0,l_2(x^0,y^0)\rangle,\\
\langle\mathrm{ad1}^*_{x^1}\alpha^1,y^0\rangle &= -\langle\alpha^1,l_2(x^1,y^0)\rangle, & \langle\mathrm{ad3}^*_{x^0,y^0}\alpha^1,z^0\rangle &= -\langle\alpha^1,l_3(x^0,y^0,z^0)\rangle.
\end{aligned}$$

Remark 4. *For any Lie algebra* $(\mathfrak{h},[\cdot,\cdot]_\mathfrak{h})$*, we have the semidirect product Lie algebra* $(\mathfrak{h}\ltimes_{\mathrm{ad}^*}\mathfrak{h}^*,[\cdot,\cdot]_{\mathrm{ad}^*})$*, which is a quadratic Lie algebra naturally. Consequently, one can construct the corresponding Lie 2-algebra* $(\mathbb{R},\mathfrak{h}\ltimes_{\mathrm{ad}^*}\mathfrak{h}^*,l_1=0,l_2=[\cdot,\cdot]_{\mathrm{ad}^*},l_3)$*, where* l_3 *is given by*

$$l_3(x+\alpha,y+\beta,z+\gamma)=\langle\gamma,[x,y]_\mathfrak{h}\rangle+\langle\beta,[z,x]_\mathfrak{h}\rangle+\langle\alpha,[y,z]_\mathfrak{h}\rangle,\quad\forall x,y,z\in\mathfrak{h},\alpha,\beta,\gamma\in\mathfrak{h}^*.$$

This Lie 2-algebra is called the Lie 2-algebra of string type in [51]. On the other hand, associated to a Lie 2-algebra $(\mathfrak{g}_{-1},\mathfrak{g}_0,\widetilde{l_1},\widetilde{l_2},\widetilde{l_3})$*, there is a naturally a quadratic Lie 2-algebra structure on* $(\mathfrak{g}_{-1}\oplus\mathfrak{g}_0^*)\oplus(\mathfrak{g}_0\oplus\mathfrak{g}_{-1}^*)$ *([34, Example 4.8]). Thus, the Lie 3-algebra given in the above example can be viewed as the higher analogue of the Lie 2-algebra of string type.*

Motivated by the above example, we show that one can obtain a Lie 3-algebra associated to a quadratic Lie 2-algebra in the sequel. This result is the higher analogue of the fact that there is a Lie 2-algebra, called the string Lie 2-algebra, associated to a quadratic Lie algebra.

A **quadratic Lie 2-algebra** is a Lie 2-algebra $(\mathfrak{g}_{-1},\mathfrak{g}_0,l_1,l_2,l_3)$ equipped with a degree 1 graded symmetric nondegenerate bilinear form S which induces an isomorphism between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0^*$, such that the following invariant conditions hold:

$$S(l_1(x^1),y^1) = S(l_1(y^1),x^1), \tag{7.7}$$

$$S(l_2(x^0,y^0),z^1) = -S(l_2(x^0,z^1),y^0), \tag{7.8}$$
$$S(l_3(x^0,y^0,z^0),u^0) = -S(l_3(x^0,y^0,u^0),z^0), \tag{7.9}$$

for all $x^0,y^0,z^0,u^0 \in \mathfrak{g}_0$, $x^1,y^1 \in \mathfrak{g}_{-1}$.

Let $(\mathfrak{g}_{-1},\mathfrak{g}_0,l_1,l_2,l_3,S)$ be a quadratic Lie 2-algebra. On the 3-term complex of vector spaces $\mathbb{R}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$, where $\mathbb{R}$ is of degree -2, we define $\mathfrak{l}_i$, $i=1,2,3,4$, by

$$\begin{cases} \mathfrak{l}_1(r) = 0, & \mathfrak{l}_1(x^1) = l_1(x^1),\\ \mathfrak{l}_2(x^0,y^0) = l_2(x^0,y^0), & \mathfrak{l}_2(x^0,y^1) = l_2(x^0,y^1),\\ \mathfrak{l}_2(x^0,r) = 0, & \mathfrak{l}_2(x^1,y^1) = 0,\\ \mathfrak{l}_3(x^0,y^0,z^0) = l_3(x^0,y^0,z^0), & \mathfrak{l}_3(x^0,y^0,z^1) = \frac{1}{2}S(z^1,l_2(x^0,y^0)),\\ \mathfrak{l}_4(x^0,y^0,z^0,u^0) = S(l_3(x^0,y^0,z^0),u^0), & \end{cases} \tag{7.10}$$

for all $x^0,y^0,z^0,u^0 \in \mathfrak{g}_0$, $x^1,y^1,z^1 \in \mathfrak{g}_{-1}$ and $r\in\mathbb{R}$.

Theorem 7.1. *With above notations,* $(\mathbb{R},\mathfrak{g}_{-1},\mathfrak{g}_0,\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_4)$ *is a Lie* 3*-algebra, called* **the higher analogue of the string Lie 2-algebra**.

Proof. It follows from direct verification of the coherence conditions for $\mathfrak{l}_3$ and $\mathfrak{l}_4$ using the invariant conditions (7.7)-(7.9). We omit details.

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