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Research article

Sustainable production model with stochastic machine breakdown using smart manufacturing under circular economy

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Abstract: Machine breakdown usually implies unexpected physical damage to machinery due to any reason which requires fixing or replacement to continue the process. This article presentsan investigation of a sustainable model with stochastic machine breakdown. To reduce the risk of disruption, a smart manufacturing system is used. Considering the environmental issues faced by people, the model is developed under a circular economy through end-of-life treatment to recapture the value of the product, labor and resources. The used buyback products are collected, out of which items in good condition are remanufactured and sold in another market while the rest are salvaged. As the production process is not perfectly reliable, the serviceable products go through an automated inspection process, and imperfect items are reworked. A mathematical model is developed for deterioration of items to analyze the optimal replenishment policies, and the results have been illustrated with numerical verification. Based on the analysis, some managerial insights have been provided for decision-makers.

Keywords: stochastic machine breakdown; automation policy with smart production; circular economy; deterioration; imperfect production

JEL Codes: M11, O14, P23

Abbreviations: EPQ: Economic production quantity; RL: Reverse logistics (RL); JIT: Just-in-time; EMQ: Economic manufacturing quantity; CE: Circular economy; EOQ: Economic order quantity

1. Introduction

The main focuses of manufacturing firms are production, quality and maintenance of the product. In a cutthroat competitive atmosphere, the decision makers face different challenges day to day to manufacture a superior product and to give better service than others. Technological advancement and growing scientific research change manufacturing infrastructure rapidly. While manufacturing services are gradually becoming more complicated, current facilities can deteriorate. Consequently, machines can randomly shift from an in-control state to an out-of-control state, resulting in machine breakdown. Hence, the study of economic production quantity (EPQ) models for unreliable systems is quite considerable and significant.

Most of the studies developed with EPQ models pay no attention to the reliability factor, thus implicitly assuming completely reliable systems. Machine failures, however, do arise and could be really troublesome, mostly in an extremely mechanized manufacturing environment. Current study efforts attempt to integrate the shortcomings of the manufacturing process and equipment into the classical sizing conclusion structure. This article deals with a manufacturing inventory system for an unreliable system under the assumptions that the machine is subjected to stochastic random breakdown.

1.1. Research gap

The research gap in this context can be articulated as follows:

In the appropriate literature, a substantial body of research has been dedicated to addressing the challenges posed by unreliable systems, particularly those subjected to machine breakdowns. Similarly, extensive work has been presented in the realm of green supply chain inventory, with a focus on sustainability and environmental considerations. However, a significant research gap exists where the joint effects of these two critical issues, namely, unreliable systems due to machine breakdowns and green supply chain inventory control, have not received adequate attention.

While prior research has recognized the impact of machine breakdowns on inventory management and explored strategies to mitigate their disruptive effects, it often overlooks the potential for proactive measures to prevent machine breakdowns. Specifically, there is a lack of investigation into how assessments and maintenance protocols can be employed to transition of a machine from an “in-control” state to an “out-of-control” state. Such proactive measures have the potential to not only reduce the number of imperfect products but also delay the occurrence of machine breakdowns during production runs.

The motivation behind this paper is to bridge this research gap by developing integrated production, inventory and maintenance models that comprehensively consider the joint effects of process decay, machine breakdowns and proactive assessments on optimal lot sizing decisions. This holistic approach acknowledges the dynamic nature of production systems and aims to optimize decision-making in a manner that enhances both system reliability and sustainability, aligning with the principles of green supply chain management. By exploring the synergy between these factors, this research seeks to contribute valuable insights to the field of production and supply chain management, offering a more comprehensive understanding of how to effectively manage inventory under conditions of uncertainty and unreliability while minimizing environmental impact.

1.2. Objective of the paper

The objective of this paper is to address a critical research gap in the field of production and supply chain management by investigating the joint impacts of unreliable systems due to machine breakdowns and green supply chain inventory control. While previous research has extensively examined each of these issues in isolation, there is a lack of comprehensive understanding regarding how they interact and influence inventory control decisions. Therefore, the primary objective of this paper is to develop integrated models that encompass production, inventory management and maintenance, taking into account the combined effects of process decay, machine breakdowns and proactive assessments on optimal lot sizing decisions.

One key objective is to create a holistic framework that considers the dynamic nature of production systems. Specifically, we aim to explore how proactive assessments can transition a machine from an "in-control" state to an "out-of-control" state in a controlled manner, thereby reducing the likelihood of machine breakdowns during production runs. This proactive approach holds the potential to not only enhance system reliability but also optimize inventory management by minimizing the production of imperfect products.

Another central objective of this paper is to align our research with the principles of green supply chain management. We seek to develop strategies that not only improve system reliability but also contribute to sustainability and environmental responsibility. By exploring the synergy between mitigating machine breakdowns and green inventory practices, we aim to provide decision-makers with insights on how to make inventory control decisions that are not only efficient but also environmentally conscious.

Furthermore, this research aims to validate the proposed models and strategies through numerical verification. By applying our models to practical scenarios and conducting numerical analyses, we aim to demonstrate the feasibility and effectiveness of the integrated approach in real-world applications. This verification process is essential in providing empirical support for our proposed strategies and ensuring their practical relevance.

In summary, the primary objectives of this paper are to develop integrated models that consider the joint impacts of process decay, machine breakdowns and proactive assessments on optimal lot sizing decisions. Additionally, we aim to align our research with green supply chain principles and validate our models through numerical verification. By achieving these objectives, we seek to provide a comprehensive framework for enhancing both the reliability and sustainability of inventory control systems in manufacturing and supply chain operations.

1.3. Novelty

This study's novelty lies in its unique approach that combines machine breakdown management, green supply chain principles and proactive assessments within a single framework. It introduces proactive measures to prevent machine breakdowns, aligns inventory control with sustainability goals and validates its strategies through practical testing. This integrated approach offers valuable insights and practical solutions for real-world production and supply chain challenges, making it distinct from existing research. Items are presented in this paper subjected to the stochastic machine breakdown. The failure rate of the production machine is taken to be probabilistic in nature. It is assumed that the manufacturing process is not perfectly reliable. Thus, some non-conforming units are also produced during the manufacturing.

2. Literature review

2.1. Machine breakdown

Machine unavailability is an unavoidable phenomenon in many production industries, resulting in three main things: preventive maintenance, corrective maintenance and machine breakdown. A lot of investigation has been made by researchers in this line of research. Groenevelt et al., (1992) developed the EPQ model subjected to stochastic machine breakdown. They studied their model with two replenishment policies. In the first policy, they considered that the machine cannot be repaired after breakdown. In the second one, they assumed that the machine will be fixed after the breakdown. The effects of machine breakdowns and corrective maintenance on the optimal policies were discussed. Liu and Cao (1999) presented an EPQ model with machine failure assuming that the demand rate is probabilistic in nature. They considered the period of failure by Gamma and Weibull distributions and resolved it rapidly and consistently. Al-khateeb (1999) studied an unreliable EPQ model under machine breakdown with general lifetime distribution. It was considered that the manufacturing process is subjected to a stochastic decay from a controllable situation to an uncontrollable situation, follow a stochastic breakdown. Chung (2003) developed a production lot sizing model for a failure-prone machine. He obtained the lower and upper bounds for the optimal lot size. He demonstrated that the optimal cost function is neither convex nor concave. In a later study, Giri and Dohi (2005) also presented a mathematical model of a stochastic economic manufacturing quantity (EMQ) framework. Zhang et al., (2007) formulated inventory levels as a major focus for supply chain management. In their model, they assumed that demand and production rates are constant. Lin (2010) studied a stochastic integrated supplier-retailer inventory problem. Singh and Prasher (2014) examined the role of inventory in a production process with flexible manufacturing, random breakdowns and stochastic repair time. Darma Wangsa and Wee (2018) developed a two-echelon supply chain involving a single vendor and single buyer with stochastic demand. They studied transportation cost, which is a function of shipping weight, distance and transportation modes. Poursoltan et al., (2020) derived an economic production quantity (EPQ) model with partial backlogging. Stochastic inventory deterioration and stochastic machine breakdown were considered. Jauhari et al., (2021) established a two-echelon supply chain inventory model containing a manufacturer and a retailer under a stochastic environment with carbon emission reductions. Emissions from transportation, production and storage activities are included in the model with reductions through carbon tax regulation. Pal and Adhikari (2021) developed an economic production quantity (EPQ) model where production is performed mainly by the original machine. Its buffer production continues when the system encounters a disruption. Deiranlou et al., (2022) introduced the general machine breakdown and repair time distributions. Shortages caused by long exponential repair time were considered. For instance, the manufacturer may procure certain quantities from an available supplier with a non-zero lead time. Sana's (2022) model consisted of a production-inventory model where preventive maintenance takes place at the end of a cycle to ensure smooth performance in the next cycle. The study examined how production systems produce defective products at different rates when “in-control” and when “out-of-control.” The defective products are then reworked for costs in parallel systems.

2.2. Reliability

Kelle and Schneider (1992) formulated reliability-type inventory models with minimal safety stock. They studied just-in-time (JIT) production. Panda and Maiti (2009) developed multi-item EPQ models. They studied flexibility and reliability in production process. Demand selling price and production stock dependence were considered. Manna et al., (2018) examined an imperfect production inventory model with production system reliability under two-layer supply chain management. In the model, the reliability parameter depends on the production rate. Islam et al., (2022) worked an inventory model to solve problems of supply uncertainty in response to a demand which expresses a Poisson distribution. One positive aspect of this model is the consideration of random inventory, delivery efficiencies and supplier reliability.

2.3. Smart production

Increasingly, smart production is receiving attention, and its importance is paramount in smooth functioning of the production process in order to ensure the quality of the final product, which directly affects the business. Due to the manufacturer's preference for handling products manually during the entire production process from the raw material to the final product, finding defective products is not uncommon. Automating the process of smart manufacturing reduces the need for manual handling, and monitoring the production processes leaves no room for human error. By minimizing human errors, manufacturers are able to ensure food safety and quality through the use of digital technologies to improve their systems. Moreover, smart production processes improve plant efficiency by providing real-time material, sourcing, production and human resources. With smart manufacturing, all aspects of the system can be analyzed to ensure that the energy efficiency of the whole process is maximized. All of the factory’s operations, from turning on and off the lights to scheduling production, can be guided by real-time information regarding energy consumption. Smart manufacturing has been gaining attention in recent years. De Giovanni (2011) presented a vendor managed inventory model with revenue sharing. Artificial intelligence was used for the production process, thereby increasing reliance on and improving performance of the supply chain. A vendor-buyer integration for a probabilistic inventory model was presented by Dey et al., (2021a). The supply chain model was developed for stochastic demand considering smart manufacturing and uniformly distributed lead time. In a subsequent study, Dey et al., (2021b) investigated a cost-effective production policy for a centralized production and remanufacturing model. Smart production was considered to deal with the imperfect production, which is a random variable. Recently, a mathematical model was provided by Kugele et al., (2022) with smart manufacturing. The model was developed with controllable carbon emissions and optimized by using geometric programming. Sarkar et al., (2022a) investigated the use of smart manufacturing processes for energy optimization during biofuel production.

2.4. Circular economy

Product life cycles are gradually shrinking, and the shorter life spans force practitioners to adapt to a circular economy (CE). Bringing the product back into the system allows the organization to recapture the value of the product, labor, material and resources. The circular economy has been identified as one of the most effective methods for an industry to ensure environmental sustainability through recycling and remanufacturing (Haupt and Hellweg, 2019). Practices and research have demonstrated that reverse logistics is an intricate procedure that complicates administrative choices trying to accomplish more noteworthy financial advantages (Srivastava, 2007). One of the complexities of remanufacturing is managing the two-supply arrangement of parts, i.e., the forward flow of material and the reverse flow. Reverse Logistics (RL) activities deal with the collection of used material; however, the quantity of buyback products is highly uncertain, which particularly influences the accumulation and replenishment policies (Dekker et al., 2004). The life cycle designing of a consumable product is vital at its earliest phase in order to maximize the life-cycle value of a product. The review presented by Sasikumar and Kannan (2009) emphasized the growing awareness of legislation and customers toward environmental issues. They portrayed two sorting proposals and recommended the research potential of end-of-life treatment. Alinovi et al., (2012) evaluated a stochastic order quantity inventory control model with a mixed manufacturing and remanufacturing strategy such that all manufacturing and procurement processes are integrated with reuse and remanufacturing to meet the demand in a timely and complete manner. The replenishment policies for the production and remanufacturing were provided to get the optimum results. Agrawal et al., (2015) provided a literature review and perspectives in RL. Up to 242 distributed articles were chosen, classified and broken down, and holes in the literature were identified to propose future research openings. Agrawal et al., (2015) additionally strengthened the finding of Sasikumar and Kannan (2009) with respect to the growth in significance of RL in environmental and sustainability concerns. Govindan et al., (2015) developed a review paper for RL and a closed-loop system for exploring future aspects. The point of the paper was to audit recent distributed papers in reverse strategic and closed-loop systems. In the review paper, the holes in the literature were identified to elucidate and to recommend future research openings. Also, a literature review of mathematical modeling was given by Bazan et al., (2016) for RL inventory models. The paper surveyed the literature on the RL stock frameworks that depend on the economic order quantity (EOQ) and EPQ with breakdown which are the principal attributes of related procedures. The literature was studied and reviewed by the particular issues confronted and modeling presumptions. Modak et al., (2018) introduced a two-tier RL model to assess the effects of reprocessing and product quality on pricing strategies. Chen et al., (2019) presented a supply chain model with a closed-loop structure involving a manufacturer, retailer and remanufacturer. They proposed a long-term collection strategy wherein the manufacturer must collaborate with the retailer to improve the remanufacturing process. Recently, Sanni et al., (2020) proposed a model for maximizing profit by tracing the reverse flow of used items. Lu et al., (2021) developed a green manufacturing-recycling network by developing a closed-loop logistics dual-objective optimization model. The study examined several methods to minimize the adverse effects on the environment by implementing environmentally friendly production techniques. To quantify the impact on the environment, the model incorporates a pollution equivalent number. Furthermore, Rentizelas et al., (2021) investigated whether wind turbine blades at the end of their life may be recycled to manufacture composite materials.

In practice, a machine cannot always work well because its spare parts may do malfunctioning at some point of time. It may cause out of stock amid its working time if breakdown period is longer. At the point when the stock level becomes less than the demand, the management unit completely sits out of gear. As a result, a green supply chain inventory model is essential for continuation of manufacturing systems without disruption of supply.

3. Assumptions and notation

The model is developed here based on the following presumptions and notations.

3.1. Assumptions

1. Demand rate is assumed to be constant (Zhang et al., 2007).
2. Machine failure takes place at random during a production run (Poursoltan et al., 2020). We have assumed that the random number of breakdowns per unit time follows the Poisson distribution, with the mean equal to β per unit time. Thus, the random production time to break down should obey the exponential distribution, with the density function $f(t) = \beta e^{-\beta t}$ and the cumulative density function $F(t) = \left(1 - e^{-\beta t}\right)$.
3. Various studies have found that the reliability of manufacturing processes can be improved through the adoption of smart production technologies, which is regarded as being highly durable (Sarkar and Sarkar, 2022b). Therefore, to reduce the risk of disruption, smart technology is used with technology investment. Smart production is considered, and the cost of smart production is assumed to be

$$C(P, \beta, x) = \left\{\frac{\pi + \gamma\beta}{P} + (r_1 + r_2\beta) + (s_1 + s_2 xy) + C_p P\right\}$$

4. Imperfect production is taken into account and *x* portion of produced items and *y* portion of remanufactured items are assumed to be defective. These items are reworked at a rate P_1.
5. Among these imperfect items, a small portion λ of the remanufactured items are totally scraped and will not be remanufactured further. Hence, scraped of the first time products and the other imperfect quality products can be repaired.

3.2. Notation

p	Production rate
P_r	Remanufacturing rate
D_1	Demand rate
D_2	Demand rate for remanufactured items
P_1	Repairing rate of the imperfect items
R	Returned rate
x	Reliability of the production process
y	Reliability of the remanufacturing process
λ_1	Scrap from the produced items
λ_2	Scrap from the remanufactured items
θ	Deterioration rate

K_m	Set up cost \$ per unit cycle for the production process
K_r	Set up cost \$ per unit cycle for the remanufacturing process
h_m	Holding cost \$ per unit per month for the newly produced items
h_r	Holding cost \$ per unit per month for the remanufactured items
h_R	Holding cost \$ per unit per month for the returned items
h_1	Holding cost \$ per unit per month for the reworked items
S_m	Production cost\$ per unit item
S_r	Remanufacturing cost\$ per unit item
S_{rw}	Rework cost\$ per unit item
C_m	Procurement cost\$ per unit item
C_r	Remanufacturing cost \$ per unit item
C_i	Inspection cost \$ per unit item
C_{rw}	Rework cost\$ per unit item
C_R	Acquisition cost \$ per unit item
C_d	Disposable cost\$ per unit item
t_1	Production time
t'	Time at which machine fails
t_s	Machine repairing time

4. Model development

This paper is intended to capture and describe a reverse logistics model in which the productivity of a manufacturer is directly related to the degree of its reliability. As is widely recognized in the manufacturing sector, the physical output realized by the firm is determined by a variety of factors that are associated with both performance and quality. In order to measure the effectiveness of a manufacturing system, it must be assessed in terms of its reliability and productivity. Various studies have found that the reliability of manufacturing processes can be improved through the adoption of smart production technologies, which is regarded as being highly durable. Whether a manufacturing system will fail or not will probably depend on its inherent reliability.

4.1. (Case 1) Reverse logistics inventory model with stochastic machine breakdown

In this case, it is assumed that the machine breakdown takes place at the time t' before completing the actual production time t_1, i.e., in this case we have taken that $t' < t_1$. Apart from the breakdown, the production process is also not perfectly reliable. Hence, some imperfect items are also produced during this period and stored separately, out of which a small portion is scrapped. The machine is repaired by the time t_2, and the machine starts to repair the defective items to the timet_3. Thereafter, the level of inventory starts to decrease due to the demand and deterioration, at the same time remanufacturing starts. Imperfect items also are generated during the remanufacturing run and repaired in the period of t_4 to t_5. Returned items are collected from the starting of the cycle to the time when remanufacturing stops. Figure1 depicts the behavior of the on-hand inventory level.

The differential equations governing the stock level during the period $0 \le t \le T$ can be written as follows.

$$\frac{dI_m(t)}{dt} + \theta I_m(t) = xP - D_1 \qquad 0 \le t \le t' \qquad I_m(0) = 0 \tag{1}$$

$$\frac{dI_m(t)}{dt} + \theta I_m(t) = -D_1 \qquad t' \le t \le t_2 \qquad I_m(t'^-) = I_m(t'^+) \tag{2}$$

$$\frac{dI_m(t)}{dt} + \theta I_m(t) = P_1 - D_1 \qquad t_2 \le t \le t_3 \qquad I_m(t_3^-) = I_m(t_3^-) \tag{3}$$

$$\frac{dI_m(t)}{dt} + \theta I_m(t) = -D_1 \qquad t_3 \le t \le t_4 \qquad I_m(t_4) = 0 \tag{4}$$

$$\frac{dI_r(t)}{dt} + \theta I_r(t) = yP_r - D_2 \qquad t_3 \le t \le t_4 \qquad I_r(t_3) = 0 \tag{5}$$

$$\frac{dI_r(t)}{dt} + \theta I_r(t) = P_1 - D_2 \qquad t_4 \le t \le t_5 \qquad I_r(t_4^-) = I_r(t_4^+) \tag{6}$$

$$\frac{dI_r(t)}{dt} + \theta I_r(t) = -D_2 \qquad t_5 \le t \le T \qquad I_r(T) = 0 \tag{7}$$

$$\frac{dI_R(t)}{dt} + \theta I_R(t) = R \qquad 0 \le t \le t_3 \qquad I_R(0) = 0 \tag{8}$$

$$\frac{dI_R(t)}{dt} + \theta I_R(t) = R - P_r \qquad t_3 \le t \le t_4 \qquad I_R(t_4) = 0 \tag{9}$$

The solutions of the above differential equations are

$$I_m(t) = \left(\frac{xP - D_1}{\theta}\right)\left(1 - e^{-\theta t}\right) \qquad 0 \le t \le t' \tag{10}$$

$$I_m(t) = \frac{xP}{\theta}\left(e^{-\theta(t-t')} - e^{-\theta t}\right) - \frac{D_1}{\theta}\left(1 - e^{-\theta t}\right) \qquad t' \le t \le t_2 \tag{11}$$

$$I_m(t) = \left(\frac{P_1}{\theta}\right)\left(1 - e^{\theta(t_3-t)}\right) - \frac{D_1}{\theta}\left(1 - e^{\theta(t_4-t)}\right) \qquad t_2 \le t \le t_3 \tag{12}$$

$$I_m(t) = \frac{D_1}{\theta}\left(e^{\theta(t_4-t)} - 1\right) \qquad t_3 \le t \le t_4 \tag{13}$$

$$I_r(t) = \frac{yP_r}{\theta}\left(1 - e^{\theta(t_3-t)}\right) \qquad t_3 \le t \le t_4 \tag{14}$$

$$I_r(t) = \frac{P_1 - D_2}{\theta}\left(1 - e^{\theta(t_4-t)}\right) + \frac{yP_r}{\theta}\left(e^{\theta(t_4-t)} - e^{\theta(t_3-t)}\right) \qquad t_4 \le t \le t_5 \tag{15}$$

$$I_r(t) = \frac{D_2}{\theta}\left(e^{\theta(T-t)} - 1\right) \qquad t_5 \le t \le T \tag{16}$$

$$I_R(t) = \frac{R}{\theta}\left(1 - e^{-\theta t}\right) \qquad 0 \le t \le t_3 \tag{17}$$

$$I_R(t) = \frac{(R - P_r)}{\theta}\left(1 - e^{\theta(t_4-t)}\right) \qquad t_3 \le t \le t_4 \tag{18}$$

The per cycle cost components for the given inventory model are as follows.

i). Expected procurement and acquisition cost

The cost of purchasing the raw material (included deterioration cost) from the supplier can be calculated as

$$(C_m P t' + C_R R t_4) \tag{19}$$

Thus, the expected purchasing cost would be

$$\int_0^{t_1} \beta(C_m P t' + C_R R t_4) e^{-\beta x} dt' \tag{20}$$

ii). Expected production cost

Productivity of an automation-based production process fluctuates with replenishment strategies. In the event of a failure in an integral component of the machine, the component malfunctions adversely affect the system's productivity, product quality and production costs. For a manufacturing system, reliability is the measurement of output per unit of time in terms of the productivity of the whole system, whereas for a service system, productivity is the measure of output per unit of time. A controllable manufacturing system is one that has the ability to vary production rate to meet the demands of the market so that the productivity of manufacturing systems can be increased. Due to the fact that this model allows flexibility in the production rate of an item, the unit production cost becomes a function of three variables: the variable production rate *m*, the variable manufacturing yield *Y* and the variable manufacturing performance equality *l*.

The per unit production cost is

$$C(P,\beta,x) = \left\{\frac{\pi+\gamma\beta}{P} + (r_1 + r_2\beta) + (s_1 + s_2xy) + C_pP\right\} \tag{21}$$

Therefore, the expected production would be

$$\int_0^{t_1} \beta\left\{\frac{\pi+\gamma\beta}{P} + (r_1 + r_2\beta) + (s_1 + s_2xy) + C_pP\right\}Pt'e^{-\beta x}dt' \tag{22}$$

iii). Expected remanufacturing cost

One of the complexities of remanufacturing is completely managing the two-supply arrangement of parts, i.e.,the forward flow of material and the reverse flow. RL activities deal with the collection of used material; however, the quantity of buyback products is highly uncertain, which particularly influences the accumulation and replenishment policies. The remanufacturing starts at time t_3, to the time t_4. Therefore, the expected cost of remanufacturing is as follows:

$$\int_0^{t_1} \beta C_rP_r(t_4 - t_3)e^{-\beta x}dt' \tag{23}$$

iv). Expected inspection cost

During the production and remanufacturing process, items are inspected via automation policy. Therefore, they can be removed from the stock of serviceable items instantly. The expected cost of inspection is as follows:

$$\int_0^{t_1} \beta C_i\{Pt' + P_r(t_4 - t_3)\}e^{-\beta x}dt' \tag{24}$$

v). Expected rework cost

x portion of produced items and *y* portion of remanufactured items are assumed to be defective per unit time of production. These items are reworked at a rate P_1. The expected cost to rework those items can be calculated as follows:

$$\int_0^{t_1} \beta C_{rw}P_1(t_3 - t_2 + t_5 - t_4)e^{-\beta x}dt' \tag{25}$$

vi). Expected holding cost for newly produced material

The items are stored in the warehouse, and the manufacturer needs to spend some investment. The holding cost of carrying the newly produced inventory during the time period zero to t_4 can be calculated as follows:

$$h_m\left[\int_0^{t'}\left(\frac{xP-D_1}{\theta}\right)\left(1-e^{-\theta t}\right)dt+\int_{t'}^{t_2}\left\{\frac{xP}{\theta}\left(e^{-\theta(t-t')}-e^{-\theta t}\right)-\frac{D_1}{\theta}\left(1-e^{-\theta t}\right)\right\}dt\right.$$
$$\left.+\int_{t_2}^{t_3}\left\{\left(\frac{P_1}{\theta}\right)\left(1-e^{\theta(t_3-t)}\right)-\frac{D_1}{\theta}\left(1-e^{\theta(t_4-t)}\right)\right\}dt+\int_{t_3}^{t_4}\frac{D_1}{\theta}\left(e^{\theta(t_4-t)}-1\right)dt\right]$$
$$=h_m\left[\left(\frac{xP-D_1}{\theta}\right)\left(\frac{\left(e^{-\theta t'}-1\right)}{\theta}+t'\right)+\frac{xP}{\theta^2}\left(1-e^{-\theta t'}-e^{-\theta(t_2-t')}+e^{-\theta t_2}\right)-\frac{D_1}{\theta}\left(\frac{e^{-\theta t_2}-e^{-\theta t'}}{\theta}+\right.\right.$$
$$\left.(t_2-t')\right)+\frac{D_1}{\theta}\left(\frac{e^{\theta(t_4-t_2)}-e^{\theta(t_4-t_3)}}{\theta}+(t_3-t_2)\right)+\frac{P_1}{\theta}\left((t_3-t_2)+\frac{\left(1-e^{\theta(t_3-t_2)}\right)}{\theta}\right)-\frac{D_1}{\theta}\left((t_4-t_3)+\right.$$
$$\left.\left.\frac{\left(1-e^{\theta(t_4-t_3)}\right)}{\theta}\right)\right] \tag{26}$$

The expected holding cost of carrying the newly produced inventory during the time period zero to t_4 can be calculated as follows:

$$\int_0^{t_1}\beta h_m\left[\left(\frac{xP-D_1}{\theta}\right)\left(\frac{\left(e^{-\theta t'}-1\right)}{\theta}+t'\right)+\frac{xP}{\theta^2}\left(1-e^{-\theta t'}-e^{-\theta(t_2-t')}+e^{-\theta t_2}\right)-\frac{D_1}{\theta}\left(\frac{e^{-\theta t_2}-e^{-\theta t'}}{\theta}+\right.\right.$$
$$\left.(t_2-t')\right)+\frac{D_1}{\theta}\left(\frac{e^{\theta(t_4-t_2)}-e^{\theta(t_4-t_3)}}{\theta}+(t_3-t_2)\right)+\frac{P_1}{\theta}\left((t_3-t_2)+\frac{\left(1-e^{\theta(t_3-t_2)}\right)}{\theta}\right)-\frac{D_1}{\theta}\left((t_4-t_3)+\right.$$
$$\left.\left.\frac{\left(1-e^{\theta(t_4-t_3)}\right)}{\theta}\right)\right]e^{-\beta x}dt' \tag{27}$$

vii). Expected holding cost for remanufactured material

The cost of carrying the remanufactured inventory during the time period t_3 to T is

$$h_r\left[\int_{t_3}^{t_4}\frac{yP_r}{\theta}\left(1-e^{\theta(t_3-t)}\right)dt+\int_{t_4}^{t_5}\left\{\frac{P_1-D_2}{\theta}\left(1-e^{\theta(t_4-t)}\right)+\frac{yP_r}{\theta}\left(e^{\theta(t_4-t)}-e^{\theta(t_3-t)}\right)\right\}dt\right.$$
$$\left.+\int_{t_5}^{T}\frac{D_2}{\theta}\left(e^{\theta(T-t)}-1\right)dt\right]$$

$$=h_r\left[\frac{yP_r}{\theta}\left(\frac{e^{\theta(t_3-t_4)}-1}{\theta}+(t_4-t_3)\right)+\frac{(P_1-D_2)}{\theta}\left(\frac{e^{-\theta(t_5-t_4)}-1}{\theta}+(t_5-t_4)\right)+\right.$$

$$\frac{yP_r}{\theta}\left(\frac{1-e^{-\theta(t_5-t_4)}+e^{-\theta(t_5-t_3)}-e^{-\theta(t_4-t_3)}}{\theta}\right)-\frac{D_2}{\theta}\left(\frac{1-e^{\theta(T-t_5)}}{\theta}+(T-t_5)\right)\Bigg] \tag{28}$$

The expected cost of carrying the remanufactured inventory during the time period t_3 to T is

$$\int_0^{t_1}\beta h_r\left[\frac{yP_r}{\theta}\left(\frac{e^{\theta(t_3-t_4)}-1}{\theta}+(t_4-t_3)\right)+\frac{(P_1-D_2)}{\theta}\left(\frac{e^{-\theta(t_5-t_4)}-1}{\theta}+(t_5-t_4)\right)+\right.$$

$$\left.\frac{yP_r}{\theta}\left(\frac{1-e^{-\theta(t_5-t_4)}+e^{-\theta(t_5-t_3)}-e^{-\theta(t_4-t_3)}}{\theta}\right)-\frac{D_2}{\theta}\left(\frac{1-e^{\theta(T-t_5)}}{\theta}+(T-t_5)\right)\right]e^{-\beta x}dt' \tag{29}$$

viii). Expected holding cost for returned items

The holding cost for carrying the collected returned items during the time period zero to t_4 is

$$h_R\left[\int_0^{t_3}\frac{R}{\theta}\left(1-e^{-\theta t}\right)dt+\int_{t_3}^{t_4}\frac{(R-P_r)}{\theta}\left(1-e^{\theta(t_4-t)}\right)dt\right]$$

$$=h_R\left\{\frac{R}{\theta}\left(t_3+\frac{\left(e^{-\theta t_3}-1\right)}{\theta}\right)+\frac{(R-P_r)}{\theta}\left((t_4-t_3)+\frac{\left(1-e^{\theta(t_4-t_3)}\right)}{\theta}\right)\right\} \tag{30}$$

The expected holding cost for carrying the collected returned items during the time period zero to t_4 is

$$\int_0^{t_1}\beta h_R\left\{\frac{R}{\theta}\left(t_3+\frac{\left(e^{-\theta t_3}-1\right)}{\theta}\right)+\frac{(R-P_r)}{\theta}\left((t_4-t_3)+\frac{\left(1-e^{\theta(t_4-t_3)}\right)}{\theta}\right)\right\}e^{-\beta x}dt' \tag{31}$$

ix). Expected holding cost for imperfect items

Due to the automation policy of inspection, imperfect items are instantly removed from the serviceable items and carried at a separated place. The holding cost for carrying the imperfect items during the time period zero to t_5 is

$$h_{rw}\left\{(1-x)P\left(t't_2-\frac{t'^2}{2}+t'\lambda_1\frac{(t_3-t_2)}{2}\right)+(1-y)P_r\left(\frac{(t_4-t_3)^2}{2}+\lambda_2(t_4-t_3)(t_5-t_4)\right)\right\} \tag{32}$$

The expected holding cost for carrying the imperfect items during the time period zero to t_5 is.

$$\int_0^{t_1}\beta h_{rw}\left\{(1-x)P\left(t't_2-\frac{t'^2}{2}+t'\lambda_1\frac{(t_3-t_2)}{2}\right)+(1-y)P_r\left(\frac{(t_4-t_3)^2}{2}+\lambda_2(t_4-t_3)(t_5-\right.\right.$$

$$\left.\left.t_4)\right)\right\}e^{-\beta x}dt' \tag{33}$$

x). Expected setup cost

The system has to invest at the starting of the cycle to setup the manufacturing process, and the cost increases as the number of failures increases. The failure rate dependent setup cost can be calculated as follows:

$$(K_m(1+R_s\beta)+K_r) \tag{34}$$

The expected setup cost can be calculated as follows:

$$\int_0^{t_1} \beta(K_m(1+R_s\beta)+K_r)e^{-\beta x}dt' \tag{35}$$

xi). Expected disposal cost

As the production and remanufacturing process are not perfectly reliable, imperfect items are found, but all the items are not supposed to be in the condition to be reworked. Those items are disposed of. The cost of disposing these items is

$$c_d\{(1-x)(1-\lambda_1)Pt'+(1-y)(1-\lambda_2)P_r(t_4-t_3)\} \tag{36}$$

Thus, the expected cost of disposing these items is

$$\int_0^{t_1} \beta c_d\{(1-x)(1-\lambda_1)Pt'+(1-y)(1-\lambda_2)P_r(t_4-t_3)\}e^{-\beta x}dt' \tag{37}$$

Hence, the total cost per unit time of the given inventory model during the cycle $[0,T]$ as a function oft_1,t_2,t_3,t_4,t_5,T and $TC_1(t_1,t_2,t_3,t_4,t_5,T)$ is given by

$$
\begin{aligned}
TC_1 = &\int_0^{t_1} \beta(C_m P t' + R t_4) e^{-\beta x} dt' + \int_0^{t_1} \beta \left\{ \frac{\pi + \gamma\beta}{P} + (r_1 + r_2\beta) + (s_1 + s_2 xy) + C_p P \right\} P t' e^{-\beta x} dt' \\
&+ \int_0^{t_1} \beta C_r P_r (t_4 - t_3) e^{-\beta x} dt' + \int_0^{t_1} \beta C_{rw} P_1 (t_3 - t_2 + t_5 - t_4) e^{-\beta x} dt' \\
&+ \int_0^{t_1} \beta C_i \{P t' + P_r (t_4 - t_3)\} e^{-\beta x} dt' \\
&+ \int_0^{t_1} \beta h_m \left[\left(\frac{xP - D_1}{\theta} \right) \left(\frac{\left(e^{-\theta t'} - 1\right)}{\theta} + t' \right) + \frac{xP}{\theta^2} \left(1 - e^{-\theta t'} - e^{-\theta(t_2 - t')} + e^{-\theta t_2}\right) \right. \\
&- \frac{D_1}{\theta} \left(\frac{e^{-\theta t_2} - e^{-\theta t'}}{\theta} + (t_2 - t') \right) + \frac{D_1}{\theta} \left(\frac{e^{\theta(t_4 - t_2)} - e^{\theta(t_4 - t_3)}}{\theta} + (t_3 - t_2) \right) \\
&\left. + \frac{P_1}{\theta} \left((t_3 - t_2) + \frac{\left(1 - e^{\theta(t_3 - t_2)}\right)}{\theta} \right) - \frac{D_1}{\theta} \left((t_4 - t_3) + \frac{\left(1 - e^{\theta(t_4 - t_3)}\right)}{\theta} \right) \right] e^{-\beta x} dt' \\
&+ \int_0^{t_1} \beta h_r \left[\frac{y P_r}{\theta} \left(\frac{e^{\theta(t_3 - t_4)} - 1}{\theta} + (t_4 - t_3) \right) + \frac{(P_1 - D_2)}{\theta} \left(\frac{e^{-\theta(t_5 - t_4)} - 1}{\theta} + (t_5 - t_4) \right) \right. \\
&+ \frac{y P_r}{\theta} \left(\frac{1 - e^{-\theta(t_5 - t_4)} + e^{-\theta(t_5 - t_3)} - e^{-\theta(t_4 - t_3)}}{\theta} \right) \\
&\left. - \frac{D_2}{\theta} \left(\frac{1 - e^{\theta(T - t_5)}}{\theta} + (T - t_5) \right) \right] e^{-\beta x} dt' \\
&+ \int_0^{t_1} \beta h_R \left\{ \frac{R}{\theta} \left(t_3 + \frac{\left(e^{-\theta t_3} - 1\right)}{\theta} \right) + \frac{(R - P_r)}{\theta} \left((t_4 - t_3) + \frac{\left(1 - e^{\theta(t_4 - t_3)}\right)}{\theta} \right) \right\} e^{-\beta x} dt' \\
&+ \int_0^{t_1} \beta h_{rw} \left\{ (1 - x) P \left(t' t_2 - \frac{t'^2}{2} + t' \lambda_1 \frac{(t_3 - t_2)}{2} \right) \right. \\
&\left. + (1 - y) P_r \left(\frac{(t_4 - t_3)^2}{2} + \lambda_2 (t_4 - t_3)(t_5 - t_4) \right) \right\} e^{-\beta x} dt' \\
&+ \int_0^{t_1} \beta (K_m (1 + R_s \beta) + K_r) e^{-\beta x} dt' \\
&+ \int_0^{t_1} \beta c_d \{(1 - x)(1 - \lambda_1) P t' + (1 - y)(1 - \lambda_2) P_r (t_4 - t_3)\} e^{-\beta x} dt'
\end{aligned}
\tag{38}
$$

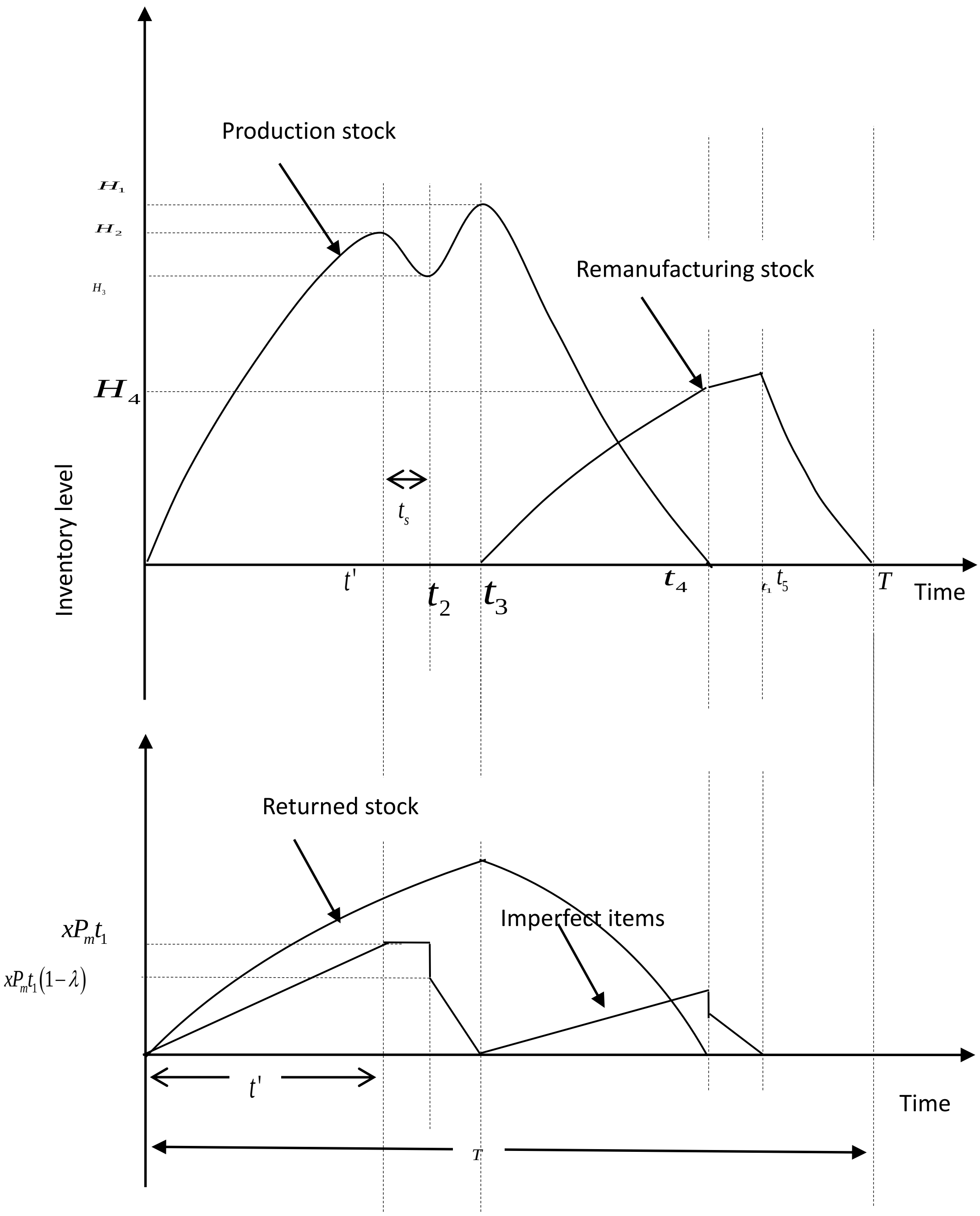

Figure 1. Behavior of inventory in a reverse logistics inventory model with random machine breakdown.

Based on the arrangement of inventory depicted in Figure 1, certain relations between the variables can derived, such as

$$t_s = (t_2 - t') \tag{39}$$

$$\frac{xP}{\theta}\left(e^{-\theta(t_2-t')} - e^{-\theta t_2}\right) - \frac{D_1}{\theta}\left(1 - e^{-\theta t_2}\right) = \left(\frac{P_1}{\theta}\right)\left(1 - e^{\theta(t_3-t_2)}\right) - \frac{D_1}{\theta}\left(1 - e^{\theta(t_4-t_2)}\right) \tag{40}$$

$$\frac{P_1-D_2}{\theta}\left(1 - e^{-\theta(t_5-t_4)}\right) + \frac{yP_r}{\theta}\left(e^{-\theta(t_5-t_4)} - e^{-\theta(t_5-t_3)}\right) = \frac{-D_2}{\theta}\left(1 - e^{\theta(T-t_5)}\right) \tag{41}$$

$$\frac{R}{\theta}\left(1-e^{-\theta t_3}\right)=\frac{(R-P_r)}{\theta}\left(1-e^{\theta(t_4-t_3)}\right) \tag{42}$$

$$(1-x)\lambda_1 P t' = P_1(t_3-t_2) \tag{43}$$

$$(1-y)\lambda_2 P_r(t_4-t_3) = P_1(t_5-t_4) \tag{44}$$

Therefore, from the above relations, the values of t_2, t_3, t_4, t_5 and T can be determined as functions of t', say,

$$t_2 = f_{12}(t') = (t_s+t') \tag{45}$$

$$t_3 = f_{13}(t') = \frac{(1-x)\lambda_1 P t' + P_1\left(t_s+t'\right)}{P_1} \tag{46}$$

$$t_4 = f_{14}(t') = \frac{P P_r t\prime(-1+x)\lambda_1 - P_1 P_r\left(t_s+t'\right)}{(P_r-R)\left\{P_1\left((t_s+t')\theta-1\right)-Pt\prime(x-1)\theta\lambda_1\right\}} \tag{47}$$

$$t_5 = f_{15}(t') = -\frac{\left\{P_r\left(P_1\left(t_s+t'\right)-Pt\prime(x-1)\lambda_1\right)\left\{{P_1}^2+(1-y)\left\{-P_1R-P_1(P_r-R)\left(t_s+t'\right)\theta+P(P_r-R)t\prime(x-1)\theta\lambda_1\right\}\lambda_2\right\}\right\}}{\left\{{P_1}^2(P_r-R)\left\{P_1\left((t_s+t')\theta-1\right)-Pt\prime(x-1)\theta\lambda_1\right\}\right\}} \tag{48}$$

$$T = f_{16}(t') = (P_r(-P(x-1)\lambda_1 t' + P_1(t_s+t'))(-P^3(x-1)^3(-1+y)y\theta^3 P_r(-R+P_r)^2\lambda_1^3\lambda_2(t')^3 + \theta P_1^5 P_r(t_s+t') + P^2(x-1)^2(y-1)y\theta^2 P_1 P_r(-R+P_r)\lambda_1^2\lambda_2(t')^2(2R+3\theta(-R+P_r)(t'_s+t')) - P(x-1)\theta P_1^2\lambda_1 t'(P(x-1)\theta(-R+P_r)\lambda_1(-Ry+2yP_r+R(-1+y)\lambda_2)t' + (y-1)yP_r\lambda_2(R+\theta(-R+P_r)(t_s+t'))(R+3\theta(-R+P_r)(t_s+t'))) + D_2P_1(P^2(x-1)^2(-1+y)\theta^2P_r(-R+P_r)\lambda_1^2\lambda_2(t')^2 - P(x-1)(-1+y)\theta P_1P_r\lambda_1\lambda_2 t'(R+2\theta(-R+P_r)(t_s+t')) + P_1^3(P_r-2\theta P_r(t_s+t')+R(-1+\theta(t_s+t'))) + \theta P_1^2(P(x-1)(-R+2P_r)\lambda_1 t' + (-1+y)P_r\lambda_2(t_s+t')(R+\theta(-R+P_r)(t_s+t')))) + \theta P_1^3((-1+y)yP_r\lambda_2(t_s+t')(R+\theta(-R+P_r)(t_s+t'))^2 + P(-1+x)\lambda_1 t'((-1+y)\lambda_2(2R^2-2RP_r+P_r^2+2R\theta(-R+P_r)(t_s+t')) + y(-2R^2+4RP_r-P_r^2+2\theta(R^2-3RP_r+2P_r^2)(t_s+t')))) + P_1^4(R^2(y(-1+\lambda_2)-\lambda_2)(-1+\theta(t_s+t'))^2 - \theta P_r^2(t_s+t')(-\lambda_2+y(-1+\lambda_2+2\theta(t_s+t'))) + P_r(-P(-1+x)\theta\lambda_1 t' - R(-1+\theta(t_s+t'))(y-3y\theta(t_s+t')+(-1+y)\lambda_2(-1+\theta(t_s+t')))))))/(D_2P_1(-R+P_r)(-P(-1+x)\theta\lambda_1 t' + P_1(-1+\theta(t_s+t')))(-(-1+y)\theta P_r\lambda_2(-P(-1+x)\lambda_1 t' + P_1(t_s+t'))(P(-1+x)\theta(R-P_r)\lambda_1 t' + P_1(R+\theta(-R+P_r)(t_s+t'))) + P_1^2(-P(-1+x)\theta(-R+2P_r)\lambda_1 t' + P_1(R-P_r+\theta(-R+2P_r)(t_s+t'))))) \tag{49}$$

Based on the above results, the expected cost function can be written as

$$
\begin{aligned}
TC_1 = {} & \int_0^{t_1} \beta(C_m Pt' + C_R R f_{14}) e^{-\beta x} dt' + \int_0^{t_1} \beta \left\{ \frac{\pi + \gamma\beta}{P} + (r_1 + r_2\beta) + (s_1 + s_2 xy) + C_p P \right\} Pt' e^{-\beta x} dt' \\
& + \int_0^{t_1} \beta C_r P_r (f_{14} - f_{13}) e^{-\beta x} dt' + \int_0^{t_1} \beta C_{rw} P_1 (f_{13} - f_{12} + f_{15} - f_{14}) e^{-\beta x} dt' \\
& + \int_0^{t_1} \beta C_i \{Pt' + P_r(f_{14} - f_{13})\} e^{-\beta x} dt' \\
& + \int_0^{t_1} \beta h_m \left[\left(\frac{xP - D_1}{\theta} \right) \left(\frac{\left(e^{-\theta t'} - 1\right)}{\theta} + t' \right) + \frac{xP}{\theta^2} \left(1 - e^{-\theta t'} - e^{-\theta(f_{12} - t')} + e^{-\theta f_{12}}\right) \right. \\
& - \frac{D_1}{\theta} \left(\frac{e^{-\theta f_{12}} - e^{-\theta t'}}{\theta} + (f_{12} - t') \right) + \frac{D_1}{\theta} \left(\frac{e^{\theta(f_{14} - f_{12})} - e^{\theta(f_{14} - f_{13})}}{\theta} + (f_{13} - f_{12}) \right) \\
& \left. + \frac{P_1}{\theta} \left((f_{13} - f_{12}) + \frac{\left(1 - e^{\theta(f_{13} - f_{12})}\right)}{\theta} \right) - \frac{D_1}{\theta} \left((f_{14} - f_{13}) + \frac{\left(1 - e^{\theta(f_{14} - f_{13})}\right)}{\theta} \right) \right] e^{-\beta x} dt' \\
& + \int_0^{t_1} \beta h_r \left[\frac{yP_r}{\theta} \left(\frac{e^{\theta(f_{13} - f_{14})} - 1}{\theta} + (f_{14} - f_{13}) \right) \right. \\
& + \frac{(P_1 - D_2)}{\theta} \left(\frac{e^{-\theta(f_{15} - f_{14})} - 1}{\theta} + (f_{15} - f_{14}) \right) \\
& + \frac{yP_r}{\theta} \left(\frac{1 - e^{-\theta(f_{15} - f_{14})} + e^{-\theta(f_{15} - f_{13})} - e^{-\theta(f_{14} - f_{13})}}{\theta} \right) \qquad (50) \\
& \left. - \frac{D_2}{\theta} \left(\frac{1 - e^{\theta(f_{16} - f_{15})}}{\theta} + (f_{16} - f_{15}) \right) \right] e^{-\beta x} dt' \\
& + \int_0^{t_1} \beta h_R \left\{ \frac{R}{\theta} \left(f_{13} + \frac{\left(e^{-\theta f_{13}} - 1\right)}{\theta} \right) + \frac{(R - P_r)}{\theta} \left((f_{14} - f_{13}) + \frac{\left(1 - e^{\theta(f_{14} - f_{13})}\right)}{\theta} \right) \right\} e^{-\beta x} dt' \\
& + \int_0^{t_1} \beta h_{rw} \left\{ (1 - x) P \left(t' f_{12} - \frac{t'^2}{2} + t' \lambda_1 \frac{(f_{13} - f_{12})}{2} \right) \right. \\
& \left. + (1 - y) P_r \left(\frac{(f_{14} - f_{13})^2}{2} + \lambda_2 (f_{14} - f_{13})(f_{15} - f_{14}) \right) \right\} e^{-\beta x} dt' \\
& + \int_0^{t_1} \beta(K_m(1 + R_s\beta) + K_r) e^{-\beta x} dt' \\
& + \int_0^{t_1} \beta c_d \{(1 - x)(1 - \lambda_1) Pt' + (1 - y)(1 - \lambda_2) P_r (f_{14} - f_{13})\} e^{-\beta x} dt'
\end{aligned}
$$

4.2. (Case 2) Reverse logistics inventory model without machine breakdown

In this case, it is considered that the machine breakdown does not occur during the production run. The behavior of the on-hand inventory is depicted in Figure 2.

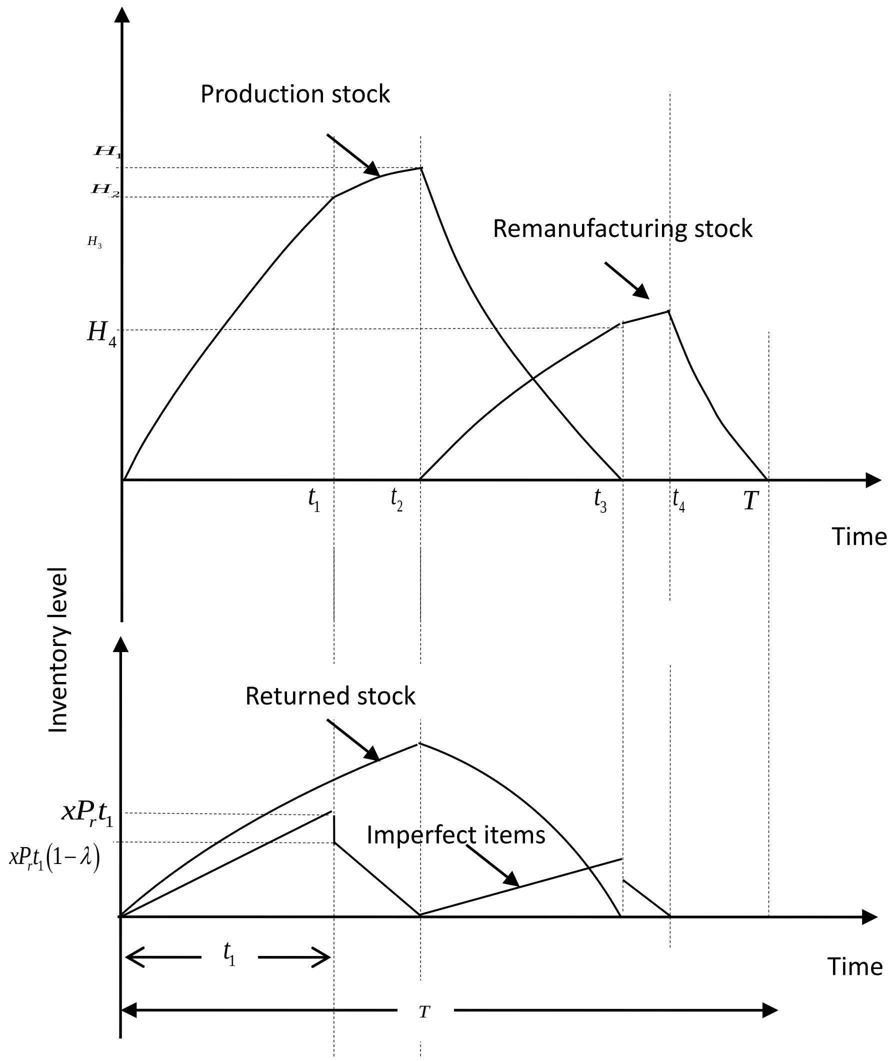

Figure 2. The behavior of inventory in a reverse logistics inventory model without machine breakdowns.

The differential equations governing the stock level during the period $0 \le t \le T$ can be written as

$$\frac{dI_m(t)}{dt} + \theta I_m(t) = xP - D_1 \qquad 0 \le t \le t_1 \qquad I_m(0) = 0 \qquad (51)$$

$$\frac{dI_m(t)}{dt} + \theta I_m(t) = P_1 - D_1 \qquad t_1 \le t \le t_2 \qquad I_m(t_1^-) = I_m(t_1^-) \qquad (52)$$

$$\frac{dI_m(t)}{dt} + \theta I_m(t) = -D_1 \qquad t_2 \le t \le t_3 \qquad I_m(t_3) = 0 \qquad (53)$$

$$\frac{dI_r(t)}{dt} + \theta I_r(t) = yP_r - D_2 \qquad t_2 \le t \le t_3 \qquad I_r(t_2) = 0 \qquad (54)$$

$$\frac{dI_r(t)}{dt}+\theta I_r(t)=P_1-D_2 \qquad t_3\le t\le t_4 \qquad I_r(t_3^-)=I_r(t_3^+) \tag{55}$$

$$\frac{dI_r(t)}{dt}+\theta I_r(t)=-D_2 \qquad t_4\le t\le t_5 \qquad I_r(T)=0 \tag{56}$$

$$\frac{dI_R(t)}{dt}+\theta I_R(t)=R \qquad 0\le t\le t_2 \qquad I_R(0)=0 \tag{57}$$

$$\frac{dI_R(t)}{dt}+\theta I_R(t)=R-P_r \qquad t_2\le t\le t_3 \qquad I_R(t_3)=0 \tag{58}$$

The solutions of the above differential equations are

$$I_m(t)=\left(\frac{xP-D_1}{\theta}\right)\left(1-e^{-\theta t}\right) \qquad 0\le t\le t_1 \tag{59}$$

$$I_m(t)=\left(\frac{P_1-D_1}{\theta}\right)\left(1-e^{\theta(t_1-t)}\right)+\left(\frac{xP-D_1}{\theta}\right)\left(e^{\theta(t_1-t)}-e^{-\theta t}\right) \qquad t_1\le t\le t_2 \tag{60}$$

$$I_m(t)=\frac{D_1}{\theta}\left(e^{\theta(t_3-t)}-1\right) \qquad t_2\le t\le t_3 \tag{61}$$

$$I_r(t)=\frac{yP_r}{\theta}\left(1-e^{\theta(t_2-t)}\right) \qquad t_2\le t\le t_3 \tag{62}$$

$$I_r(t)=\frac{P_1-D_2}{\theta}\left(1-e^{\theta(t_3-t)}\right)+\frac{yP_r}{\theta}\left(e^{\theta(t_3-t)}-e^{\theta(t_2-t)}\right) \qquad t_3\le t\le t_4 \tag{63}$$

$$I_r(t)=\frac{D_2}{\theta}\left(e^{\theta(T-t)}-1\right) \qquad t_4\le t\le t_5 \tag{64}$$

$$I_R(t)=\frac{R}{\theta}\left(1-e^{-\theta t}\right) \qquad 0\le t\le t_2 \tag{65}$$

$$I_R(t)=\frac{(R-P_r)}{\theta}\left(1-e^{\theta(t_3-t)}\right) \qquad t_2\le t\le t_3 \tag{66}$$

The per cycle cost components for the given inventory model are as follows.

i). Expected procurement and acquisition cost

The cost of purchasing the raw material (including deterioration cost) from the supplier can be calculated as

$$(C_mPt_1+C_RRt_3) \tag{67}$$

Thus, the expected purchasing cost would be

$$\int_{t_1}^{\infty}\beta(C_mPt_1+Rt_3)e^{-\beta x}dt' \tag{68}$$

ii). Expected production cost

The per unit expected production cost is

$$\int_{t_1}^{\infty}\beta\left\{\frac{\pi+\gamma\beta}{P}+(r_1+r_2\beta)+(s_1+s_2xy)+C_pP\right\}Pt_1e^{-\beta x}dt' \tag{69}$$

iii). Expected repairing cost

The per unit expected repairing cost is

$$\int_{t_1}^{\infty} \beta C_r P_r (t_3 - t_2) e^{-\beta x} dt' \tag{70}$$

iv). Expected rework cost

The per unit expected rework cost is

$$\int_{t_1}^{\infty} \beta C_{rw} P_1 (t_2 - t_1 + t_4 - t_3) e^{-\beta x} dt' \tag{71}$$

v). Expected inspection cost

The per unit expected inspection cost is

$$\int_{t_1}^{\infty} \beta C_i \{P t_1 + P_r (t_3 - t_2)\} e^{-\beta x} dt' \tag{72}$$

vi). Expected holding cost for newly produced material

The items are stored in the warehouse, and the manufacturer needs to spend some investment. The expected holding cost of carrying the newly produced inventory during the time period zero to t_3 can be calculated as follows:

$$h_m \left[\int_0^{t_1} \left(\frac{xP - D_1}{\theta} \right) \left(1 - e^{-\theta t} \right) dt \right.$$

$$+ \int_{t_1}^{t_2} \left\{ \left(\frac{P_1 - D_1}{\theta} \right) \left(1 - e^{\theta (t_1 - t)} \right) + \left(\frac{xP - D_1}{\theta} \right) \left(e^{\theta (t_1 - t)} - e^{-\theta t} \right) \right\} dt$$

$$\left. + \int_{t_2}^{t_3} \frac{D_1}{\theta} \left(e^{\theta (t_3 - t)} - 1 \right) dt \right]$$

$$= h_m \left[\left(\frac{xP - D_1}{\theta} \right) \left(\frac{\left(e^{-\theta t_1} - 1 \right)}{\theta} + t_1 \right) + \left(\frac{xP - D_1}{\theta^2} \right) \left(1 - e^{-\theta t_1} - e^{-\theta (t_2 - t_1)} + e^{-\theta t_2} \right) + \frac{(P_1 - D_1)}{\theta} \left(\frac{e^{-\theta (t_2 - t_1)} - 1}{\theta} + \right. \right.$$

$$\left. \left. (t_2 - t_1) \right) - \frac{D_1}{\theta} \left(\frac{1 - e^{\theta (t_3 - t_2)}}{\theta} + (t_3 - t_2) \right) \right] \tag{73}$$

The expected holding cost of carrying the newly produced inventory during the time period zero to t_3 can be calculated as follows:

$$\int_{t_1}^{\infty} \beta h_m \left[\left(\frac{xP - D_1}{\theta} \right) \left(\frac{\left(e^{-\theta t_1} - 1 \right)}{\theta} + t_1 \right) + \left(\frac{xP - D_1}{\theta^2} \right) \left(1 - e^{-\theta t_1} - e^{-\theta (t_2 - t_1)} + e^{-\theta t_2} \right) + \right.$$

$$\left. \frac{(P_1 - D_1)}{\theta} \left(\frac{e^{-\theta (t_2 - t_1)} - 1}{\theta} + (t_2 - t_1) \right) - \frac{D_1}{\theta} \left(\frac{1 - e^{\theta (t_3 - t_2)}}{\theta} + (t_3 - t_2) \right) \right] e^{-\beta x} dt' \tag{74}$$

vii). Expected holding cost for remanufactured material

$$h_r\left[\int_{t_2}^{t_3}\frac{yP_r}{\theta}\left(1-e^{\theta(t_2-t)}\right)dt+\int_{t_3}^{t_4}\left\{\frac{P_1-D_2}{\theta}\left(1-e^{\theta(t_3-t)}\right)+\frac{yP_r}{\theta}\left(e^{\theta(t_3-t)}-e^{\theta(t_2-t)}\right)\right\}dt+\int_{t_4}^{T}\frac{D_2}{\theta}\left(e^{\theta(T-t)}-1\right)dt\right]$$

$$=h_r\left[\frac{yP_r}{\theta}\left(\frac{e^{-\theta(t_3-t_2)}-1}{\theta}+(t_3-t_2)\right)+\frac{(P_1-D_2)}{\theta}\left(\frac{e^{-\theta(t_4-t_3)}-1}{\theta}+(t_4-t_3)\right)+\frac{yP_r}{\theta}\left(\frac{1-e^{-\theta(t_4-t_3)}+e^{-\theta(t_4-t_2)}-e^{-\theta(t_3-t_2)}}{\theta}\right)-\frac{D_2}{\theta}\left(\frac{1-e^{\theta(T-t_4)}}{\theta}+(T-t_4)\right)\right]=\int_{t_1}^{\infty}\beta h_r\left[\frac{yP_r}{\theta}\left(\frac{e^{-\theta(t_3-t_2)}-1}{\theta}+(t_3-t_2)\right)+\frac{(P_1-D_2)}{\theta}\left(\frac{e^{-\theta(t_4-t_3)}-1}{\theta}+(t_4-t_3)\right)+\frac{yP_r}{\theta}\left(\frac{1-e^{-\theta(t_4-t_3)}+e^{-\theta(t_4-t_2)}-e^{-\theta(t_3-t_2)}}{\theta}\right)-\frac{D_2}{\theta}\left(\frac{1-e^{\theta(T-t_4)}}{\theta}+(T-t_4)\right)\right]e^{-\beta x}dt' \tag{75}$$

viii). Expected holding cost for returned items

$$h_R\left[\int_0^{t_2}\frac{R}{\theta}\left(1-e^{-\theta t}\right)dt+\int_{t_2}^{t_3}\frac{(R-P_r)}{\theta}\left(1-e^{\theta(t_3-t)}\right)dt\right]$$

$$=h_R\left\{\frac{R}{\theta}\left(t_2+\frac{\left(e^{-\theta t_2}-1\right)}{\theta}\right)+\frac{(R-P_r)}{\theta}\left((t_3-t_2)+\frac{\left(1-e^{\theta(t_3-t_2)}\right)}{\theta}\right)\right\}$$

$$=\int_{t_1}^{\infty}\beta h_R\left\{\frac{R}{\theta}\left(t_2+\frac{\left(e^{-\theta t_2}-1\right)}{\theta}\right)+\frac{(R-P_r)}{\theta}\left((t_3-t_2)+\frac{\left(1-e^{\theta(t_3-t_2)}\right)}{\theta}\right)\right\}e^{-\beta x}dt' \tag{76}$$

ix). Expected holding cost for reworked items

$$h_{rw}\left\{(1-x)P\left(\frac{t_1^2}{2}+t_1\lambda_1\frac{(t_2-t_1)}{2}\right)+(1-y)P_r\left(\frac{(t_3-t_2)^2}{2}+\lambda_2(t_3-t_2)(t_4-t_3)\right)\right\} \tag{77}$$

$$=\int_{t_1}^{\infty}\beta h_{rw}\left\{(1-x)P\left(\frac{t_1^2}{2}+t_1\lambda_1\frac{(t_2-t_1)}{2}\right)+(1-y)P_r\left(\frac{(t_3-t_2)^2}{2}+\lambda_2(t_3-t_2)(t_4-t_3)\right)\right\}e^{-\beta x}dt'$$

x). Expected setup cost

$$K_m(1+R_s\beta)+K_r=\int_{t_1}^{\infty}\beta(K_m(1+R_s\beta)+K_r)e^{-\beta x}dt' \tag{78}$$

xi). Expected disposal cost

$$c_d\{(1-x)(1-\lambda_1)Pt_1+(1-y)(1-\lambda_2)P_r(t_3-t_2)\}$$

$$=\int_{t_1}^{\infty}\beta c_d\{(1-x)(1-\lambda_1)Pt_1+(1-y)(1-\lambda_2)P_r(t_3-t_2)\}e^{-\beta x}dt' \tag{79}$$

Hence, the total cost per unit time of the given inventory model during the cycle $[0,T]$ as a function of t_1,t_2,t_3,t_4 and T, i.e., $TC_2(t_1,t_2,t_3,t_4,T)$, is Total cost, given by

Total cost= Procurement and acquisition cost+ Production and remanufacturing cost+ Holding cost for remanufactured material + Holding cost for produced material + Holding cost for returned items+ Holding cost for reworked items+ Disposal cost +Setup cost

$$=\int_{t_1}^{\infty}\beta(C_mPt_1+C_RRt_3)e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta\left\{\frac{\pi+\gamma\beta}{P}+(r_1+r_2\beta)+(s_1+s_2xy)+C_pP\right\}Pt_1e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta C_rP_r(t_3-t_2)e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta C_{rw}P_1(t_2-t_1+t_4-t_3)e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta C_i\{Pt_1+P_r(t_3-t_2)\}e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta h_m\left[\left(\frac{xP-D_1}{\theta}\right)\left(\frac{(e^{-\theta t_1}-1)}{\theta}+t_1\right)+\left(\frac{xP-D_1}{\theta^2}\right)\left(1-e^{-\theta t_1}-e^{-\theta(t_2-t_1)}+e^{-\theta t_2}\right)+\frac{(P_1-D_1)}{\theta}\left(\frac{e^{-\theta(t_2-t_1)}-1}{\theta}+(t_2-t_1)\right)-\frac{D_1}{\theta}\left(\frac{1-e^{\theta(t_3-t_2)}}{\theta}+(t_3-t_2)\right)\right]e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta h_r\left[\frac{yP_r}{\theta}\left(\frac{e^{-\theta(t_3-t_2)}-1}{\theta}+(t_3-t_2)\right)+\frac{(P_1-D_2)}{\theta}\left(\frac{e^{-\theta(t_4-t_3)}-1}{\theta}+(t_4-t_3)\right)+\frac{yP_r}{\theta}\left(\frac{1-e^{-\theta(t_4-t_3)}+e^{-\theta(t_4-t_2)}-e^{-\theta(t_3-t_2)}}{\theta}\right)-\frac{D_2}{\theta}\left(\frac{1-e^{\theta(T-t_4)}}{\theta}+(T-t_4)\right)\right]e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta h_R\left\{\frac{R}{\theta}\left(t_2+\frac{(e^{-\theta t_2}-1)}{\theta}\right)+\frac{(R-P_r)}{\theta}\left((t_3-t_2)+\frac{(1-e^{\theta(t_3-t_2)})}{\theta}\right)\right\}e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta h_{rw}\left\{(1-x)P\left(\frac{t_1^2}{2}+t_1\lambda_1\frac{(t_2-t_1)}{2}\right)+(1-y)P_r\left(\frac{(t_3-t_2)^2}{2}+\lambda_2(t_3-t_2)(t_4-t_3)\right)\right\}e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta(K_m(1+R_s\beta)+K_r)e^{-\beta x}dt'+\int_{t_1}^{\infty}\beta c_d\{(1-x)(1-\lambda_1)Pt_1+(1-y)(1-\lambda_2)P_r(t_3-t_2)\}e^{-\beta x}dt' \tag{80}$$

The above can be rewritten as

$$TC_2 = \Big[(C_m P t_1 + C_R R t_3) + \Big\{\frac{\pi+\gamma\beta}{P} + (r_1 + r_2\beta) + (s_1 + s_2 xy) + C_p P\Big\} P t_1 + C_r P_r (t_3 - t_2) + C_{rw} P_1 (t_2 - t_1 + t_4 - t_3) + C_i\{P t_1 + P_r(t_3 - t_2)\} + h_m \Big[\Big(\frac{xP-D_1}{\theta}\Big)\Big(\frac{(e^{-\theta t_1}-1)}{\theta} + t_1\Big) + \Big(\frac{xP-D_1}{\theta^2}\Big)\big(1 - e^{-\theta t_1} - e^{-\theta(t_2-t_1)} + e^{-\theta t_2}\big) + \frac{(P_1-D_1)}{\theta}\Big(\frac{e^{-\theta(t_2-t_1)}-1}{\theta} + (t_2 - t_1)\Big) - \frac{D_1}{\theta}\Big(\frac{1-e^{\theta(t_3-t_2)}}{\theta} + (t_3 - t_2)\Big)\Big] + h_r\Big[\frac{yP_r}{\theta}\Big(\frac{e^{-\theta(t_3-t_2)}-1}{\theta} + (t_3 - t_2)\Big) + \frac{(P_1-D_2)}{\theta}\Big(\frac{e^{-\theta(t_4-t_3)}-1}{\theta} + (t_4 - t_3)\Big) + \frac{yP_r}{\theta}\Big(\frac{1-e^{-\theta(t_4-t_3)}+e^{-\theta(t_4-t_2)}-e^{-\theta(t_3-t_2)}}{\theta}\Big) - \frac{D_2}{\theta}\Big(\frac{1-e^{\theta(T-t_4)}}{\theta} + (T - t_4)\Big)\Big] + h_R\Big\{\frac{R}{\theta}\Big(t_2 + \frac{(e^{-\theta t_2}-1)}{\theta}\Big) + \frac{(R-P_r)}{\theta}\Big((t_3 - t_2) + \frac{(1-e^{\theta(t_3-t_2)})}{\theta}\Big)\Big\} + h_{rw}\Big\{(1-x)P\Big(\frac{t_1^2}{2} + t_1\lambda_1\frac{(t_2-t_1)}{2}\Big) + (1-y)P_r\Big(\frac{(t_3-t_2)^2}{2} + \lambda_2(t_3 - t_2)(t_4 - t_3)\Big)\Big\} + K_m(1 + R_s\beta) + K_r + c_d\{(1-x)(1-\lambda_1)P t_1 + (1-y)(1-\lambda_2)P_r(t_3 - t_2)\}\Big]\int_{t_1}^{\infty} \beta e^{-\beta x}\,dt \tag{81}$$

Here, the cost function of the system is given in terms of $t_1, t_2, t_3, t_4$and T. To find the optimum solution, we have to find the optimum values of t_1, t_2, t_3, t_4 and T that minimize $TC_2(t_1, t_2, \dots, T)$, but we have some relations between the variables as follows.

$$\Big(\frac{P_1-D_1}{\theta}\Big)\big(1 - e^{-\theta(t_2-t_1)}\big) + \Big(\frac{xP-D_1}{\theta}\Big)\big(e^{-\theta(t_2-t_1)} - e^{-\theta t_2}\big) = \frac{D_1}{\theta}\big(e^{\theta(t_3-t_2)} - 1\big) \tag{82}$$

$$\frac{P_1-D_2}{\theta}\big(1 - e^{-\theta(t_4-t_3)}\big) + \frac{yP_r}{\theta}\big(e^{-\theta(t_4-t_3)} - e^{-\theta(t_4-t_2)}\big) = \frac{D_2}{\theta}\big(e^{\theta(T-t_4)} - 1\big) \tag{83}$$

$$\frac{R}{\theta}\big(1 - e^{-\theta t_2}\big) = \frac{(R-P_r)}{\theta}\big(1 - e^{\theta(t_3-t_2)}\big) \tag{84}$$

$$(1-x)\lambda_1 P t_1 = P_1(t_2 - t_1) \tag{85}$$

$$(1-y)\lambda_2 P_r(t_3 - t_2) = P_1(t_4 - t_3) \tag{86}$$

Therefore, from the above equations, the values of$t_2, t_3, t_4$and$T$can be determined as functions of t_1, say,

$$t_2 = f_{21}(t_1) = \Big(\frac{(1-x)\lambda_1 P}{P_1} + 1\Big) t_1 \tag{87}$$

$$t_3 = f_{22}(t_1) = \left(\frac{P_r}{P_r - \mathrm{R}}\right)\left(\frac{(1-x)\lambda_1 P}{P_1} + 1\right) t_1 \tag{88}$$

$$t_4 = f_{23}(t_1) = \left(\frac{P_r}{P_r - \mathrm{R}}\right)\{(1-y)\lambda_2 R + P_1\}\left(\frac{(1-x)\lambda_1 P}{P_1} + 1\right) t_1 \tag{89}$$

$$t_4 = f_{24}(t_1) = \frac{(1-x)\lambda_1 P t' + P_1(t_s - t')}{P_1} \tag{90}$$

$$T = f_{25}(t_1) = -\frac{t_1 P_r(P_1 - P(-1+x)\lambda_1)(D_2 + P_1^2 + y(R+P_r) - P_1(R(-1+y)\lambda_2))}{D_2 P_1(R - P_r)} \tag{91}$$

Therefore, the total variable cost function will be a function of a single variable, say, t_1. The problem will be converted into the following problem with one variable $Z_2(t_1)$:

$$TC_2 = \Bigg[(C_m P t_1 + C_R R f_{23}) + \left\{\frac{\pi+\gamma\beta}{P} + (r_1 + r_2\beta)(s_1 + s_2\beta) + C_p P\right\} P t_1 + C_r P_r(f_{23} - f_{22}) +$$

$$C_{rw} P_1(f_{22} - t_1 + f_{24} - f_{23}) + C_i\{P t_1 + P_r(f_{23} - f_{22})\} + h_m\Bigg[\left(\frac{xP - D_1}{\theta}\right)\left(\frac{(e^{-\theta t_1} - 1)}{\theta} + t_1\right) +$$

$$\left(\frac{xP - D_1}{\theta^2}\right)\left(1 - e^{-\theta t_1} - e^{-\theta(f_{22} - t_1)} + e^{-\theta f_{22}}\right) + \frac{(P_1 - D_1)}{\theta}\left(\frac{e^{-\theta(f_{22} - t_1)} - 1}{\theta} + (f_{22} - t_1)\right) -$$

$$\frac{D_1}{\theta}\left(\frac{1 - e^{\theta(f_{23} - f_{22})}}{\theta} + (f_{23} - f_{22})\right)\Bigg] + h_r\Bigg[\frac{yP_r}{\theta}\left(\frac{e^{-\theta(f_{23} - f_{22})} - 1}{\theta} + (f_{23} - f_{22})\right) + \frac{(P_1 - D_2)}{\theta}\left(\frac{e^{-\theta(f_{24} - f_{23})} - 1}{\theta} +\right.$$

$$\left.(f_{24} - f_{23})\right) + \frac{yP_r}{\theta}\left(\frac{1 - e^{-\theta(f_{24} - f_{23})} + e^{-\theta(f_{24} - f_{22})} - e^{-\theta(f_{23} - f_{22})}}{\theta}\right) - \frac{D_2}{\theta}\left(\frac{1 - e^{\theta(f_{25} - f_{24})}}{\theta} + (f_{25} - f_{24})\right)\Bigg] +$$

$$h_R\left\{\frac{R}{\theta}\left(f_{22} + \frac{(e^{-\theta f_{22}} - 1)}{\theta}\right) + \frac{(R - P_r)}{\theta}\left((f_{23} - f_{22}) + \frac{(1 - e^{\theta(f_{23} - f_{22})})}{\theta}\right)\right\} + h_{rw}\left\{(1-x)P\left(\frac{t_1^2}{2} +\right.\right.$$

$$\left.\left. t_1\lambda_1\frac{(f_{22} - t_1)}{2}\right) + (1-y)P_r\left(\frac{(f_{23} - f_{22})^2}{2} + \lambda_2(f_{23} - f_{22})(f_{24} - f_{23})\right)\right\} + K_m(1 + R_s\beta) + K_r +$$

$$c_d\{(1-x)(1-\lambda_1)P t_1 + (1-y)(1-\lambda_2)P_r(f_{23} - f_{22})\}\Bigg]\int_{t_1}^{\infty}\beta e^{-\beta x}\,dt \tag{92}$$

5. Solution approach

In this model, we have developed a green supply chain model with stochastic machine breakdown, that is, the failure can take place anytime during the production process. In the last section, we have derived the cost functions for both cases: The inventory cost function for case 1 when breakdown occurs is $E[Z_1(t')]$, and the cost function for case 2 when the breakdown does not occur is $E[Z_2(t_1)]$. Here, we have assumed that the random number of breakdowns per unit time follows the Poisson distribution, with the mean equal to β per unit time. Thus, the random production time to breakdown should obey the exponential distribution, with the density function $f(t) = \beta e^{-\beta t}$ and the cumulative density function $F(t) = \left(1 - e^{-\beta t}\right)$. Then, the expected production-inventory cost per unit time (whether a breakdown takes place or not), $E[Z(t_1)]$is

$$E\left[Z(t_1)\right] = \frac{\int_0^{t_1} E\left[Z_1(t')\right] f(t)\,dt + \int_{t_1}^{\infty} E\left[Z_2(t_1)\right] f(t)\,dt}{E[T]} \tag{93}$$

where

$$E[T] = \int_0^{t_1} E(T')\, f(t)\,dt + \int_{t_1}^{\infty} E(T)\, f(t)\,dt \tag{94}$$

Alternatively, it can be written as

$$E[T] = \int_0^{t_1} f_{15}(t')\, f(t)\,dt + \int_{t_1}^{\infty} f_{24}(t_1)\, f(t)\,dt \tag{95}$$

Now, substituting$E[Z_1(t')]$, $E[Z_2(t_1)]$ and $E[T]$ from Eqs. (50), (92) and (95), in Eq. (93), one can obtain the expected inventory cost per unit time.

Now, the necessary conditions for having a minimum for the problem is $\frac{dE\left[Z(t_1)\right]}{dt_1} = 0$ (96)

Using Equation 96, one can find the optimal value of t_1 and hence evaluate the minimum inventory cost $E[Z(t_1)]$.

6. Numerical example and sensitivity analysis

To illustrate the proposed model, we have considered the following input parameters in appropriate units:

$P_r = 400$ items per month, $P = 500$ items per month, $D_1 = 200$ items per month, $D_2 = 100$ items per month, $P_1 = 200$ items per month, $R = 50$ items per month, $x = 0.2$, $y = 0.15$, $\lambda_1 = 0.25$, $\lambda_2 = 0.2$, $\theta = 0.01$, $K_m = 5000$ \$ per cycle, $K_r = 2000$ \$ per cycle, $h_m = 1$\$ per items per month, $h_r = 0.5$ \$ per items per month, $h_R = 0.5$\$ per items per month, $h_1 = 0.5$ \$ per items per month, $S_m = 5$ items per month, $S_r = 1$ items per month, $S_{rw} = 0.2$ items per month, $C_m = 2$ items per month, $C_R = 0.5$ items per month, $C_s = 5$ items per month, $C_d = 0.1$ items per month, $\beta = 0.5$, $t_s = 0.5$ items per month, $\pi = 1500$ \$ per production lot, $r_1 = 12.5$ \$ per cycle,

$r_2 = 18.5$ \$ per items, $s_1 = 15$ \$ per cycle, $s_2 = 10$ \$ per item, $C_p = 20$ \$ per item, $C_r = 45$ \$ per item, $C_{rw} = 10$ \$ per item, $C_i = 5$\$ per item, $R_s = 150$\$ per item, $\gamma = 45$

Applying the procedure proposed in the last section (with the help of the software Mathematica) we find the optimal value of the production time period $t_1 = 7.36573$ and hence the corresponding minimum total expected cost $E[Z(t_1)] = 817840$

The convexity of the inventory model is shown in Figure 3. The two-dimensional graph shows that the integrated expected total annual cost is convex.

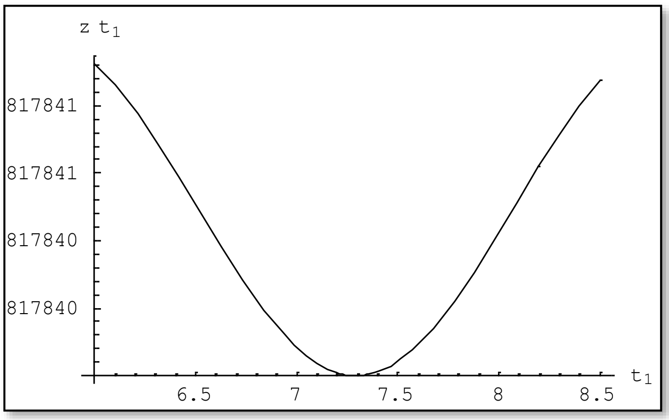

Figure 3. Behavior of the inventory cost function with or without breakdown.

6.1.1. Sensitivity analysis

As one can see from the example above, the numerical technique can be used to analyze the effect of parameters in order to determine the optimal values of the parameters as well as the minimum cost to be incurred by the system. The results of the investigation are obtained by changing the values of one parameter in turn while leaving the other parameters at their original values.

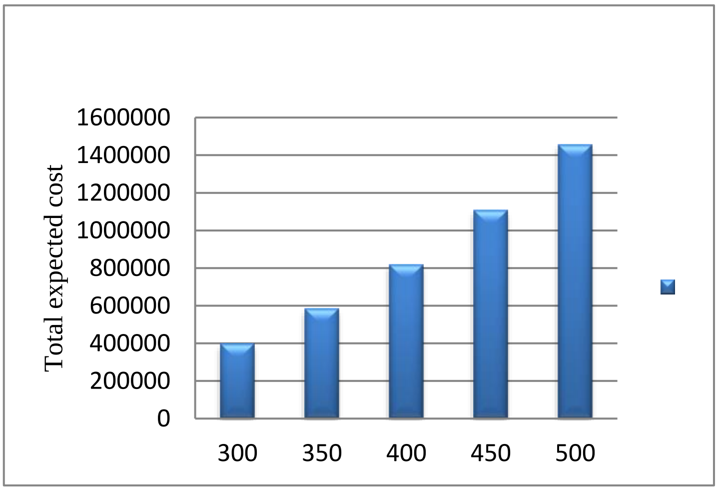

Figure 4. The behavior of total profit for varying remanufacturing rate.

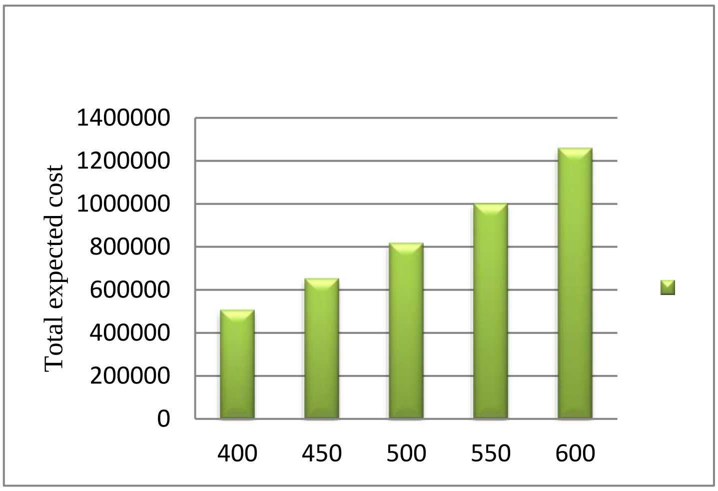

Figure 5. The behavior of total profit for varying production rate.

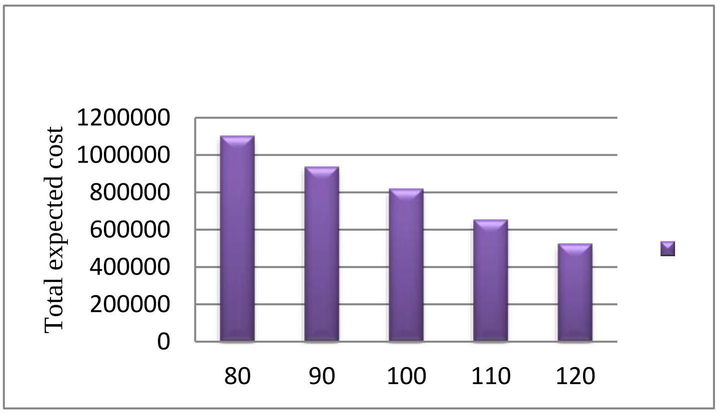

Figure 6. The behavior of expected cost for varying demand rate.

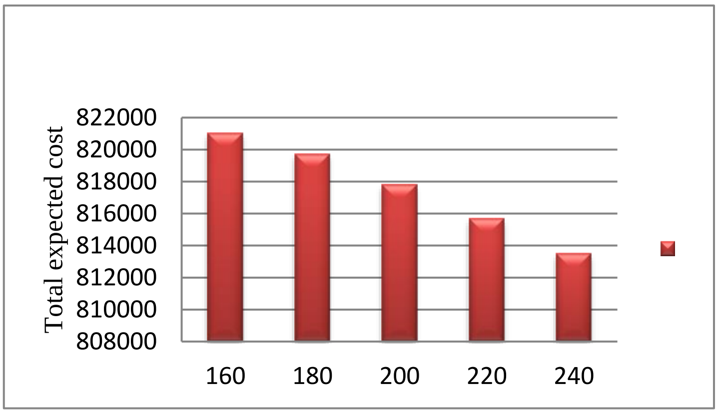

Figure 7. The behavior of expected cost for repairing rate.

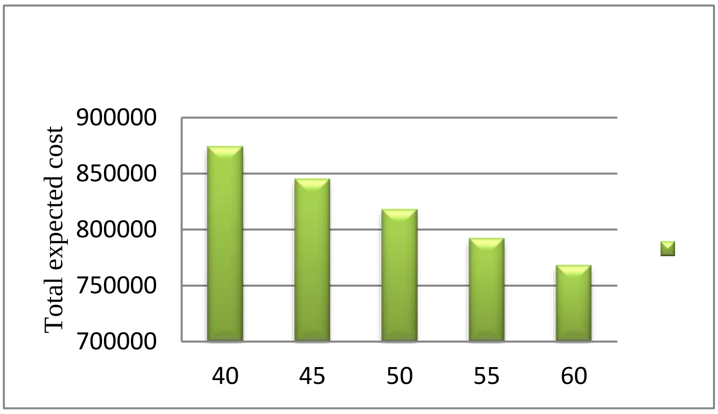

Figure 8. The behavior of expected cost for varying return rate.

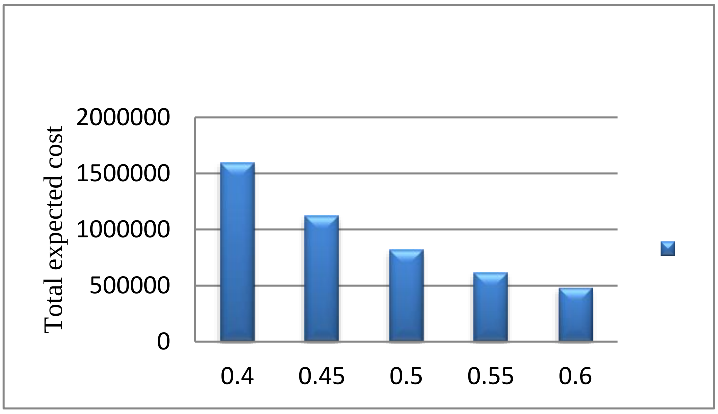

Figure 9. The behavior of expected cost w.r.t. β .

6.2. Observations

1. The analysis, as shown in Table 1 and Figures 4 to 9, highlights that the total cost exhibits a decreasing trend as the demand rate, return rate, $\beta$and repairing rate increase. This trend is reasonable, as it corresponds to a reduction in holding costs due to higher values of these parameters.
2. Conversely, the results demonstrate that the total cost increases with an increment in the production rate and remanufacturing rate. This is attributed to the fact that higher production and remanufacturing rates lead to elevated on-hand inventory levels per unit time, subsequently increasing holding costs and, consequently, the total cost.
3. The study reveals that the total cost is highly sensitive to variations in the production rate, remanufacturing rate, demand rate and β. Small changes in these parameters can significantly impact the total cost.
4. In contrast, the total cost displays only moderate sensitivity to changes in the return rate.

Table 1. Sensitivity analysis with respect to the different parameters.

Parameter		t_1	TC
P_r	300	7.37257	399069
	350	7.36863	583824
	400	7.36573	817840
	450	7.36350	1106890
	500	7.36173	1456730
P	400	7.39605	507074
	450	7.37931	653559
	500	7.36573	817840
	550	7.35451	1001650
	600	7.34985	1258420
D	80	7.32658	1102455
	90	7.34665	936542
	100	7.36573	817840
	110	7.38787	651225
	120	7.40995	523109
P_r	160	7.37722	821022
	180	7.37085	819733
	200	7.36573	817840
	220	7.36153	815702
	240	7.35801	813503
R	40	7.37233	874138
	45	7.36889	845055
	50	7.36573	817840
	55	7.36284	792320
	60	7.36019	768341
β	0.40	9.20792	1595220
	0.45	8.18451	1121040
	0.50	7.36573	817840
	0.55	6.69576	615000
	0.60	5.36548	483235

7. Managerial insights

Decision-makers should consider optimizing the demand rate, return rate and repairing rate to reduce total costs. Increasing these parameters can lead to more efficient inventory management and lower holding costs. On the other hand, managers need to carefully assess and balance production and remanufacturing rates. Incremental increases in these rates may boost production but can also lead to higher inventory holding costs. Therefore, a well-calibrated production and remanufacturing strategy is essential. When making decisions related to production, remanufacturing and its demand, the managers should be aware of the high sensitivity of total costs to changes in these parameters. This underscores the importance of precise parameter estimation and management to minimize overall costs. While the return rate has a relatively moderate impact on total costs compared to other factors, it should not be neglected. Careful management of the return rate can still contribute to cost savings and improved efficiency in inventory control.

In summary, the observations and managerial insights from this analysis provide valuable guidance for optimizing inventory management strategies in the context of varying demand, returns, production and remanufacturing rates. Effective decision-making in these areas can significantly impact the overall cost efficiency of the system.

8. Conclusions

In this paper, we have introduced a reverse logistics inventory model that addresses the management of imperfect materials in the presence of stochastic machine breakdowns. Furthermore, we have considered the deterioration of products over time, adding another layer of complexity to the inventory control problem. Our analysis focused on two distinct scenarios: one where breakdowns occur and another where they do not. For each scenario, we derived mathematical formulas representing the total cost.

However, it is important to acknowledge certain limitations in our current study. We have assumed a Poisson distribution for the random number of breakdowns per unit time, simplifying the stochastic aspect of the breakdowns. Future research could explore more complex scenarios involving multiple breakdowns during both production and the remanufacturing cycle, providing a more realistic representation of the operational challenges faced in manufacturing and supply chain contexts.

Additionally, this model does not account for volume flexibility or probabilistic demand, which is prevalent in dynamic production environments. Investigating how our model can be extended to accommodate these factors would be a valuable avenue for future research. The incorporation of volume flexibility could enhance the model's adaptability to varying production needs, while addressing probabilistic demand would further improve its applicability to real-world situations.

In conclusion, while this paper offers a foundational framework for reverse logistics inventory management in the presence of machine breakdowns and product deterioration, it is by no means exhaustive. Future research endeavors should explore the nuances of multiple breakdown occurrences, volume flexibility and probabilistic demand, providing a more comprehensive and practical understanding of inventory control in dynamic and uncertain manufacturing environments. By addressing these limitations and extending the model, researchers and practitioners can make more informed decisions to enhance the efficiency and sustainability of their supply chain operations.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

Data Availability Statement

All data used to justify the proposed model are given in the manuscript.

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