Electronic
Research Archive

https://www.aimspress.com/journal/era

ERA, 34(4): 2157–2177.
DOI: 10.3934/era.2026097
Received: 30 January 2026
Revised: 01 March 2026
Accepted: 05 March 2026
Published: 10 March 2026

Research article

Zero dissipation limit to rarefaction waves with a vacuum for one-dimensional viscous compressible two-phase flows

Yixuan Zhao[1] **and Shuzhen Zhang**[2,*]

[1] School of Mathematics and Big data, Anhui University of Science and Technology, Huainan 232001, China

[2] School of Mathematics and Statistics, Shandong University of Technology, Zibo 255000, China

* **Correspondence:** Email: szzhang@sdut.edu.cn.

Abstract: We are concerned with the zero dissipation limit to rarefaction waves with vacuum for a non-conservative viscous compressible two-fluid system in one dimension. In this paper, given a rarefaction wave with one-side vacuum state, we establish the existence of a solution sequence for the system that approaches this rarefaction wave with vacuum as viscosity vanishes and we derive a uniform convergence rate for this approximation. The result is proved by a scaling argument and an energy method.

Keywords: two-phase flow; zero dissipation limit; compressible Navier-Stokes equation; rarefaction wave; vacuum

---

1. Introduction and main result

This paper studies a two-fluid model derived via the Chapman-Enskog expansion from a Vlasov-Fokker-Planck equation coupled with the compressible Navier-Stokes equations [1]. Specifically, we focus on the following system:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + [\rho u^2 + p(\rho)]_x = \nu u_{xx} + n(v-u), \\ n_t + (nv)_x = 0, \\ (nv)_t + [nv^2 + \tilde{p}(n)]_x = (\eta(n)v_x)_x - n(v-u), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$ is the spatial variable and $t > 0$ is the time variable. The unknown functions ρ and u stand for the density and velocity of a dense phase, respectively n and v represent the macroscopic density and velocity of the disperse phase. The pressure-density functions of two phases take the form

$$p(\rho) = \frac{\rho^\gamma}{\gamma}, \ \ \tilde{p}(n) = \frac{n^\beta}{\beta},$$

with $\gamma \geq 1$ and $\beta \geq 1$ being positive constants. Here, we restrict ourselves to the case $\gamma = \beta = 1$. The viscosities of two phases ν and $\eta(n)$ are given by

$$\nu = \epsilon, \ \ \eta(n) = \epsilon n,$$

where $\epsilon > 0$ is the viscosity coefficient.

The study of vacuum states in gas dynamics presents considerable challenges, primarily due to the analytical difficulties arising from degeneracies and singularities. Within the framework of Riemann solutions, it is established that only rarefaction waves, as opposed to shock waves, can connect to vacuum states in the one-dimensional isentropic compressible Euler equations [2]. This theoretical understanding motivates the investigation of viscous approximations to such vacuum-rarefaction waves, as seen in the analysis of time-asymptotic behavior for the Navier-Stokes equations [3, 4]. Furthermore, the zero-dissipation limit from the Navier-Stokes equations to these Euler rarefaction waves with a one-side vacuum has been rigorously justified through the construction of convergent solution sequences [5]. Liang [6] considered the density-dependent viscosity case (i.e., $\mu(\rho) = \epsilon\rho^\alpha$) under the conditions on the coefficient α and the gas constant

$$1 < \gamma \leq 2, \quad \alpha \leq \gamma, \quad \frac{3}{5} \leq \alpha \leq \frac{3}{2},$$

and later Li and Wang [7] generalized the result to the non-isentropic case. Wang [8] generalized the result for all $\gamma > 1$ and $\alpha > 0$.

There has been extensive research on two-phase flows [9, 10], etc. Several results have been established for the two-fluid system (1.1). Li et al. [11, 12] investigated the inflow problem and the outflow problem for the full two-phase model for the system (1.1) in a half line. Wang and Zhao [13, 14] succeeded in the applying time-asymptotic stability of planar rarefaction wave to a two-fluid system in two dimensions.

Our work departs from the aforementioned studies by targeting the zero-dissipation limit for a rarefaction wave with vacuum in the one-dimensional compressible two-fluid system (1.1). The core analytical difficulty stems from the presence of vacuum, which induces degeneracies and singularities. This challenge is compounded by the system's coupled dynamics: two compressible, viscous fluids (one with constant viscosity and the other with density-dependent viscosity) interact via drag and Vlasov forces within the same phase space. Unlike the single-fluid case, obtaining uniform estimates near vacuum is significantly harder due to these inter-fluid interactions. Nonetheless, by leveraging a scaling argument combined with an elementary energy analysis that capitalizes on the intrinsic relaxation damping of the rarefaction wave's structure, we are able to overcome these obstacles.

We now proceed to present the precise formulation of the problem. For the two-phase flow system (1.1), the corresponding inviscid Euler equations read as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + \rho)_x = 0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} n_t + (nv)_x = 0, \\ (nv)_t + (nv^2 + n)_x = 0. \end{cases} \tag{1.3}$$

For ρ, $n > 0$, systems (1.2) and (1.3) exhibit strict hyperbolicity, with all characteristic fields being genuinely nonlinear. Note that the eigenvalues of the Jacobi matrix of the Euler systems (1.2) and (1.3) for (ρ, u) and (n, v) are

$$\lambda_i(u) = u + (-1)^i, \quad \lambda_i(v) = v + (-1)^i, \quad i = 1, 2.$$

Corresponding right eigenvectors are

$$\vec{r}_i(\rho) = ((-1)^i\rho,\ 1),\ \ \vec{r}_i(n) = ((-1)^i n,\ 1),\ \ i = 1, 2,$$

such that for $i = 1, 2$, we have

$$\vec{r}_i(\rho) \cdot \nabla_{(\rho,u)}\lambda_i(u) \neq 0,\ \vec{r}_i(n) \cdot \nabla_{(n,v)}\lambda_i(v) \neq 0.$$

The i-Riemann invariants ($i = 1, 2$) can be defined by

$$Z_i(\rho, u) = u + (-1)^{i+1} \int^{\rho} \frac{1}{\xi} d\xi,\ \ Z_i(n, v) = v + (-1)^{i+1} \int^{n} \frac{1}{\xi} d\xi,\ \ i = 1, 2,$$

such that

$$\nabla_{(\rho,u)}Z_i(\rho, u) \cdot \vec{r}_i(\rho) \equiv 0,\ \ \nabla_{(n,v)}Z_i(n, v) \cdot \vec{r}_i(n) \equiv 0,\ \ i = 1, 2.$$

The primary objective of this paper is to construct, for a given rarefaction wave with a one-side vacuum state solving the isentropic Euler equations (1.2) and (1.3), a sequence of solutions to system (1.1) that converges uniformly to this wave as the viscosity vanishes, and to establish the corresponding convergence rate. Several key aspects of our analysis merit further commentary. First, analogous to the approach in [5], we construct the 2-rarefaction waves that connect to the vacuum on one side. This construction proceeds the from given Riemann initial data

$$\begin{cases} \rho(0, x) = 0, & x < 0, \\ (\rho, u)(0, x) = (\rho_+, u_+), & x > 0, \end{cases} \tag{1.4}$$

and

$$\begin{cases} n(0, x) = 0, & x < 0, \\ (n, v)(0, x) = (n_+, u_+), & x > 0. \end{cases} \tag{1.5}$$

Here, the left-hand state is a vacuum, while the right-hand state is defined by the prescribed constants $\rho_+, n_+ > 0$ and u_+. Consequently, the Riemann problem (1.2) and (1.4) yields a 2-rarefaction wave connecting to the vacuum on the left. The velocities $u_- = Z_2(\rho_+, u_+)$ and $v_- = Z_2(n_+, u_+)$, which represent the speeds at which the two fluids enter the vacuum from the rarefaction waves, are directly determined by the definitions of the 2-Riemann invariants $Z_2(\rho, u)$ and $Z_2(n, v)$. We now get a 2-rarefaction wave $(\rho^{r_2}, u^{r_2})(\frac{x}{t})$ of (1.2) connecting the vacuum state $\rho = 0$ to (ρ_+, u_+), which can be expressed by

$$\lambda_2(\rho^{r_2}(\frac{x}{t}), u^{r_2}(\frac{x}{t})) = \begin{cases} \rho^{r_2}(\frac{x}{t}), & \text{if } \frac{x}{t} < \lambda_2(0, u_-), \\ \frac{x}{t}, & \text{if } u_- \leq \frac{x}{t} \leq \lambda_2(\rho_+, u_+), \\ \lambda_2(\rho_+, u_+), & \text{if } \frac{x}{t} > \lambda_2(\rho_+, u_+), \end{cases} \tag{1.6}$$

and

$$Z_2(\rho^{r_2}(\frac{x}{t}), u^{r_2}(\frac{x}{t})) = Z_2(0, u_-) = Z_2(\rho_+, u_+). \tag{1.7}$$

The momentum of the 2-rarefaction wave m^{r_2} can be defined by

$$m^{r_2}(\frac{x}{t}) = \begin{cases} \rho^{r_2}(\frac{x}{t})u^{r_2}(\frac{x}{t}), & \text{if } \rho^{r_2} > 0, \\ 0, & \text{if } \rho^{r_2} = 0. \end{cases} \tag{1.8}$$

Similarly, the Riemann problems (1.3) and (1.5) possesses a 2-rarefaction wave connected to the left-hand-side vacuum. From this, we obtain the inflow velocity $v_- = Z_2(n_+, u_+)$ for the fluids entering the vacuum from the rarefaction wave. Thus, we can also get a 2-rarefaction wave $(n^{r_2}, v^{r_2})(\frac{x}{t})$ of (1.3) connecting the vacuum state $n = 0$ to (n_+, u_+), and the momentum of 2-rarefaction wave $\tilde{m}^{r_2}$ can be defined by

$$\tilde{m}^{r_2}(\frac{x}{t}) = \begin{cases} n^{r_2}(\frac{x}{t})v^{r_2}(\frac{x}{t}), & \text{if } n^{r_2} > 0, \\ 0, & \text{if } n^{r_2} = 0. \end{cases} \tag{1.9}$$

In the interest of clarity and brevity, the notation $\|\cdot\|_{L^2}$ is replaced by $\|\cdot\|$ throughout the following discussion. The constant C in this paper can represent several positive generic constants, which is dependent on time t and ϵ without special remarks. We now state our main result as follows:

Theorem 1.1. *Let* $(\rho^{r_2}, m^{r_2})(\frac{x}{t})$ *and* $(n^{r_2}, \tilde{m}^{r_2})(\frac{x}{t})$ *be the 2-rarefaction waves with one-side vacuum state defined by* (1.2)–(1.5). *Then a small positive constant* ϵ_0 *such that if* $\epsilon \in (0, \epsilon_0)$, *the compressible two-phase flow system* (1.1) *admits a family of smooth solutions* $(\rho^\epsilon, m^\epsilon = \rho^\epsilon u^\epsilon)(x,t)$ *and* $(n^\epsilon, \tilde{m}^\epsilon = n^\epsilon v^\epsilon)(x,t)$ *satisfying*

$$\begin{aligned} &(\rho^\epsilon - \rho^{r_2}, m^\epsilon - m^{r_2}, n^\epsilon - n^{r_2}, \tilde{m}^\epsilon - \tilde{m}^{r_2}) \in C^0((0,+\infty); L^2(\mathbb{R})), \\ &(\rho^\epsilon, m^\epsilon, n^\epsilon, \tilde{m}^\epsilon)_x \in C^0((0,+\infty); L^2(\mathbb{R})), \ (u^\epsilon_{xx}, v^\epsilon_{xx}) \in L^2(0,+\infty; L^2(\mathbb{R})). \end{aligned}$$

Furthermore, when $\epsilon \to 0$, *for any given positive constant* $h > 0$, *a constant* $C_h > 0$, *independent of* ϵ, *such that*

$$\begin{aligned} &\sup_{t\geq h} \|\rho^\epsilon(\cdot,t) - \rho^{r_2}(\frac{\cdot}{t})\|_{L^\infty} \leq C_h \epsilon^a |\ln \epsilon|, \quad \sup_{t\geq h} \|m^\epsilon(\cdot,t) - m^{r_2}(\frac{\cdot}{t})\|_{L^\infty} \leq C_h \epsilon^a |\ln \epsilon|, \\ &\sup_{t\geq h} \|n^\epsilon(\cdot,t) - n^{r_2}(\frac{\cdot}{t})\|_{L^\infty} \leq C_h \epsilon^a |\ln \epsilon|, \quad \sup_{t\geq h} \|\tilde{m}^\epsilon(\cdot,t) - \tilde{m}^{r_2}(\frac{\cdot}{t})\|_{L^\infty} \leq C_h \epsilon^a |\ln \epsilon|. \end{aligned} \tag{1.10}$$

with the positive constant a given by $a = \frac{1}{63}$.

The structure of this paper is outlined below. Section 2 is devoted to the construction of smooth 2-rarefaction waves via the inviscid Burgers equation, which serve as approximations the cut-off rarefaction waves in the Euler system. The proof of Theorem 1.1 is presented in Section 3.

2. Rarefaction wave

In this section, for the sake of completeness, we restate the approximation of the cut-off rarefaction wave and summarize key estimates for this approximate cut-off rarefaction wave. Following a similar methodology to that in [5] for the Navier-Stokes equations, we construct the approximate rarefaction wave.

Given any constant $\mu > 0$, there a unique state $(\rho, u) = \left(\mu, u_\mu\right)$ lying on the 2-rarefaction wave curve. The explicit expression for u_μ, namely $u_\mu = u_+ - \ln \frac{\rho_+}{\mu}$, is derived from the fundamental property that the Riemann invariant $Z_2(\rho, u)$ remains constant along the rarefaction wave.

Thus, we obtain a cut-off 2-rarefaction wave $(\rho_\mu^{r_2}, u_\mu^{r_2})\left(\frac{x}{t}\right)$ that connects the state (μ, u_μ) to (ρ_+, u_+) and satisfies Eqs (1.2) and (1.4), which is given by

$$\lambda_2(u_\mu^{r_2})(\frac{x}{t}) = \begin{cases} \lambda_2\left(u_\mu\right), & \frac{x}{t} < \lambda_2(u_\mu), \\ \frac{x}{t}, & \lambda_2(u_\mu) \le \frac{x}{t} \le \lambda_2(u_+), \\ \lambda_2(u_+), & \frac{x}{t} > \lambda_2(u_+). \end{cases}$$

Following this pattern, it is possible to obtain a cut-off 2-rarefaction wave $(n_\mu^{r_2}, u_\mu^{r_2})(\frac{x}{t})$ to (1.3) and (1.5) that connects the state $(\frac{n_+}{\rho_+}\mu, u_\mu)$ to the state (n_+, u_+).

Correspondingly, we can define the momentum functions $m_\mu^{r_2} = \rho_\mu^{r_2} u_\mu^{r_2}$ and $\tilde{m}_\mu^{r_2} = n_\mu^{r_2} u_\mu^{r_2}$. It can be shown that the cut-off 2-rarefaction waves $(\rho_\mu^{r_2}, m_\mu^{r_2})(\frac{x}{t})$ and $(n_\mu^{r_2}, \tilde{m}_\mu^{r_2})(\frac{x}{t})$ converge to the original 2-rarefaction waves $(\rho^{r_2}, m^{r_2})(\frac{x}{t})$ and $(n^{r_2}, \tilde{m}^{r_2})(\frac{x}{t})$ in the sup-norm the convergence rate μ tends to zero. Stated more precisely, the following lemma is a direct consequence of the explicit solution formula for a rarefaction wave.

Lemma 2.1. *([5]) Let $\mu_0 \in (0, 1)$ be the constant defined such that for all $\mu \in (0, \mu_0]$, $t > 0$, it holds*

$$\|(\rho_\mu^{r_2}, m_\mu^{r_2}, n_\mu^{r_2}, \tilde{m}_\mu^{r_2})(\frac{\cdot}{t}) - (\rho^{r_2}, m^{r_2}, n^{r_2}, \tilde{m}^{r_2})(\frac{\cdot}{t})\|_{L^\infty} \le C\mu.$$

In this paper, it is defined that $w_\delta^r(x, t)$ is the solution to the Burgers equation with the Cauchy initial value

$$\begin{cases} w_t + w w_x = 0, \\ w(x, 0) = w_0(x) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} tanh \frac{x}{\delta}, \end{cases} \tag{2.1}$$

where $\delta > 0$ is a small parameter to be determined. In fact, we choose $\delta = \epsilon^a$ with a given constant a. Subsequently, the approximate rarefaction waves $(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t)$ and $(\bar{n}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t)$ of the cut-off 2-rarefaction waves $(\rho_\mu^{r_2}, u_\mu^{r_2})(\frac{x}{t})$ and $(n_\mu^{r_2}, u_\mu^{r_2})(\frac{x}{t})$ to the two-phase flow system (1.1) can be determined via the following computation:

$$\begin{cases} w_+ = \lambda_2(u_+), \ w_- = \lambda_2(u_\mu), \\ w_\delta^r(x, t) = \lambda_2(\bar{u}_{\mu,\delta})(x, t), \\ Z_2(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t) = Z_2(\rho_+, u_+) = Z_2(\mu, u_\mu), \\ Z_2(\bar{n}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t) = Z_2(n_+, u_+) = Z_2(\frac{n_+}{\rho_+}\mu, u_\mu), \end{cases} \tag{2.2}$$

where

$$w_\delta^r(x, t) = w_\delta(x_0(x, t)), \quad x = x_0(x, t) + w_\delta(x_0(x, t))t.$$

The properties of $w_\delta^r(x, t)$ reported in [15, 16], the properties of $(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t)$ and $(\bar{n}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t)$ are derived. For notational simplicity, $(\bar{\rho}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t)$ and $(\bar{n}_{\mu,\delta}, \bar{u}_{\mu,\delta})(x, t)$ abbreviated as $(\bar{\rho}, \bar{u})(x, t)$ and $(\bar{n}, \bar{u})(x, t)$. In summary, $(\bar{\rho}, \bar{u})(x, t)$ and $(\bar{n}, \bar{u})(x, t)$, as defined above, satisfy

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0, \\ (\bar{\rho}\bar{u})_t + (\bar{\rho}\bar{u}^2 + \bar{\rho})_x = 0, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \bar{n}_t + (\bar{n}\bar{u})_x = 0, \\ (\bar{n}\bar{u})_t + (\bar{n}\bar{u}^2 + \bar{n})_x = 0. \end{cases} \tag{2.4}$$

Lemma 2.2. *([5]) The following properties hold for the approximate cut-off 2-rarefaction waves* $(\bar{\rho}, \bar{u})$ *and* $(\bar{n}, \bar{u})$ *given in* (2.2).

(1) $\bar{u}_x(x,t) = (w^r_\delta)_x > 0$, *for* $x \in \mathbb{R},\ t \geq 0$.

$\bar{\rho}_x = \bar{\rho}\bar{u}_x$, $\bar{n}_x = \bar{n}\bar{u}_x$, $\bar{\rho}_{xx} = \bar{\rho}\bar{u}_{xx} + \bar{\rho}(\bar{u}_x)^2$ *and* $\bar{n}_{xx} = \bar{n}\bar{u}_{xx} + \bar{n}(\bar{u}_x)^2$.

(2) It holds that for all $t > 0, \delta > 0$, *and* $p \in [1, \infty]$ *:*

$$\|\bar{u}_x(\cdot, t)\|_{L^p} \leq C(w_+ - w_-)^{1/p}(\delta + t)^{-1+1/p},$$
$$\|\bar{u}_{xx}(\cdot, t)\|_{L^p} \leq C(\delta + t)^{-1}\delta^{-1+1/p}.$$

(3) Let $\delta \in (0, \delta_0]$ *and* $t > 0$, *where* $\delta_0 \in (0, 1)$ *is a fixed constant. Let* $(\rho^{r_2}_\mu, u^{r_2}_\mu)$ *and* $(n^{r_2}_\mu, u^{r_2}_\mu)$ *be the cut-off* 2*-rarefaction waves.*

$$\|(\bar{\rho} - \rho^{r_2}_\mu, \bar{n} - n^{r_2}_\mu, \bar{u} - u^{r_2}_\mu)(\cdot, t)\|_{L^\infty} \leq C\delta t^{-1}[\ln(1+t) + |\ln\delta|].$$

3. Proof of Theorem 1.1

We now proceed to prove Theorem 1.1, and the solutions $(\rho^\epsilon, u^\epsilon)(x,t)$ and $(n^\epsilon, v^\epsilon)(x,t)$ are constructed as the perturbations around the approximate rarefaction waves $(\bar{\rho}, \bar{u})(x,t)$ and $(\bar{n}, \bar{u})(x,t)$. Consider the Cauchy problem for (1.1) with smooth initial data

$$(\rho^\epsilon, u^\epsilon)(x, t=0) = (\bar{\rho}, \bar{u})(x, 0),\ (n^\epsilon, v^\epsilon)(x, t=0) = (\bar{n}, \bar{u})(x, 0). \tag{3.1}$$

The perturbations are defined as

$$(\phi, \psi)(y, \tau) = (\rho^\epsilon, u^\epsilon)(x,t) - (\bar{\rho}, \bar{u})(x,t),\ (\tilde{\phi}, \tilde{\psi})(y, \tau) = (n^\epsilon, v^\epsilon)(x,t) - (\bar{n}, \bar{u})(x,t). \tag{3.2}$$

Here, the scaled variables y and ϵ are denoted by

$$y = \frac{x}{\epsilon}, \quad \tau = \frac{t}{\epsilon}, \tag{3.3}$$

The quadruple $(\rho^\epsilon, u^\epsilon, n^\epsilon, v^\epsilon)$ denotes the solution to (1.1). For simplicity, we omit the superscript ϵ and write (ρ, u, n, v).

By (3.2) and (3.3), we have $\rho = \bar{\rho} + \phi,\ u = \bar{u} + \psi,\ n = \bar{n} + \tilde{\phi},\ v = \bar{v} + \tilde{\psi}$, and $\partial_t = \epsilon^{-1}\partial_\tau,\ \partial_x = \epsilon^{-1}\partial_y$. Substituting these relations into (1.1), using (2.3) and (2.4) to cancel the background wave terms, and collecting the remaining terms, we obtain

$$\begin{cases} \phi_\tau + \rho\psi_y + u\phi_y = -f, \\ \rho\psi_\tau + \rho u\psi_y + \phi_y - \psi_{yy} = -g + n(\tilde{\psi} - \psi), \\ \tilde{\phi}_\tau + n\tilde{\psi}_y + v\tilde{\phi}_y = -\tilde{f}, \\ n\tilde{\psi}_\tau + nv\tilde{\psi}_y + \tilde{\phi}_y - \bar{n}\tilde{\psi}_{yy} = -\tilde{g} + \bar{n}_y v_y + (\tilde{\phi}v_y)_y - n(\tilde{\psi} - \psi), \end{cases} \tag{3.4}$$

where

$$\begin{cases} f = \bar{u}_y\phi + \bar{\rho}_y\psi, \\ g = \rho\psi\bar{u}_y - \bar{\rho}_y\frac{\phi}{\bar{\rho}} - \bar{u}_{yy}, \\ \tilde{f} = \bar{u}_y\tilde{\phi} + \bar{n}_y\tilde{\psi}, \\ \tilde{g} = n\tilde{\psi}\bar{u}_y - \bar{n}_y\frac{\tilde{\phi}}{\bar{n}} - \bar{n}\bar{u}_{yy}. \end{cases} \tag{3.5}$$

Initial data

$$(\phi, \psi, \tilde{\phi}, \tilde{\psi})(y, 0) = (0, 0, 0, 0). \tag{3.6}$$

Additionally, the solution space for the problem (3.4)–(3.6) is defined as follows:

$$\begin{aligned} X(0, \tau_1(\epsilon)) = \{(\phi, \psi, \tilde{\phi}, \tilde{\psi}) | (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \in C^0([0, \tau_1]; H^1(\mathbb{R})), (\phi_y, \tilde{\phi}_y) \in L^2(0, \tau_1; L^2(\mathbb{R})), \\ (\psi_y, \tilde{\psi}_y) \in L^2(0, \tau_1; H^1(\mathbb{R}))\}, \end{aligned} \tag{3.7}$$

where $0 \le \tau_1(\epsilon) \le +\infty$.

At this point, the result concerning the perturbation solution $(\phi, \psi, \tilde{\phi}, \tilde{\psi})$ is formulated in the following theorem:

Theorem 3.1. *Let* $\epsilon_1 > 0$ *be a constant independent of* ϵ*. Then another constant* $C > 0$ *exists, also independent of* ϵ*, such that for any* $0 < \epsilon < \epsilon_1$*, the problem in* (3.4)–(3.6) *admits a unique global-in-time solution* $(\phi, \psi, \tilde{\phi}, \tilde{\psi}) \in X(0, +\infty)$ *satisfying*

$$\begin{aligned} & \sup_{\tau\in[0,+\infty]} \int_{\mathbb{R}} (\phi^2 + \phi_y^2 + \bar{\rho}\psi^2 + \psi_y^2 + \tilde{\phi}^2 + \tilde{\phi}_y^2 + \bar{n}\tilde{\psi}^2 + \tilde{\psi}_y^2)dy \\ & + \int_0^{+\infty} \int_{\mathbb{R}} (\phi_y^2 + \bar{\rho}\bar{u}_y\psi^2 + \bar{u}_y\psi_y^2 + \psi_y^2 + \psi_{yy}^2 + \tilde{\phi}_y^2 + \bar{n}\bar{u}_y\tilde{\psi}^2 + \bar{u}_y\tilde{\psi}_y^2 + \bar{n}\tilde{\psi}_y^2 + \tilde{\psi}_{yy}^2)dyds \\ & + \int_0^{+\infty} \int_{\mathbb{R}} [\bar{n}(\tilde{\psi} - \psi)^2 + (\tilde{\psi}_y - \psi_y)^2]dyds \le C\epsilon^{\frac{1}{2}-\frac{21}{2}a}|\ln\epsilon|^{-9}. \end{aligned} \tag{3.8}$$

with a is being given in Theorem 1.1.

Here, we proceed with our analysis guided by the following a priori assumptions:

$$\sup_{\tau\in[0,\tau_1]} \|(\phi, \psi, \tilde{\phi}, \tilde{\psi})\|_{L^\infty} \le \epsilon^a, \quad \sup_{\tau\in[0,\tau_1]} \|(\phi_y, \psi_y, \tilde{\phi}_y, \tilde{\psi}_y)\| \le 1 \tag{3.9}$$

with a given in Theorem 1.1. We choose

$$\mu = \epsilon^a|\ln\epsilon|, \quad \delta = \epsilon^a. \tag{3.10}$$

If $\epsilon \ll 1$, then one has

$$\begin{aligned}
&\rho = \bar{\rho} + \phi \geq \bar{\rho} - \|\phi\|_{L^\infty} \geq \bar{\rho} - \epsilon^a \geq \bar{\rho} - \frac{1}{2}\mu \geq \frac{\bar{\rho}}{2},\\
&\rho = \bar{\rho} + \phi \leq \bar{\rho} + \|\phi\|_{L^\infty} \leq \bar{\rho} + \epsilon^a \leq \bar{\rho} + \frac{1}{2}\mu \leq \frac{3\bar{\rho}}{2},\\
&n = \bar{n} + \phi \geq \bar{n} - \|\phi\|_{L^\infty} \geq \bar{n} - \epsilon^a \geq \bar{n} - \frac{n_+}{2\rho_+}\mu \geq \frac{\bar{n}}{2},\\
&n = \bar{n} + \phi \leq \bar{n} + \|\phi\|_{L^\infty} \leq \bar{n} + \epsilon^a \leq \bar{n} + \frac{n_+}{2\rho_+}\mu \leq \frac{3\bar{n}}{2}.
\end{aligned} \tag{3.11}$$

where $\mu \geq \max\{2, \frac{2\rho_+}{n_+}\}\epsilon^a$ if $\epsilon \ll 1$. By integrating (3.9) and (3.11), one can derive

$$\frac{\bar{\rho}}{2} < \rho < \frac{3\bar{\rho}}{2}, \ \frac{\bar{n}}{2} < n < \frac{3\bar{n}}{2}. \tag{3.12}$$

The standard proof for the local existence of solutions to (3.4)–(3.6) is omitted here; see [17] for details. To establish the convergence rate of the local solution with respect to ϵ as given in (3.9), we denote the local existence time interval as $[0, \tau_0] = [0, \tau_0(\epsilon)]$. The subsequent proof strategy for Theorem 3.1 involves extending this local solution to a global one on $(0, +\infty)$, for a small but fixed viscosity coefficient ϵ. This extension relies on establishing the following a priori estimates for a fixed ϵ satisfying $0 < \epsilon \ll 1$.

Theorem 3.2. *(a priori estimate) Take a solution* $(\phi, \psi, \tilde{\phi}, \tilde{\psi}) \in X(0, \tau_1(\epsilon))$ *to* (3.4)–(3.6)*, with* $\tau_1(\epsilon)$ *being its maximum existence time under the a priori assumptions* (3.9)*. One can find a constant* $\epsilon_0 > 0$ *such that whenever* $0 \leq \epsilon \leq \epsilon_0$*, the following holds:*

$$\begin{aligned}
&\sup_{\tau\in[0,\tau_1(\epsilon)]} \int_{\mathbb{R}} (\phi^2 + \phi_y^2 + \bar{\rho}\psi^2 + \psi_y^2 + \tilde{\phi}^2 + \tilde{\phi}_y^2 + \bar{n}\tilde{\psi}^2 + \tilde{\psi}_y^2) dy\\
&+ \int_0^{\tau_1(\epsilon)} \int_{\mathbb{R}} (\phi_y^2 + \bar{\rho}\bar{u}_y\psi^2 + \bar{u}_y\psi_y^2 + \psi_y^2 + \psi_{yy}^2 + \tilde{\phi}_y^2 + \bar{n}\bar{u}_y\tilde{\psi}^2 + \bar{u}_y\tilde{\psi}_y^2 + \bar{n}\tilde{\psi}_y^2 + \tilde{\psi}_{yy}^2) dyds\\
&+ \int_0^{\tau_1(\epsilon)} \int_{\mathbb{R}} [\bar{n}(\tilde{\psi} - \psi)^2 + (\tilde{\psi}_y - \psi_y)^2] dyds \leq C\frac{\epsilon^{\frac{1}{2}}}{\mu^{10}\delta^{\frac{1}{2}}} |\ln \epsilon| = C\epsilon^{\frac{1}{2} - \frac{21}{2}a} |\ln \epsilon|^{-9} \leq C\epsilon^{\frac{1}{3}} |\ln \epsilon|^{-9},
\end{aligned} \tag{3.13}$$

with a as given in Theorem 1.1, and the constant C is independent of ϵ *and* $\tau_1(\epsilon)$*.*

Next, Lemmas 3.1–3.3 are shown to give the proof of Theorem 3.2.

Lemma 3.1. *Positive constants C and* ϵ_0 *exist such that for* $0 \leq \tau \leq \tau_1(\epsilon)$ *and* $\epsilon < \epsilon_0$,

$$\begin{aligned}
&\sup_{\tau\in[0,\tau_1(\epsilon)]} \int_{\mathbb{R}} (\phi^2 + \bar{\rho}\psi^2 + \tilde{\phi}^2 + \bar{n}\tilde{\psi}^2) dy + \int_0^{\tau_1(\epsilon)} \int_{\mathbb{R}} [\bar{\rho}\bar{u}_y\psi^2 + \psi_y^2 + \bar{n}\bar{u}_y\tilde{\psi}^2 + \bar{n}\tilde{\psi}_y^2] dyds\\
&+ \int_0^{\tau_1(\epsilon)} \int_{\mathbb{R}} \bar{n}(\tilde{\psi} - \psi)^2 dyds \leq C\frac{\epsilon^{\frac{1}{2}}}{\mu\delta^{\frac{1}{2}}}.
\end{aligned} \tag{3.14}$$

Proof. To begin with, define

$$E := \Phi(\rho, \bar{\rho}) + \frac{\psi^2}{2},$$

where

$$\Phi(\rho, \bar{\rho}) := \int_{\bar{\rho}}^{\rho} \frac{\xi - \bar{\rho}}{\xi^2} d\xi = \frac{\bar{\rho}}{\rho} - \ln \frac{\bar{\rho}}{\rho} - 1. \tag{3.15}$$

Direct computations yield

$$(\rho E)_\tau + \psi^2 \rho \bar{u}_y + \psi_y^2 - n(\tilde{\psi} - \psi)\psi + (\rho u E - \psi\psi_y + \phi\psi)_y = \bar{u}_{yy}\psi. \tag{3.16}$$

Similarly, define

$$\tilde{E} := \tilde{\Phi}(n, \bar{n}) + \frac{\tilde{\psi}^2}{2},$$

where

$$\tilde{\Phi}(n, \bar{n}) := \int_{\bar{n}}^{n} \frac{\xi - \bar{n}}{\xi^2} d\xi = \frac{\bar{n}}{n} - \ln \frac{\bar{n}}{n} - 1. \tag{3.17}$$

Direct computations yield

$$(\rho\tilde{E})_\tau + \tilde{\psi}^2 n \bar{u}_y + n\tilde{\psi}_y^2 + n(\tilde{\psi} - \psi)\tilde{\psi} + (nv\tilde{E} - n\tilde{\psi}\tilde{\psi}_y + \tilde{\phi}\tilde{\psi} - \tilde{\phi}\bar{u}_y\tilde{\psi})_y = \bar{n}\bar{u}_{yy}\tilde{\psi} + \bar{n}_y\tilde{\psi}\bar{u}_y - \tilde{\phi}\tilde{\psi}_y\bar{u}_y. \tag{3.18}$$

Adding (3.16) and (3.18) together and integrating the resulted equation over $\mathbb{R} \times [0, \tau]$ implies

$$\begin{aligned} &\int_{\mathbb{R}} (\phi^2 + \bar{\rho}\psi^2 + \tilde{\phi}^2 + \bar{n}\tilde{\psi}^2) dy + \int_0^\tau \int_{\mathbb{R}} [\bar{\rho}\bar{u}_y\psi^2 + \psi_y^2 + \bar{n}\bar{u}_y\tilde{\psi}^2 + \bar{n}\tilde{\psi}_y^2 + \bar{n}(\tilde{\psi} - \psi)^2] dyds \\ \le& C \int_0^\tau \int_{\mathbb{R}} (|\bar{n}\bar{u}_{yy}\tilde{\psi}| + |\bar{u}_{yy}\psi| + |\bar{n}_y\bar{u}_y\tilde{\psi}| + |\tilde{\phi}\tilde{\psi}_y\bar{u}_y|) dyds := \sum_{i=1}^{4} I_i. \end{aligned} \tag{3.19}$$

We now estimate the terms on the right hand side of (3.19) one by one. By Hölder's inequality, Sobolev's inequality, Young's inequality and Lemma 2.2, the following holds:

$$\begin{aligned} I_1 \le& C \int_0^\tau \|\bar{u}_{yy}\|_{L^1} \|\tilde{\psi}\|^{\frac{1}{2}} \|\tilde{\psi}_y\|^{\frac{1}{2}} ds \le C\mu^{\frac{1}{2}} \int_0^\tau \frac{1}{s + \frac{\delta}{\epsilon}} \|\bar{n}\tilde{\psi}\|^{\frac{1}{2}} \|\bar{n}\tilde{\psi}_y\|^{\frac{1}{2}} ds \\ \le& \frac{1}{80} \int_0^\tau \int_{\mathbb{R}} \bar{n}\tilde{\psi}_y^2 dyds + C\mu^{-\frac{2}{3}} \int_0^\tau \frac{1}{(s + \frac{\delta}{\epsilon})^{\frac{4}{3}}} \|\bar{n}^{\frac{1}{2}}\tilde{\psi}\|^{\frac{2}{3}} ds \\ \le& \frac{1}{80} \int_0^\tau \int_{\mathbb{R}} \bar{n}\tilde{\psi}_y^2 dyds + C\mu^{-\frac{2}{3}} \sup_{\tau \in [0, \tau_1]} \|\bar{n}^{\frac{1}{2}}\tilde{\psi}\|^{\frac{2}{3}} \int_0^\tau \frac{1}{(s + \frac{\delta}{\epsilon})^{\frac{4}{3}}} ds \\ \le& \frac{1}{80} \int_0^\tau \int_{\mathbb{R}} \bar{n}\tilde{\psi}_y^2 dyds + \frac{1}{80} \sup_{\tau \in [0, \tau_1]} \int_{\mathbb{R}} \bar{n}\tilde{\psi}^2 dy + C\frac{\epsilon^{\frac{1}{2}}}{\mu\delta^{\frac{1}{2}}}. \end{aligned} \tag{3.20}$$

Similarly, the following holds:

$$I_2 \le \frac{1}{80} \int_0^\tau \int_{\mathbb{R}} \psi_y^2 dyds + \frac{1}{80} \sup_{\tau \in [0, \tau_1]} \int_{\mathbb{R}} \bar{\rho}\psi^2 dy + C\frac{\epsilon^{\frac{1}{2}}}{\mu^{\frac{1}{2}}\delta^{\frac{1}{2}}}. \tag{3.21}$$

From Lemma 2.2, one can get

$$\begin{aligned} I_3 &\leq C\int_0^\tau\int_{\mathbb{R}}|\bar{n}_y\tilde{\psi}\bar{u}_y|dyds \leq C\int_0^\tau\int_{\mathbb{R}}|\bar{n}^{\frac{1}{2}}\bar{u}_y^{\frac{1}{2}}\tilde{\psi}\bar{u}_y^{\frac{3}{2}}|dyds \\ &\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\bar{u}_y\tilde{\psi}^2dyds + C\int_0^\tau\int_{\mathbb{R}}\bar{u}_y^3dyds \\ &\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\bar{u}_y\tilde{\psi}^2dyds + C\frac{\epsilon}{\delta}. \end{aligned} \tag{3.22}$$

Similarly, the following holds:

$$\begin{aligned} I_4 &\leq C\int_0^\tau\int_{\mathbb{R}}|\bar{n}^{\frac{1}{2}}\tilde{\psi}_y\tilde{\phi}\bar{u}_y\bar{n}^{-\frac{1}{2}}|dyds \\ &\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\tilde{\psi}_y^2dyds + C\mu^{-1}\int_0^\tau\int_{\mathbb{R}}\bar{u}_y^2\tilde{\phi}^2dyds \\ &\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\tilde{\psi}_y^2dyds + C\epsilon^2\mu^{-1}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy\int_0^\tau\|\bar{u}_x\|_{L^\infty}^2ds \\ &\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\tilde{\psi}_y^2dyds + C\frac{\epsilon}{\mu\delta}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy \\ &\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\tilde{\psi}_y^2dyds + \frac{1}{80}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy. \end{aligned} \tag{3.23}$$

Here, we use the fact that

$$C\frac{\epsilon}{\mu\delta} \leq C\epsilon^{1-2a}|\ln\epsilon|^{-1} \leq \frac{1}{80},\ if\ \epsilon \ll 1,$$

which will be omitted for a similar reason in the following part. Combining (3.19)–(3.23) yields

$$\begin{aligned} &\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbb{R}}(\phi^2+\bar{\rho}\psi^2+\tilde{\phi}^2+\bar{n}\tilde{\psi}^2)dy + \int_0^{\tau_1(\epsilon)}\int_{\mathbb{R}}[\bar{\rho}\bar{u}_y\psi^2+\psi_y^2+\bar{n}\bar{u}_y\tilde{\psi}^2+\bar{n}\tilde{\psi}_y^2]dyds \\ &+\int_0^{\tau_1(\epsilon)}\int_{\mathbb{R}}\bar{n}(\tilde{\psi}-\psi)^2dyds \leq C\frac{\epsilon^{\frac{1}{2}}}{\mu\delta^{\frac{1}{2}}}. \end{aligned} \tag{3.24}$$

With this, the proof of Lemma 3.1 is finished. □

Lemma 3.2. *There exists a positive constants C and ϵ_0 such that for* $0 \leq \tau \leq \tau_1(\epsilon)$ *and* $\epsilon < \epsilon_0$,

$$\begin{aligned} &\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbb{R}}(\tilde{\phi}_y^2+\phi_y^2)dy + \int_0^{\tau_1(\epsilon)}\int_{\mathbb{R}}(\phi_y^2+\tilde{\phi}_y^2)dyds \\ \leq &C\frac{\epsilon^{\frac{1}{2}}}{\mu^8\delta^{\frac{1}{2}}}|\ln\epsilon| + C(\mu^2|\ln\epsilon|^{-1} + \frac{\epsilon^{\frac{1}{4}}}{\mu^{\frac{5}{2}+}\delta^{\frac{1}{4}}})\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds. \end{aligned} \tag{3.25}$$

Proof. Upon substituting $(3.4)_1$ into $(3.4)_2$, we multiply the resulting equation by $\frac{\bar{\rho}}{\rho^2}\phi_y$, which yields

$$\begin{aligned}
&(\frac{\bar{\rho}}{2\rho^3}\phi_y^2+\frac{\bar{\rho}}{\rho}\psi\phi_y)_\tau+(\frac{\bar{\rho}}{2\rho^3}u\phi_y^2-\frac{\bar{\rho}}{\rho}\psi\phi_\tau)_y+\frac{\bar{\rho}}{\rho^2}\phi_y^2\\
=&\bar{\rho}[\frac{-\bar{\rho}_y\psi^2}{\rho^2}(\phi_y+\bar{\rho}_y)+\bar{u}_y\frac{\psi_y\phi}{\rho}-\bar{\rho}_y\bar{u}_y\frac{\psi\phi}{\rho^2}+\bar{\rho}\bar{u}_y\frac{\psi\phi_y}{\rho^2}]+\bar{\rho}\psi_y^2-\frac{\bar{\rho}}{\rho^3}\phi_y(\bar{u}_{yy}\phi+\bar{\rho}_{yy}\psi+2\bar{\rho}_y\psi_y)\\
&+\frac{\phi_y^2}{2\rho^3}(\bar{\rho}_\tau+\bar{\rho}_y u)-\frac{\bar{\rho}}{\rho^2}g\phi_y+\frac{\psi\phi_y}{\rho}\bar{\rho}_\tau-\frac{\psi\phi_\tau}{\rho}\bar{\rho}_y+\frac{\bar{\rho}n}{\rho^2}(\tilde{\psi}-\psi)\phi_y\\
:=&M(y,\tau).
\end{aligned} \tag{3.26}$$

Substituting $(3.4)_3$ into $(3.4)_4$ and multiplying the equation by $\frac{\bar{n}}{n^2}\tilde{\phi}_y$

$$\begin{aligned}
&(\frac{\bar{n}^2}{2n^3}\tilde{\phi}_y^2+\frac{\bar{n}}{n}\tilde{\psi}\tilde{\phi}_y)_\tau+(\frac{\bar{n}^2}{2n^3}v\tilde{\phi}_y^2-\frac{\bar{n}}{n}\tilde{\psi}\tilde{\phi}_\tau)_y+\frac{\bar{n}}{n^2}\tilde{\phi}_y^2\\
=&\bar{n}[\frac{-\bar{n}_y\tilde{\psi}^2}{n^2}(\tilde{\phi}_y+\bar{n}_y)+\bar{u}_y\frac{\tilde{\psi}_y\tilde{\phi}}{n}-\bar{n}_y\bar{u}_y\frac{\tilde{\psi}\tilde{\phi}}{n^2}+\bar{n}\bar{u}_y\frac{\tilde{\psi}\tilde{\phi}_y}{n^2}]+\bar{n}\tilde{\psi}_y^2-\frac{\bar{n}^2}{n^3}\tilde{\phi}_y(\bar{u}_{yy}\tilde{\phi}+\bar{n}_{yy}\tilde{\psi}+2\bar{n}_y\tilde{\psi}_y)\\
&+\frac{\tilde{\phi}_y^2}{2n^3}[(\bar{n}^2)_\tau+(\bar{n}^2)_y v]-\frac{\bar{n}}{n^2}\tilde{g}\tilde{\phi}_y+\frac{\tilde{\psi}\tilde{\phi}_y}{n}\bar{n}_\tau-\frac{\tilde{\psi}\tilde{\phi}_\tau}{n}\bar{n}_y+\bar{n}\bar{n}_y\frac{v_y\tilde{\phi}_y}{n^2}+(\tilde{\phi}v_y)_y\frac{\bar{n}}{n^2}\tilde{\phi}_y-\frac{\bar{n}}{n}(\tilde{\psi}-\psi)\tilde{\phi}_y\\
:=&N(y,\tau).
\end{aligned} \tag{3.27}$$

The combined integration of (3.26) + (3.27) over $\mathbb{R}\times[0,\tau]$ leads to

$$\begin{aligned}
&\int_{\mathbb{R}}(\tilde{\phi}_y^2+\phi_y^2)dy+\int_0^\tau\int_{\mathbb{R}}(\phi_y^2+\tilde{\phi}_y^2)dyds\\
\leq &C\int_{\mathbb{R}}(|\psi\phi_y|+|\tilde{\psi}\tilde{\phi}_y|)dy+C\int_0^\tau\int_{\mathbb{R}}[|M(y,\tau)|+|N(y,\tau)|]dyds.
\end{aligned} \tag{3.28}$$

Our next task is to provide estimates for all terms on the right-hand side of (3.28). By Cauchy's inequality, the following holds:

$$\int_{\mathbb{R}}(|\psi\phi_y|+|\tilde{\psi}\tilde{\phi}_y|)dy\leq\frac{1}{80}\int_{\mathbb{R}}(\tilde{\phi}_y^2+\phi_y^2)dy+C\mu^{-1}\int_{\mathbb{R}}(\bar{\rho}\psi^2+\bar{n}\tilde{\psi}^2)dy. \tag{3.29}$$

Since the method used in estimating $C\int_0^\tau\int_{\mathbb{R}}|M(y,\tau)|dyds$ is exactly same to $C\int_0^\tau\int_{\mathbb{R}}|N(y,\tau)|dyds$, the details can be omitted. We start to estimate $C\int_0^\tau\int_{\mathbb{R}}|N(y,\tau)|dyds$. From (3.27), one has

$$\begin{aligned}
C\int_0^\tau\int_{\mathbb{R}}|N(y,\tau)|dyds\leq&C\int_0^\tau\int_{\mathbb{R}}[|\bar{n}\bar{n}_y\frac{\tilde{\psi}^2}{n^2}\tilde{\phi}_y|+|\bar{n}\bar{n}_y^2\frac{\tilde{\psi}^2}{n^2}|+|\bar{n}\bar{u}_y\frac{\tilde{\psi}_y\tilde{\phi}}{n}|+|\bar{n}\bar{n}_y\bar{u}_y\frac{\tilde{\psi}\tilde{\phi}}{n^2}|+|\bar{n}^2\bar{u}_y\frac{\tilde{\psi}\tilde{\phi}_y}{n^2}|\\
&+|\frac{\bar{n}^2}{n^3}\tilde{\phi}_y\bar{u}_{yy}\tilde{\phi}|+|\frac{\bar{n}^2}{n^3}\tilde{\phi}_y\bar{\rho}_{yy}\tilde{\psi}|+|\frac{\bar{n}^2}{n^3}\tilde{\phi}_y\bar{\rho}_y\tilde{\psi}_y|+|\frac{\tilde{\phi}_y^2}{2n^3}(\bar{n}^2)_\tau|+|\frac{\tilde{\phi}_y^2}{2n^3}(\bar{n}^2)_y v|\\
&+|\bar{n}\tilde{g}\frac{\tilde{\phi}_y}{n^2}|+|\bar{n}\bar{n}_y\frac{v_y\tilde{\phi}_y}{n^2}|+|\frac{\tilde{\psi}\tilde{\phi}_y}{n}\bar{n}_\tau|+|\frac{\tilde{\psi}\tilde{\phi}_\tau}{n}\bar{n}_y|+|\frac{\bar{n}}{n^2}\tilde{\phi}_y^2\bar{u}_y|+|\frac{\bar{n}}{n^2}\tilde{\phi}_y^2\tilde{\psi}_y|\\
&+|\frac{\bar{n}}{n^2}\tilde{\phi}\tilde{\phi}_y\bar{u}_{yy}|+|\frac{\bar{n}}{n^2}\tilde{\phi}\tilde{\phi}_y\tilde{\psi}_{yy}|+|\frac{\bar{n}}{n}(\tilde{\psi}-\psi)\tilde{\phi}_y|]dyds\\
:=&\sum_{i=1}^{19}N_i.
\end{aligned}$$

Based on the a priori assumptions (3.9), it can be deduced that

$$\begin{aligned} N_1 &\le C\int_0^\tau\int_{\mathbb{R}}|\bar{u}_y\tilde{\psi}^2\tilde{\phi}_y|dyds \le C\int_0^\tau\|\tilde{\psi}\|_{L^\infty}\int_{\mathbb{R}}|\bar{n}^{\frac{1}{2}}\bar{u}_y^{\frac{1}{2}}\tilde{\psi}\tilde{\phi}_y\bar{u}_y^{\frac{1}{2}}\bar{n}^{-\frac{1}{2}}|dyds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + C\frac{\epsilon^{1+2a}}{\mu\delta}\int_0^\tau\int_{\mathbb{R}}\bar{n}\bar{u}_y\tilde{\psi}^2dyds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}(\tilde{\phi}_y^2+\bar{n}\bar{u}_y\tilde{\psi}^2)dyds. \end{aligned} \tag{3.30}$$

It follows from Lemma 2.2 that

$$N_2 \le C\int_0^\tau\int_{\mathbb{R}}|\bar{n}\bar{u}_y^2\tilde{\psi}^2|dyds \le C\frac{\epsilon}{\delta}\int_0^\tau\int_{\mathbb{R}}\bar{n}\bar{u}_y\tilde{\psi}^2dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\bar{u}_y\tilde{\psi}^2dyds. \tag{3.31}$$

It is similar to (3.23) that

$$N_3 \le C\int_0^\tau\int_{\mathbb{R}}|\bar{u}_y\tilde{\psi}_y\tilde{\phi}|dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\tilde{\psi}_y^2dyds + \frac{1}{80}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy. \tag{3.32}$$

By Hölder's inequality, Cauchy's inequality and Lemma 2.2, the following holds:

$$\begin{aligned} N_4 &\le C\int_0^\tau\int_{\mathbb{R}}|\bar{u}_y^2\tilde{\psi}\tilde{\phi}|dyds \le C\int_0^\tau\int_{\mathbb{R}}|\bar{n}^{\frac{1}{2}}\bar{u}_y^{\frac{1}{2}}\tilde{\psi}\bar{n}^{-\frac{1}{2}}\bar{u}_y^{\frac{3}{2}}\tilde{\phi}|dyds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\bar{u}_y\tilde{\psi}^2dyds + C\epsilon^3\mu^{-1}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy\int_0^\tau\|\bar{u}_x\|_{L^\infty}^3ds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\bar{u}_y\tilde{\psi}^2dyds + C\frac{\epsilon^2}{\mu\delta^2}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}\bar{u}_y\tilde{\psi}^2dyds + \frac{1}{80}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy. \end{aligned} \tag{3.33}$$

Similarly, the following holds:

$$\begin{aligned} N_5 &\le C\int_0^\tau\int_{\mathbb{R}}|\bar{u}_y\tilde{\psi}\tilde{\phi}_y|dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}(\tilde{\phi}_y^2+\bar{n}\bar{u}_y\tilde{\psi}^2)dyds, \\ N_6 &\le C\int_0^\tau\int_{\mathbb{R}}|\bar{n}^{-1}\bar{u}_{yy}\tilde{\phi}\tilde{\phi}_y|dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + \frac{1}{80}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy. \end{aligned} \tag{3.34}$$

From Lemma 2.2, one can get

$$\begin{aligned} N_7 &\le C\int_0^\tau\int_{\mathbb{R}}|\bar{n}^{-1}\tilde{\phi}_{yy}\bar{n}_{yy}\tilde{\psi}|dyds \le C\int_0^\tau\int_{\mathbb{R}}(|\bar{u}_{yy}\tilde{\phi}_y\tilde{\psi}| + |\bar{u}_y^2\tilde{\phi}_y\tilde{\psi}|)dyds \\ &\le C\frac{\epsilon}{\delta}\int_0^\tau\int_{\mathbb{R}}|\tilde{\phi}_y\bar{n}^{\frac{1}{2}}\bar{u}^{\frac{1}{2}}\tilde{\psi}\bar{u}_y^{\frac{1}{2}}\bar{n}^{-\frac{1}{2}}|dyds + C\int_0^\tau\int_{\mathbb{R}}|\tilde{\phi}_y\bar{n}^{\frac{1}{2}}\bar{u}^{\frac{1}{2}}\tilde{\psi}\bar{u}_y^{\frac{3}{2}}\bar{n}^{-\frac{1}{2}}|dyds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + C\frac{\epsilon^3}{\mu\delta^3}\int_0^\tau\int_{\mathbb{R}}\bar{n}\bar{u}_y\tilde{\psi}^2dyds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}(\tilde{\phi}_y^2+\bar{n}\bar{u}_y\tilde{\psi}^2)dyds. \end{aligned} \tag{3.35}$$

From Cauchy's inequality combined with Lemma 2.2, it can be concluded that

$$\begin{aligned}
N_8 &\le C\int_0^\tau\int_{\mathbb{R}} |\bar u_y\tilde\phi_y\tilde\psi_y|dyds \le C\int_0^\tau\int_{\mathbb{R}} |\tilde\phi_y\bar n^{\frac{1}{2}}\tilde\psi_y\bar u_y\bar n^{-\frac{1}{2}}|dyds\\
&\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde\phi_y^2dyds + C\frac{\epsilon^2}{\mu\delta^2}\int_0^\tau\int_{\mathbb{R}}\bar n\tilde\psi_y^2dyds\\
&\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}(\tilde\phi_y^2+\bar n\tilde\psi_y^2)dyds.
\end{aligned}\tag{3.36}$$

Since

$$\bar n_\tau = -(\bar n\bar u)_y,$$

one has

$$\begin{aligned}
N_9 &\le C\int_0^\tau\int_{\mathbb{R}} |\bar n^{-3}\tilde\phi_y^2(\bar n^2)_\tau|dyds \le C\int_0^\tau\int_{\mathbb{R}} |\bar n^{-1}\bar u_y\tilde\phi_y^2|dyds\\
&\le C\frac{\epsilon}{\mu\delta}\int_0^\tau\int_{\mathbb{R}}\tilde\phi_y^2dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde\phi_y^2dyds.
\end{aligned}\tag{3.37}$$

Similarly, one has

$$N_{10} \le C\int_0^\tau\int_{\mathbb{R}}|\bar n^{-2}\tilde\phi_y^2\bar n_y v|dyds \le C\frac{\epsilon}{\mu\delta}\int_0^\tau\int_{\mathbb{R}}\tilde\phi_y^2dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde\phi_y^2dyds.\tag{3.38}$$

According to (3.5), we have

$$N_{11} \le C\int_0^\tau\int_{\mathbb{R}}|\bar n^{-1}\tilde g\tilde\phi_y|dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde\phi_y^2dyds + C\int_0^\tau\int_{\mathbb{R}}\bar n^{-2}\tilde g^2dyds,\tag{3.39}$$

where

$$\begin{aligned}
C\int_0^\tau\int_{\mathbb{R}}\bar n^{-2}\tilde g^2dyds &\le C\int_0^\tau\int_{\mathbb{R}}(\tilde\psi^2\bar u_y^2 + \bar n^{-2}\bar u_y^2\tilde\phi^2 + \bar u_{yy}^2)dyds\\
&\le C\epsilon^2\mu^{-2}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde\phi^2dy\int_0^\tau\|\bar u_x\|_{L^\infty}^2ds + C\int_0^\tau\int_{\mathbb{R}}(\bar n\bar u\tilde\psi\bar u_y\bar n^{-1} + \bar u_{yy}^2)dyds\\
&\le C\frac{\epsilon}{\mu^2\delta}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde\phi^2dy + C\frac{\epsilon}{\mu\delta}\int_0^\tau\int_{\mathbb{R}}\bar n\bar u_y\tilde\psi^2dyds + C\frac{\epsilon^2}{\delta^2}\\
&\le \frac{1}{80}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde\phi^2dy + \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar n\bar u_y\tilde\psi^2dyds + C\frac{\epsilon^2}{\delta^2}.
\end{aligned}$$

According to the definition of perturbations (3.2) and Lemma 2.2, it follows that

$$\begin{aligned}
N_{12} &\le C\int_0^\tau\int_{\mathbb{R}}|\bar u_y v_y\tilde\phi_y|dyds \le C\int_0^\tau\int_{\mathbb{R}}(|\bar u_y^2\tilde\phi_y| + |\bar u_y\tilde\psi_y\tilde\phi|)dyds\\
&\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde\phi_y^2dyds + C\int_0^\tau\int_{\mathbb{R}}\bar n\tilde\psi_y^2dyds + C\frac{\epsilon}{\mu\delta}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde\phi^2dy + C\int_0^\tau\int_{\mathbb{R}}\bar u_y^4dyds\\
&\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde\phi_y^2dyds + C\int_0^\tau\int_{\mathbb{R}}\bar n\tilde\psi_y^2dyds + \frac{1}{80}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde\phi^2dy + C\frac{\epsilon^2}{\delta^2}.
\end{aligned}\tag{3.40}$$

From

$$\bar{n}_\tau = -(\bar{n}\bar{u})_y,$$

one has

$$\begin{aligned}
N_{13} &\leq C\int_0^\tau\int_{\mathbb{R}} |\bar{n}^{-1}\tilde{\psi}\tilde{\phi}_y\bar{n}_\tau|dyds \leq C\int_0^\tau\int_{\mathbb{R}} |\bar{n}^{-1}\tilde{\psi}\tilde{\phi}_y(\bar{n}\bar{u}_y+\bar{n}_y\bar{u})|dyds\\
&\leq C\int_0^\tau\int_{\mathbb{R}} |\tilde{\psi}\tilde{\phi}_y\bar{u}_y|dyds \leq C\int_0^\tau\int_{\mathbb{R}} |\tilde{\phi}_y\bar{n}^{\frac{1}{2}}\bar{u}_y^{\frac{1}{2}}\tilde{\psi}\bar{u}_y^{\frac{1}{2}}\bar{n}^{-\frac{1}{2}}|dyds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \tilde{\phi}_y^2dyds + C\frac{\epsilon}{\mu\delta}\int_0^\tau\int_{\mathbb{R}} \bar{n}\bar{u}_y\tilde{\psi}^2dyds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} (\tilde{\phi}_y^2+\bar{n}\bar{u}_y\tilde{\psi}^2)dyds.
\end{aligned} \tag{3.41}$$

From (3.4), one has

$$\tilde{\phi}_\tau = -n\tilde{\psi}_y - v\tilde{\phi}_y - \bar{u}_y\tilde{\phi} - \bar{n}_y\tilde{\psi},$$

and one can get

$$\begin{aligned}
N_{14} &\leq C\int_0^\tau\int_{\mathbb{R}} |\tilde{\psi}\bar{u}_y\tilde{\phi}_\tau|dyds\\
&\leq C\int_0^\tau\int_{\mathbb{R}} (|\tilde{\psi}\bar{u}_y\tilde{\psi}_y| + |\tilde{\psi}\bar{u}_y\tilde{\phi}_y| + |\tilde{\psi}\bar{u}_y^2\tilde{\phi}| + |\tilde{\psi}^2\bar{u}_y\bar{n}_y|)dyds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} (\bar{n}\tilde{\psi}_y^2+\tilde{\phi}_y^2+\bar{n}\bar{u}_y\tilde{\psi})dyds + C\epsilon^3\mu^{-1}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy\int_0^\tau \|\bar{u}_x\|_{L^\infty}^3ds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} (\bar{n}\tilde{\psi}_y^2+\tilde{\phi}_y^2+\bar{n}\bar{u}_y\tilde{\psi})dyds + C\frac{\epsilon^2}{\mu\delta^2}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} (\bar{n}\tilde{\psi}_y^2+\tilde{\phi}_y^2+\bar{n}\bar{u}_y\tilde{\psi})dyds + \frac{1}{80}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy.
\end{aligned} \tag{3.42}$$

By Lemma 2.2, the following holds:

$$N_{15} \leq C\int_0^\tau\int_{\mathbb{R}} |\bar{n}^{-1}\bar{u}_y\tilde{\phi}_y^2|dyds \leq C\frac{\epsilon}{\mu\delta}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds \leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds. \tag{3.43}$$

Combining Hölder's, Sobolev's and Cauchy's inequalities with the a priori assumption (3.9), the following holds:

$$\begin{aligned}
N_{16} &\leq C\int_0^\tau\int_{\mathbb{R}} |\bar{n}^{-1}\tilde{\psi}_y\tilde{\phi}_y^2|dyds \leq C\mu^{-1}\int_0^\tau \|\tilde{\psi}_y\|_{L^\infty}\|\tilde{\phi}_y\|^2ds\\
&\leq C\mu^{-1}\int_0^\tau \|\tilde{\psi}_y\|^{\frac{1}{2}}\|\tilde{\psi}_{yy}\|^{\frac{1}{2}}\|\tilde{\phi}_y\|^2ds \leq C\mu^{\frac{1}{2}}\mu^{-\frac{3}{2}}\int_0^\tau \|\tilde{\psi}_y\|^{\frac{1}{2}}\|\tilde{\psi}_{yy}\|^{\frac{1}{2}}\|\tilde{\phi}_y\|^2ds\\
&\leq C\mu^2|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds + C\mu^{-2}|\ln\epsilon|^{\frac{1}{3}}\sup_{\tau\in[0.\tau_1]}\|\tilde{\phi}_y\|^{\frac{4}{3}}\int_0^\tau \|\tilde{\psi}_y\|^{\frac{2}{3}}\|\tilde{\phi}_y\|^{\frac{4}{3}}ds\\
&\leq C\mu^2|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds + C\mu^{-2}|\ln\epsilon|^{\frac{1}{3}}\int_0^\tau \|\tilde{\psi}_y\|^{\frac{2}{3}}\|\tilde{\phi}_y\|^{\frac{4}{3}}ds\\
&\leq C\mu^2|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds + \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + C\mu^{-7}|\ln\epsilon|\int_0^\tau\int_{\mathbb{R}}\bar{n}\tilde{\psi}_y^2dyds.
\end{aligned} \tag{3.44}$$

It follows from Lemma 2.2 that

$$\begin{aligned}
N_{17} &\leq C\int_0^\tau\int_{\mathbb{R}}|\bar{n}^{-1}\bar{u}_{yy}\tilde{\phi}\tilde{\phi}_y|dyds \leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + C\mu^{-2}\int_0^\tau\int_{\mathbb{R}}\bar{u}_{yy}^2\tilde{\phi}^2dyds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + C\epsilon^4\mu^{-2}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy\int_0^\tau\|\bar{u}_{xx}\|_{L^\infty}^2ds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + C\frac{\epsilon^3}{\mu^2\delta^3}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + \frac{1}{80}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy.
\end{aligned}\tag{3.45}$$

Following Hölder's inequality, Sobolev's inequality, Cauchy's inequality and the a priori assumption (3.9), this gives

$$\begin{aligned}
N_{18} &\leq C\int_0^\tau\int_{\mathbb{R}}|\bar{n}^{-1}\tilde{\phi}\tilde{\phi}_y\tilde{\psi}_{yy}|dyds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + C\mu^{-2}\int_0^\tau\|\tilde{\phi}\|_{L^\infty}^2\|\tilde{\psi}_{yy}\|^2ds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + C\mu^{-2}\sup_{\tau\in[0.\tau_1]}\|\tilde{\phi}\|\sup_{\tau\in[0.\tau_1]}\|\tilde{\phi}_y\|\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{n}^{\gamma-2}\tilde{\phi}_y^2dyds + C\frac{\epsilon^{\frac{1}{4}}}{\mu^{\frac{5}{2}}\delta^{\frac{1}{4}}}\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds,
\end{aligned}\tag{3.46}$$

where the last inequality uses $\int_{\mathbb{R}}\tilde{\phi}^2dy \leq C\frac{\epsilon^{\frac{1}{2}}}{\mu\delta^{\frac{1}{2}}}$ in Lemma 3.1. According to the definition of rarefaction wave, the following holds:

$$\begin{aligned}
N_{19} &\leq C\int_0^\tau\int_{\mathbb{R}}|\tilde{\phi}_y\bar{n}^{\frac{1}{2}}(\tilde{\psi}-\psi)\bar{n}^{-\frac{1}{2}}|dyds\\
&\leq \frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\phi}_y^2dyds + C\mu^{-1}\int_0^\tau\int_{\mathbb{R}}\bar{n}(\tilde{\psi}-\psi)^2dyds.
\end{aligned}\tag{3.47}$$

Combining of (3.29)–(3.47) with (3.14), when substituted into (3.28), leads to the result that

$$\begin{aligned}
&\int_{\mathbb{R}}(\tilde{\phi}_y^2+\phi_y^2)dy + \int_0^\tau\int_{\mathbb{R}}(\phi_y^2+\tilde{\phi}_y^2)dyds\\
\leq& C\frac{\epsilon^2}{\delta^2} + C\mu^{-7}|\ln\epsilon|\{\int_0^\tau\int_{\mathbb{R}}[\bar{n}(\tilde{\psi}-\psi)^2+\bar{\rho}\bar{u}_y\psi^2+\psi_y^2+\bar{n}\bar{u}_y\tilde{\psi}^2+\bar{n}\tilde{\psi}_y^2]dyds\\
&+\int_{\mathbb{R}}(\bar{\rho}\psi^2+\bar{n}\tilde{\psi}^2)dy\} + C(\mu^2|\ln\epsilon|^{-1}+\frac{\epsilon^{\frac{1}{4}}}{\mu^{\frac{5}{2}}\delta^{\frac{1}{4}}})\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds\\
\leq& C\frac{\epsilon^{\frac{1}{2}}}{\mu^8\delta^{\frac{1}{2}}}|\ln\epsilon| + C(\mu^2|\ln\epsilon|^{-1}+\frac{\epsilon^{\frac{1}{4}}}{\mu^{\frac{5}{2}}\delta^{\frac{1}{4}}})\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds.
\end{aligned}\tag{3.48}$$

Thus the proof of Lemma 3.2 is finished. □

Lemma 3.3. *There exists positive constants C and* ϵ_0 *such that for* $0 \le \tau \le \tau_1(\epsilon)$ *and* $\epsilon < \epsilon_0$,

$$\begin{aligned}&\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbb{R}}(\psi_y^2+\tilde{\psi}_y^2)dy++\int_0^{\tau_1(\epsilon)}\int_{\mathbb{R}}[\bar{u}_y\psi_y^2+\psi_{yy}^2+\bar{u}_y\tilde{\psi}_y^2+\tilde{\psi}_{yy}^2]dyds\\&+\int_0^{\tau_1(\epsilon)}\int_{\mathbb{R}}(\tilde{\psi}_y-\psi_y)^2dyds\le C\frac{\epsilon^{\frac{1}{2}}}{\mu^{10}\delta^{\frac{1}{2}}}|\ln\epsilon|.\end{aligned}\tag{3.49}$$

Proof. Multiplying $(3.4)_2$ by $-\frac{\psi_{yy}}{\rho}$ yields

$$(\frac{\psi_y^2}{2})_\tau-(\psi_y\psi_\tau+u\frac{\psi_y^2}{2})_y+\bar{u}_y\frac{\psi_y^2}{2}+\frac{\psi_{yy}^2}{\rho}=\phi_y\frac{\psi_{yy}}{\rho}+g\frac{\psi_{yy}}{\rho}-\frac{n}{\rho}(\tilde{\psi}-\psi)\psi_{yy}-\frac{\psi_y^3}{2}.\tag{3.50}$$

Multiplying $(3.4)_4$ by $-\frac{\tilde{\psi}_{yy}}{n}$ yields

$$(\frac{\tilde{\psi}_y^2}{2})_\tau-(\tilde{\psi}_y\tilde{\psi}_\tau+v\frac{\tilde{\psi}_y^2}{2})_y+\bar{u}_y\frac{\tilde{\psi}_y^2}{2}+\tilde{\psi}_{yy}^2=\tilde{\phi}_y\frac{\tilde{\psi}_{yy}}{\rho}+\tilde{g}\frac{\tilde{\psi}_{yy}}{n}-(\tilde{\psi}-\psi)\tilde{\psi}_{yy}-\frac{\tilde{\psi}_y^3}{2}-n_yv_y\frac{\tilde{\psi}_{yy}}{n}-\tilde{\phi}v_{yy}\frac{\tilde{\psi}_{yy}}{n}.\tag{3.51}$$

The combined integration of (3.50) and (3.51) over $\mathbb{R}\times[0,\tau]$ leads to

$$\begin{aligned}&\int_{\mathbb{R}}(\psi_y^2+\tilde{\psi}_y^2)dy++\int_0^\tau\int_{\mathbb{R}}[(\tilde{\psi}_y-\psi_y)^2+\bar{u}_y\psi_y^2+\psi_{yy}^2+\bar{u}_y\tilde{\psi}_y^2+\tilde{\psi}_{yy}^2]dyds\\&\le C\int_0^\tau\int_{\mathbb{R}}[|\phi_y\frac{\psi_{yy}}{\rho}|+|g\frac{\psi_{yy}}{\rho}|+|(1-\frac{n}{\rho})(\tilde{\psi}-\psi)\psi_{yy}|+|\tilde{\phi}_y\frac{\tilde{\psi}_{yy}}{n}|+|\tilde{g}\frac{\tilde{\psi}_{yy}}{n}|\\&\qquad+|\frac{\psi_y^3}{2}|+|\frac{\tilde{\psi}_y^3}{2}|+|n_yv_y\frac{\tilde{\psi}_{yy}}{n}|+|\tilde{\phi}v_{yy}\frac{\tilde{\psi}_{yy}}{n}|]dyds\\&:=\sum_{i=1}^{9}J_i.\end{aligned}\tag{3.52}$$

Our next task is to provide estimates for all terms on the right-hand side of (3.52). By Cauchy's inequality, one has

$$J_1\le C\int_0^\tau\int_{\mathbb{R}}|\bar{\rho}^{-1}\phi_y\psi_{yy}|dyds\le\frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\psi_{yy}^2dyds+C\mu^{-2}\int_0^\tau\int_{\mathbb{R}}\phi_y^2dyds.\tag{3.53}$$

From

$$g=\rho\psi\bar{u}_y-\bar{\rho}_y\frac{\phi}{\bar{\rho}}-\bar{u}_{yy}$$

one has

$$J_2\le C\int_0^\tau\int_{\mathbb{R}}|\bar{\rho}^{-1}g\psi_{yy}|dyds\le\frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\psi_{yy}^2dyds+C\int_0^\tau\int_{\mathbb{R}}\bar{\rho}^{-2}g^2dyds,\tag{3.54}$$

where

$$\begin{aligned}C\int_0^\tau\int_{\mathbb{R}}\bar{\rho}^{-2}g^2dyds\le&C\int_0^\tau\int_{\mathbb{R}}(\bar{\rho}^{-2}\bar{u}_{yy}^2+\psi^2\bar{u}_y^2+\bar{\rho}^{-4}\bar{\rho}_y^2\phi^2)dyds\\\le&C\mu^{-2}\int_0^\tau\int_{\mathbb{R}}\bar{u}_{yy}^2dyds+C\frac{\epsilon}{\mu\delta}\int_0^\tau\int_{\mathbb{R}}\bar{\rho}\bar{u}_y\psi^2dyds+C\frac{\epsilon}{\mu^2\delta}\sup_{\tau\in[0.\tau_1]}\int_{\mathbb{R}}\phi^2dy\\\le&C\frac{\epsilon^2}{\mu^2\delta^2}+\frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\bar{\rho}\bar{u}_y\psi^2dyds+\frac{1}{80}\sup_{\tau\in[0.\tau_1]}\int_{\mathbb{R}}\phi^2dy.\end{aligned}$$

By Hölder's inequality and Cauchy's inequality, the following holds:

$$J_3 \le C\int_0^\tau\int_{\mathbb{R}} |\rho^{-1}(\tilde{\psi}-\psi)\psi_{yy}|dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \psi_{yy}^2 + C\mu^{-3}\int_0^\tau\int_{\mathbb{R}} \bar{n}(\tilde{\psi}-\psi)^2 dyds. \tag{3.55}$$

This is similar to the estimate of J_1 and J_2, where

$$\begin{aligned} J_4 &\le C\int_0^\tau\int_{\mathbb{R}} |\bar{n}^{-1}\tilde{\phi}_y\tilde{\psi}_{yy}|dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \tilde{\psi}_{yy}^2 dyds + C\mu^{-2}\int_0^\tau\int_{\mathbb{R}} \tilde{\phi}_y^2 dyds, \\ J_5 &\le C\int_0^\tau\int_{\mathbb{R}} |\tilde{\psi}_{yy}\bar{n}^{-1}\tilde{g}|dyds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \tilde{\psi}_{yy}^2 dyds + C\frac{\epsilon^2}{\delta^2} + \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \bar{n}\bar{u}_y\tilde{\psi}^2 dyds + \frac{1}{80}\sup_{\tau\in[0.\tau_1]}\int_{\mathbb{R}} \tilde{\phi}^2 dy. \end{aligned} \tag{3.56}$$

It follows from Sobolev's inequality that

$$\begin{aligned} J_6 &\le C\int_0^\tau\int_{\mathbb{R}} |\psi_y^3|dyds \le C\int_0^\tau ||\psi_y||^2||\psi_y||_{L^\infty}ds \le C\int_0^\tau ||\psi_y||^{\frac{5}{2}}||\psi_{yy}||^{\frac{1}{2}}ds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \psi_{yy}^2 dyds + C\sup_{\tau\in[0.\tau_1]}||\psi_y||^{\frac{4}{3}}\int_0^\tau\int_{\mathbb{R}} \psi_y^2 dyds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \psi_{yy}^2 dyds + C\int_0^\tau\int_{\mathbb{R}} \psi_y^2 dyds. \end{aligned} \tag{3.57}$$

Similarly, the following holds:

$$J_7 \le C\int_0^\tau\int_{\mathbb{R}} |\tilde{\psi}_y^3|dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \tilde{\psi}_{yy}^2 dyds + C\mu^{-1}\int_0^\tau\int_{\mathbb{R}} \bar{n}\tilde{\psi}_y^2 dyds. \tag{3.58}$$

According to (3.2), Lemma 2.2 and a priori assumption (3.9), one has

$$\begin{aligned} J_8 &\le C\int_0^\tau\int_{\mathbb{R}} |\bar{n}^{-1}n_y v_y\tilde{\psi}_{yy}|dyds \\ &\le C\int_0^\tau\int_{\mathbb{R}} |\bar{n}^{-1}|(|\tilde{\psi}_{yy}||\tilde{\phi}_y||\tilde{\psi}_y| + |\tilde{\psi}_{yy}||\tilde{\phi}_y||\bar{u}_y| + |\tilde{\psi}_{yy}||\bar{n}_y||\tilde{\psi}_y| + |\tilde{\psi}_{yy}||\bar{n}_y||\bar{u}_y|)dyds \\ &:= J_8^1 + J_8^2 + J_8^3 + J_8^4, \end{aligned} \tag{3.59}$$

where

$$\begin{aligned} J_8^1 &\le C\mu^{-1}\int_0^\tau ||\tilde{\psi}_{yy}||||\tilde{\phi}_y||||\tilde{\psi}_y||_{L^\infty}ds \le C\mu^{-1}\int_0^\tau ||\tilde{\psi}_{yy}||^{\frac{3}{2}}||\tilde{\phi}_y||||\tilde{\psi}_y||^{\frac{1}{2}}ds \\ &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \tilde{\psi}_{yy}^2 dyds + C\mu^{-5}\int_0^\tau\int_{\mathbb{R}} \bar{n}\tilde{\psi}_y^2 dyds, \\ J_8^2 &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \tilde{\psi}_{yy}^2 dyds + C\frac{\epsilon^2}{\mu^2\delta^2}\int_0^\tau\int_{\mathbb{R}} \tilde{\phi}_y^2 dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} (\tilde{\psi}_{yy}^2 + \tilde{\phi}_y^2)dyds, \\ J_8^3 &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \tilde{\psi}_{yy}^2 dyds + C\frac{\epsilon^2}{\mu\delta^2}\int_0^\tau\int_{\mathbb{R}} \bar{n}\tilde{\psi}_y^2 dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} (\tilde{\psi}_{yy}^2 + \bar{n}\tilde{\psi}_y^2)dyds, \\ J_8^4 &\le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \tilde{\psi}_{yy}^2 dyds + C\int_0^\tau\int_{\mathbb{R}} \bar{u}_y^4 dyds \le \frac{1}{80}\int_0^\tau\int_{\mathbb{R}} \tilde{\psi}_{yy}^2 dyds + C\frac{\epsilon^2}{\delta^2}. \end{aligned}$$

With the aid of Hölder's inequality, Cauchy's inequality, Lemma 2.2 and the a priori assumption given in (3.9), we arrive at the following conclusion:

$$\begin{aligned}
J_9 &\le C\int_0^\tau\int_{\mathbb{R}}(|\bar{n}^{-1}\tilde{\phi}\bar{u}_{yy}\tilde{\psi}_{yy}|+|\bar{n}^{-1}\tilde{\phi}\tilde{\psi}_{yy}^2|)dyds\\
&\le\frac{1}{80}\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds+C\epsilon^4\mu^{-2}\sup_{\tau\in[0.\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy\int_0^\tau\|\bar{u}_{xx}\|_{L^\infty}^2ds+C\frac{\epsilon^a}{\mu}\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds\\
&\le\frac{1}{40}\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds+C\frac{\epsilon^3}{\mu^2\delta^3}\sup_{\tau\in[0.\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy\\
&\le\frac{1}{40}\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds+\frac{1}{80}\sup_{\tau\in[0.\tau_1]}\int_{\mathbb{R}}\tilde{\phi}^2dy.
\end{aligned}\tag{3.60}$$

The combination (3.53)–(3.60) with (3.14), when substituted into (3.52), leads to the result that

$$\begin{aligned}
&\int_{\mathbb{R}}(\psi_y^2+\tilde{\psi}_y^2)dy++\int_0^\tau\int_{\mathbb{R}}[(\tilde{\psi}_y-\psi_y)^2+\bar{u}_y\psi_y^2+\psi_{yy}^2+\bar{u}_y\tilde{\psi}_y^2+\tilde{\psi}_{yy}^2]dyds\\
\le&C\frac{\epsilon^2}{\mu^2\delta^2}+C\mu^{-5}\{\int_{\mathbb{R}}(\phi^2+\tilde{\phi}^2)dy+\int_0^\tau\int_{\mathbb{R}}[\bar{\rho}\bar{u}_y\psi^2+\psi_y^2+\bar{n}\bar{u}_y\tilde{\psi}^2+\bar{n}\tilde{\psi}_y^2+\bar{n}(\tilde{\psi}-\psi)^2]dyds\}\\
&+C\mu^{-2}[\int_{\mathbb{R}}(\phi_y^2+\tilde{\phi}_y^2)dy+\int_0^\tau\int_{\mathbb{R}}(\phi_y^2+\tilde{\phi}_y^2)dyds]\\
\le&C\frac{\epsilon^{\frac{1}{2}}}{\mu^6\delta^{\frac{1}{2}}}+C\mu^{-2}[\frac{\epsilon^{\frac{1}{2}}}{\mu^8\delta^{\frac{1}{2}}}|\ln\epsilon|+(\mu^2|\ln\epsilon|^{-1}+\frac{\epsilon^{\frac{1}{4}}}{\mu^{\frac{5}{2}}\delta^{\frac{1}{4}}})\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds]\\
\le&C\frac{\epsilon^{\frac{1}{2}}}{\mu^{10}\delta^{\frac{1}{2}}}|\ln\epsilon|+C(|\ln\epsilon|^{-1}+\frac{\epsilon^{\frac{1}{4}}}{\mu^{\frac{9}{2}}\delta^{\frac{1}{4}}})\int_0^\tau\int_{\mathbb{R}}\tilde{\psi}_{yy}^2dyds,
\end{aligned}$$

which yields

$$\begin{aligned}
&\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbb{R}}(\psi_y^2+\tilde{\psi}_y^2)dy++\int_0^{\tau_1(\epsilon)}\int_{\mathbb{R}}[\bar{u}_y\psi_y^2+\psi_{yy}^2+\bar{u}_y\tilde{\psi}_y^2+\tilde{\psi}_{yy}^2]dyds\\
&+\int_0^{\tau_1(\epsilon)}\int_{\mathbb{R}}(\tilde{\psi}_y-\psi_y)^2\le C\frac{\epsilon^{\frac{1}{2}}}{\mu^{10}\delta^{\frac{1}{2}}}|\ln\epsilon|.
\end{aligned}\tag{3.61}$$

□

Proof of Theorem 3.2. Equation (3.13) follows from a combination of (3.14), (3.25), and (3.49), which completes the proof of Theorem 3.2. □

Proof of Theorem 3.1. The a priori estimates obtained here are actually stronger than the original assumptions (3.9) on the interval $[0,\tau_1]$. This allows us to conclude that $\tau_1(\epsilon)=\infty$. Indeed, if $\tau_1(\epsilon)<\infty$, one could, by reapplying the local existence theory at time $\tau=\tau_1(\epsilon)$, find a time $\tau_2(\epsilon)>\tau_1(\epsilon)$ such that the solution still satisfies (3.9) on $[0,\tau_2(\epsilon)]$, contradicting the maximality of $\tau_1(\epsilon)$. Consequently, the local solution can be globally extended to $(0,+\infty)$ for any fixed sufficiently small ϵ. □

Proof of Theorem 1.1. What remains is to establish (1.10) given the constant a. The proof of (1.8) for a fixed constant a follows from Lemma 2.2 and (3.8), together with the definitions $\mu = \epsilon^a |\ln\epsilon|$ and $\delta = \epsilon^a$. Consequently, for any positive constant h, a constant $C_h > 0$ exists, independent of ϵ, for which

$$\begin{aligned}
&\sup_{t\geq h} \|\rho(\cdot,t) - \rho^{r_2}(\cdot,t)\|_{L^\infty}\\
\leq &\sup_{\tau\in[0,+\infty)} \|\phi(\cdot,\tau)\|_{L^\infty} + \sup_{t\geq h}\|\bar{\rho}(\cdot,t) - \rho_\mu^{r_2}(\tfrac{\cdot}{t})\|_{L^\infty} + \sup_{t\geq h}\|\rho_\mu^{r_2}(\tfrac{\cdot}{t}) - \rho^{r_2}(\tfrac{\cdot}{t})\|_{L^\infty}\\
\leq &C_h(\epsilon^a + \delta|\ln\delta| + \mu) \leq C_h\epsilon^a|\ln\epsilon|,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\geq h}\|m(\cdot,t) - m^{r_2}(\cdot,t)\|_{L^\infty}\\
\leq &\sup_{t\geq h}\left(\|m(\cdot,t) - \bar{m}(\cdot,t)\|_{L^\infty} + \|\bar{m}(\cdot,t) - m_\mu^{r_2}(\tfrac{\cdot}{t})\|_{L^\infty} + \|m_\mu^{r_2}(\tfrac{\cdot}{t}) - m^{r_2}(\tfrac{\cdot}{t})\|_{L^\infty}\right)\\
\leq &\sup_{\tau\in[0,+\infty)}\left(\|\phi(\cdot,\tau)\|_{L^\infty} + \|\psi(\cdot,\tau)\|_{L^\infty}\right) + \sup_{t\geq h}\left(\|\bar{m}(\cdot,t) - m_\mu^{r_2}(\tfrac{\cdot}{t})\|_{L^\infty} + \|m_\mu^{r_2}(\tfrac{\cdot}{t}) - m^{r_2}(\tfrac{\cdot}{t})\|_{L^\infty}\right)\\
\leq &C_h(\epsilon^a + \delta|\ln\delta| + \mu) \leq C_h\epsilon^a|\ln\epsilon|.
\end{aligned}$$

Similarly, it holds that

$$\sup_{t\geq h}\|n(\cdot,t) - n^{r_2}(\cdot,t)\|_{L^\infty} \leq C_h\epsilon^a|\ln\epsilon|$$

$$\sup_{t\geq h}\|\tilde{m}(\cdot,t) - \tilde{m}^{r_2}(\cdot,t)\|_{L^\infty} \leq C_h\epsilon^a|\ln\epsilon|.$$

Therefore, the argument above establishes Theorem 1.1. □

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The research of Yixuan Zhao is supported by the Scientific Research Foundation for High-level Talents of Anhui University of Science and Technology (2023yjrc104) and Anhui Provincial Natural Science Foundation (2208085MA04). The research of Shuzhen Zhang is supported by the Scientific and Technical Innovation Team for Young Scholars in Universities of Shandong Province (Grant No. 2025KJG015).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. H. Li, T. Wang, Y. Wang, Wave phenomena to the three-dimensional fluid-particle model, *Arch. Rational Mech. Anal.*, **243** (2022), 1019–1089. https://doi.org/10.1007/s00205-021-01747-z
2. T. Liu, J. A. Smoller, On the vacuum state for the isentropic gas dynamics equations, *Adv. Appl. Math.*, **1** (1980), 345–359. https://doi.org/10.1016/0196-8858(80)90016-0
3. Q. Jiu, Y. Wang, Z. Xin, Vacuum behaviors around rarefaction waves to 1D compressible Navier-Stokes equations with density-dependent viscosity, *SIAM J. Math. Anal.*, **45** (2013), 3194–3228. https://doi.org/10.1137/120879919
4. M. Perepelitsa, Asymptotics toward rarefaction waves and vacuum for 1-D compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, **42** (2010), 1404–1412. https://doi.org/10.1137/08072855X
5. F. Huang, M. Li, Y. Wang, Zero dissipation limit to rarefaction wave with vacuum for one-dimensional compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, **44** (2012), 1742–1759. https://doi.org/10.1137/100814305
6. Z. Liang, Zero dissipation limit of the rarefaction wave with vacuum for the compressible Navier-Stokes equations with density-dependent viscosity, *Sci. Sin. Math.*, **42** (2012), 215–233. https://doi.org/10.1360/012011-401
7. M. Li, T. Wang, Zero dissipation limit to rarefaction wave with vacuum for one-dimensional full compressible Navier-Stokes equations, *Commun. Math. Sci.*, **12** (2014), 1135–1154. https://dx.doi.org/10.4310/CMS.2014.v12.n6.a6
8. T. Wang, Vanishing viscosity limit to rarefaction wave with vacuum for 1-D compressible Navier-Stokes equations with density-dependent viscosity, *Commun. Math. Sci.*, **13** (2015), 477–495. https://doi.org/10.4310/CMS.2015.V13.N2.A11
9. D. Zeidan, E. Romenski, A. Slaouti, E. F. Toro, Numerical study of wave propagation in compressible two-phase flow, *Int. J. Numer. Methods Fluids*, **54** (2007), 393–417. https://doi.org/10.1002/fld.1404
10. J. A. Carrillo, T. Goudon, Stability and asymptotic analysis of a fluid-particle interaction model, *Commun. Partial Differ. Equations*, **31** (2006), 1349–1379. https://doi.org/10.1080/03605300500394389
11. H. Li, S. Zhao, Existence and nonlinear stability of stationary solutions to the full two-phase flow model in a half line, *Appl. Math. Lett.*, **116** (2021), 107039. https://doi.org/10.1016/j.aml.2021.107039
12. H. Li, S. Zhao, H. Zuo, Existence and nonlinear stability of steady-states to the outflow problem for the full two-phase flow, *J. Differ. Equations*, **309** (2022), 350–385. https://doi.org/10.1016/j.jde.2021.11.040
13. S. Wang, Y. Zhao, Stability of planar rarefaction wave to a multi-dimensional non-conservative viscous compressible two-phase flow, *J. Math. Anal. Appl.*, **506** (2022), 125657. https://doi.org/10.1016/j.jmaa.2021.125657

14. S. Wang, Y. Zhao, Asymptotic stability of planar rarefaction wave to a multi-dimensional two-phase flow, *Discrete Contin. Dyn. Syst. Ser. B*, **28** (2023), 623–647. https://doi.org/10.3934/dcdsb.2022091
15. A. Matsumura, K. Nishihara, Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.*, **3** (1986), 1–13. https://doi.org/10.1007/BF03167088
16. Z. Xin, Zero dissipation limit to rarefaction waves for the one-dimensional Navier-Stokes equations of compressible isentropic gases, *Commun. Pure Appl. Math.*, **46** (1993), 621–665. https://doi.org/10.1002/cpa.3160460502
17. J. Sprekels, S. Zheng, Global solutions to the equations of a Ginzburg-Landau theory for structural phase transitions in shape memory alloys, *Phys. D Nonlinear Phenom.*, **39** (1989), 59–76. https://doi.org/10.1016/0167-2789(89)90039-0

AIMS Press