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Survey

From the local to the nonlocal Brezis-Nirenberg problem: A brief survey

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Abstract: This brief survey is devoted to the famous Brezis-Nirenberg problem, firstly studied in the celebrated paper [1]. Since then, critical equations have been widely studied from many perspectives and in various contexts, including, among others, general local operators, nonlocal operators, mixed local and nonlocal operators, higher-order operators, and operators in non-Euclidean contexts. It is interesting to note that, regardless of the context, most results concerning critical Dirichlet problems are obtained through suitable adaptations of the original argument of Brezis and Nirenberg. The purpose of this paper is to consider the nonlocal counterpart of the Brezis-Nirenberg problem (in which the classical Laplace operator is replaced by its fractional version). Here, we group and summarize the results obtained in [2–6], focusing on the similarities and differences between the local and nonlocal cases, highlighting the innovations and adaptations to consider in the treatment of the fractional case.

Dedicated to Patrizia, who is something like a nonlocal operator: her influence is felt not only locally, but also far away.

Keywords: nonlocal problems; fractional Laplacian; critical nonlinearities; variational methods; compactness condition

1. Introduction

A very interesting area of nonlinear analysis lies in the study of elliptic equations modeled by

$$\begin{cases} -\Delta u - \lambda u = |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $-\Delta$ is the classical Laplace operator, Ω is an open, bounded subset of $\mathbb{R}^n$, $n > 2$, λ is a real parameter and $q > 1$.

Elliptic problems with the Laplace operator are fundamental in applications, since they model physical phenomena at equilibrium, like steady-state heat distribution, electrostatics (potential fields), fluid flow, and diffusion, describing stable states where net flux is zero.

Common research areas and challenges on these topics include the following:

- existence and uniqueness: proving (weak, entropy, or classical) solutions exist and whether they are unique;
- regularity theory: determining how smooth the solutions are;
- boundary conditions: studying Dirichlet (fixed values), Neumann (fixed normal derivative), or mixed conditions, which significantly alter solution properties;
- nonlinearities: incorporating singular terms, superlinear or sublinear terms, subcritical, critical or supercritical growth, requiring advanced techniques and new ideas.

Mathematical research heavily focuses on generalizations of the Laplace operator, like the p-Laplacian and the fractional Laplacian ones, often with different boundary conditions (Dirichlet, Neumann, mixed) and nonlinearities. In these frameworks, research and challenges are again related to exploring existence, regularity, uniqueness, and multiplicity of solutions, using advanced techniques from functional analysis, variational methods and suitable adaptations of the techniques considered when the leading operator is the Laplacian. These variations introduce challenges like singular terms, variable exponents, and fractional orders, expanding the scope beyond classical linear theory.

In recent years, a great attention has been focused on the fractional nonlocal counterpart of problem (1.1), namely

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.2}$$

where $s \in (0,1)$ is fixed, and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy\,, \quad x \in \mathbb{R}^n \tag{1.3}$$

(see for instance [7] and references therein for further details on the fractional Laplacian).

The interest on this topic is both for the pure mathematical research and in view of concrete real-world applications. Indeed, the fractional Laplace operator (together with its generalizations) appears in a very natural way in different frameworks, such as the description of several phenomena like, just to name a few, the thin obstacle problem, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, jump Lévy processes, multiple scattering, minimal surfaces, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, materials science, water waves, optimization, and finance.

The current literature on these topics and on their applications is very interesting and quite large. Motivated by the increasing interest on these subjects, the present survey focuses on some fractional problems involving critical nonlinearities. Precisely, we give a selection of results on critical fractional equations, studied in the recent literature, particularly in relation to the celebrated Brezis-Nirenberg problem.

1.1. The Brezis-Nirenberg problem

The study of boundary-value problems involving critical nonlinearities goes back to the seminal paper [1] by Brezis and Nirenberg, where the authors considered the following equation:

$$\begin{cases} -\Delta u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$, $n > 2$, is open, bounded, λ is a real parameter, and $2^* = 2n/(n-2)$ is the critical Sobolev exponent.

Critical equations often arise in the study of variational problems, partial differential equations, and geometric function theory, proving the existence of solutions that achieve the best possible bounds (extremal functions) or solutions that are not the "simplest" (non-minimal) for complex geometric energy functionals. Examples in these directions are given by the existence of extremal functions for isoperimetric inequalities, Hardy-Littlewood-Sobolev inequalities, trace inequalities and embedding theorems [8–11], the existence of non-minimal solutions for Yang-Mills functionals [12,13], often tied to topological properties, the Rellich's conjecture about the existence of non-minimal solutions for H-systems [14], and the existence of physically relevant solutions in gauge field theory (see [15]).

All in all, in nonlinear analysis and geometric analysis, critical problems appear when proving the existence of important mathematical objects (extremal functions and non-minimal solutions), highlighting deep connections between analysis and geometry, particularly for problems involving curvature, energy, and partial differential equations, and also demonstrating that challenging geometric and analytical problems often possess rich solution scenarios that go beyond the obvious.

In addition to the purely mathematical interest, a motivation for the study of problem (1.4) is given by the so-called Yamabe problem, that is, the problem of the prescribed scalar curvature on a manifold.

In its classical form, first considered in [16], the Yamabe problem can be stated as follows: given a n dimensional compact Riemannian manifold $(\mathcal{M}, g)$, $n > 2$, with scalar curvature $k = k(x)$, find a metric $\tilde{g}$ conformal to g with constant scalar curvature $\tilde{k}$.

If we put

$$\tilde{g} = u^{4/(n-2)}g\,,$$

where $u > 0$ is the conformal factor, then this problem can be formulated as follows: find $u > 0$ satisfying

$$-4\frac{n-1}{n-2}\Delta_{\mathcal{M}}u = \tilde{k}u^{2^*-1} - k(x)u \quad \text{in } \mathcal{M}\,,$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$ with respect to the metric g.

The Yamabe problem can be set also in other contexts provided a notion of scalar curvature can be defined (see, for instance, [17], where a notion of nonlocal scalar curvature is considered).

In [1] Brezis and Nirenberg obtained optimal conditions on the parameter λ for the existence of positive solutions to the problem (1.4), which, after these results, was called the Brezis-Nirenberg problem.

Precisely, for a general open bounded domain $\Omega \subset \mathbb{R}^n$, Brezis and Nirenberg proved that

- if $n \geq 4$, then for any $\lambda \in (0, \lambda_1(-\Delta))$ the critical problem has a positive solution;
- if $n = 3$ then there exists a constant $\lambda_* \in (0, \lambda_1(-\Delta))$, depending on the domain Ω, such that for any $\lambda \in (\lambda_*, \lambda_1(-\Delta))$ the critical problem has a positive solution,

where $\lambda_1(-\Delta)$ is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data.

Since then, critical boundary-value problems have been extensively studied under many different aspects and in various contexts. It is impossible to cite all the bibliography on this subject. In the local setting there is a huge amount of literature on the Brezis-Nirenberg problem (even with general semilinear or quasilinear operators) in which numerous existence, non-existence, and multiplicity results have been obtained. Considering the scope of this review, among other contributions on the subject, we recall that Capozzi et al. in [18] and Gazzola and Ruf in [19] extended the original results of [1], proving that the critical problem (1.4) has a nontrivial solution provided

- $n > 4$ and $\lambda > 0$;
- $n = 4$ and $\lambda > 0$ is not an eigenvalue of the Laplace operator $-\Delta$.

We should also recall the recent contribution [20], where the authors proved that when $n = 3$, the critical problem (1.4) admits infinitely many sign-changing solutions for any $\lambda > 0$.

The Brezis-Nirenberg problem has also been considered in many other contexts, giving rise, in this case too, to a vast and interesting literature. In the framework of nonlocal operators we refer to [2–6, 21–23], for mixed local and nonlocal operators to [24, 25], for the superposition of different local operators to [26], and for higher-order local operators to [27–31]. Finally, for differential operators in non-Euclidean settings we refer to [32] for the hyperbolic spaces, [33–35] for the context of Carnot groups and [36] for the Heisenberg group.

It is interesting to observe that, independently of the setting, the major part of the results concerning critical Dirichlet problems is obtained by suitable adaptations of the original argument by Brezis and Nirenberg (their approach is related to ideas of Aubin and Trudinger; see [37, 38]). This argument is purely variational and the main ingredient seems to be the explicit knowledge of the minimizers in the classical Sobolev inequality involved with the problem (see Section 2 for the details) or, sometimes, their asymptotic behavior. We stress that in many settings it is very difficult to get the explicit expression of these minimizers as well as their asymptotic behavior, and this leads to a not always simple adaptation of the original arguments by Brezis and Nirenberg to the new context.

1.2. The nonlocal Brezis-Nirenberg problem

This survey is devoted to critical boundary-value problems in a nonlocal fractional context. The fractional nonlocal counterpart of the Brezis-Nirenberg problem (1.4) reads as follows:

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2_s^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{1.5}$$

Here, Ω is an open, bounded subset of $\mathbb{R}^n$, $n > 2s$, λ is a real parameter, and $2_s^* = 2n/(n-2s)$ is the fractional critical Sobolev exponent (notice that when $s = 1$, the above exponent reduces to the classical critical Sobolev exponent 2^*). The homogeneous Dirichlet datum in (1.5) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, as it happens in the classical case of the Laplacian, consistently with the nonlocal nature of the operator $(-\Delta)^s$.

In the works [2–6], the authors proved that the famous result by Brezis and Nirenberg for the Laplace equation continues to hold also in the nonlocal setting of (1.5), that is, they extended all the classical results known for problem (1.4) to the case of its nonlocal fractional version (1.5), adapting the strategy used in the Laplacian framework to the fractional Laplace setting (and, in general, to the setting of nonlocal integrodifferential operators). Precisely, they showed that the nonlocal Brezis-Nirenberg problem (1.5) admits a nontrivial solution provided

- $n > 4s$ and $\lambda > 0$;
- $n = 4s$ and $\lambda > 0$ is different from the eigenvalues of $(-\Delta)^s$;
- $2s < n < 4s$ and $\lambda > 0$ is sufficiently large,

extending the existence results already known in the local setting (see [1, 18, 19, 39]) to the nonlocal one.

The aim of this brief survey is to group and summarize the results from [2–6]. We will focus on the similarities and differences between the local and nonlocal cases, highlight the innovations and adaptations to be considered in the treatment of the nonlocal case, in order to offer an overview on the Brezis-Nirenberg problem in the fractional Laplacian setting.

In the next sections the ideas will be provided and the techniques used to obtain the result of existence will be discussed, without giving proofs, for which we refer to the works [2–6].

The present paper is organized as follows. In Section 2 we recall the original argument used in the seminal paper by Brezis and Nirenberg, while in Section 3 we discuss the adaptation of this strategy to the nonlocal fractional critical context.

2. The original argument by Brezis and Nirenberg

This section is devoted to the techniques and methods introduced in the seminal paper [1] for getting optimal conditions on the parameter λ for the existence of positive solutions to the critical problem (1.4).

The original argument considered in [1] by Brezis and Nirenberg is purely variational and consists in the study of the critical points of the energy functional $\mathcal{J}_\lambda : H^1_0(\Omega) \to \mathbb{R}$ defined as

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx, \tag{2.1}$$

where $H^1_0(\Omega)$ is the classical Sobolev space.

The exponent 2^* is critical in the sense that it is the limiting Sobolev exponent in the embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$, which is not compact, but just continuous. Due to this, for the functional $\mathcal{J}_\lambda$, the Palais-Smale condition does not hold true globally, and this creates problems when trying to find critical points via classical variational methods.

Precisely, for $\mathcal{J}_\lambda$ the Palais-Smale condition holds true only in a suitable range related to the best Sobolev constant in the embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$, that is, at any level c satisfying the following inequality:

$$c < \frac{1}{n} S^{n/2},$$

where S is the best Sobolev constant given by

$$S := \inf_{v \in H^1_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla v(x)|^2\,dx}{\displaystyle\Big(\int_\Omega |v(x)|^{2^*}dx\Big)^{2/2^*}}.$$

In [1], what was decisive in overcoming this lack of compactness was proving that there exists a function $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega |\nabla u_0(x)|^2 - \lambda \int_\Omega |u_0(x)|^2 \, dx < S\Big(\int_\Omega |u_0(x)|^{2^*} dx \Big)^{2/2^*} . \tag{2.2}$$

The main ingredient for the proof of (2.2) is the explicit knowledge of extremal functions for the Sobolev inequality in $\mathbb{R}^n$ given by

$$\tilde{u}(x) = \kappa \Big(\mu^2 + |x|^2 \Big)^{-(n-2)/2}, \quad x \in \mathbb{R}^n ,$$

with $\kappa \in \mathbb{R} \setminus \{0\}$ and $\mu > 0$.

Starting from $\tilde{u}$, Brezis and Nirenberg defined the family of functions u_ε as follows:

$$u_\varepsilon(x) = \eta(x) \varepsilon^{-(n-2)/2} \, \bar{u} \left(\frac{x/\varepsilon}{S^{1/(2s)}} \right), \qquad x \in \mathbb{R}^n$$

for any $\varepsilon > 0$, where

$$\bar{u}(x) = \frac{\tilde{u}(x)}{\|\tilde{u}\|_{L^{2^*}(\Omega)}} ,$$

and $\eta \in C^\infty(\Omega)$ is such that $0 \le \eta \le 1$ in Ω, $\eta \equiv 1$ in B_δ, and $\eta \equiv 0$ in $\Omega \setminus B_{2\delta}$. Here, $B_\delta = B(0, \delta)$ (without loss of generality, we may assume that $0 \in \Omega$).

They proved that the function u_ε satisfies the following crucial estimates:

$$\int_\Omega |\nabla u_\varepsilon(x)|^2 \, dx \le S^{n/2} + O(\varepsilon^{n-2}), \tag{2.3}$$

$$\int_\Omega |u_\varepsilon(x)|^2 \, dx \ge \begin{cases} C\varepsilon^2 + O(\varepsilon^{n-2}) & \text{if} \quad n > 4 \\ C\varepsilon^2 \Big| \log \varepsilon \Big| + O(\varepsilon^2) & \text{if} \quad n = 4 \\ C\varepsilon + O(\varepsilon^2) & \text{if} \quad n = 3 , \end{cases} \tag{2.4}$$

and

$$\int_\Omega |u_\varepsilon(x)|^{2^*} \, dx = S^{n/2} + O(\varepsilon^n), \tag{2.5}$$

as $\varepsilon \to 0$, for some positive constant C depending only on n.

Thanks to the estimates (2.3)–(2.5) and the application of the Mountain Pass Theorem (see [40]), Brezis and Nirenberg got the existence of positive solutions for (1.4) for any positive λ less than the first eigenvalue of the Laplace operator with homogeneous Dirichlet boundary data. Adapting the same methods and using the Linking Theorem (see [41, 42]), the existence of nontrivial solutions for (1.4) was later proved also for larger values of λ, getting the following final existence result for (1.4) (see [1, 18, 19]):

Theorem 1. *The critical problem* (1.4) *has a nontrivial solution provided*

- $n > 4$ *and* $\lambda > 0$*;*
- $n = 4$ *and* $\lambda > 0$ *is not an eigenvalue of* $-\Delta$.

For further details on these results and related topics we refer also to the monographs [43, 44].

3. The Brezis-Nirenberg strategy adapted to the nonlocal framework

In Section 2, we recalled the original arguments by Brezis and Nirenberg for the treatment of the Laplacian setting. Here, we consider its nonlocal counterpart, focusing on the similarities and differences between the local and nonlocal cases and highlighting the innovations and adaptations to consider in the study of the fractional case.

Even in the fractional context the nature of the critical problem (1.5) is variational. Indeed, weak solutions of (1.5) can be seen as critical points of the energy functional $\mathcal{J}_{s,\lambda} : X_0 \to \mathbb{R}$ defined as

$$\mathcal{J}_{s,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2_s^*}\int_\Omega |u(x)|^{2_s^*}dx\,, \tag{3.1}$$

where the functional space X_0 is the Hilbert space defined as

$$X_0 = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$$

and endowed with the norm given by (see [45, Lemma 7])

$$X_0 \ni v \mapsto \|v\|_{X_0} = \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}. \tag{3.2}$$

Here, $H^s(\mathbb{R}^n)$ is the usual fractional Sobolev space (see, for instance, [7]), endowed with the so-called *Gagliardo norm*:

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}, \tag{3.3}$$

where $\|\cdot\|_{L^q(\mathbb{R}^n)}$ denotes the usual norm in the Lebesgue space $L^q(\mathbb{R}^n)$, $q \geq 1$.

We remark that the definition of X_0 allows one to develop a functional analytical setting that is inspired by (but not equivalent to) the fractional Sobolev spaces, in order to correctly encode the Dirichlet boundary datum in the variational formulation of problem (1.5) (for the details, see [45]).

As it happens in the Laplacian context, also in the nonlocal critical setting there is a lack of compactness. Indeed, also in this case, the Palais-Smale condition does not hold true globally, but only in a suitable range related to the best fractional Sobolev constant in the embedding $X_0 \hookrightarrow L^{2_s^*}(\mathbb{R}^n)$, that is at any level c, with c satisfying

$$c < \frac{s}{n} S_s^{n/(2s)}\,,$$

where S_s is the best fractional Sobolev constant defined as

$$S_s := \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\Big(\displaystyle\int_{\mathbb{R}^n} |v(x)|^{2_s^*}dx\Big)^{2/2_s^*}}\,.$$

Obviously, this lack of compactness is an obstruction to the application of a minimax argument to the energy functional $\mathcal{J}_{s,\lambda}$. In order to overcome this difficulty, the strategy to be used can be summarized as follows: the existence of nontrivial solutions can be obtained through the Mountain Pass or the Linking Theorems (see [40–42]) applied to the functional $\mathcal{J}_{s,\lambda}$ showing the following:

- A compactness conditions for $\mathcal{J}_{s,\lambda}$: the functional $\mathcal{J}_{s,\lambda}$ satisfies the $(PS)_c$-condition for any c such that $c < \frac{s}{n}S_s^{n/(2s)}$;
- A geometric structure for $\mathcal{J}_{s,\lambda}$: the functional $\mathcal{J}_{s,\lambda}$ has the geometry required by the Mountain Pass or the Linking Theorem;
- An estimate of the critical level of $\mathcal{J}_{s,\lambda}$: the Mountain Pass or the Linking critical level of $\mathcal{J}_{s,\lambda}$ lies below the threshold where the $(PS)_c$-condition holds true.

As it happens in the local case, also in the nonlocal context, the compactness and geometry of the energy functional associated to the problem are proved using the properties of the space in which the functional is defined. Precisely, the compactness property and the geometry required by the classical minimax theorems are satisfied by $\mathcal{J}_{s,\lambda}$, thanks to the fact that we work in the functional space X_0, whose definition encodes also the Dirichlet boundary condition, and to the embeddings properties of X_0 into the classical Lebesgue spaces, that is,

$$\text{the embedding } X_0 \hookrightarrow L^\nu(\mathbb{R}^n) \text{ is compact for any } \nu \in [1, 2_s^*)$$

and

$$\text{the embedding } X_0 \hookrightarrow L^{2_s^*}(\mathbb{R}^n) \text{ is continuous.}$$

We stress that it is important here that the homogeneous Dirichlet datum $u = 0$ in $\mathbb{R}^n \setminus \Omega$ is encoded in the definition of X_0, similarly to what happens with its local counterpart $H_0^1(\Omega)$.

What is delicate in our strategy is proving the estimate of the critical level of the functional $\mathcal{J}_{s,\lambda}$.

3.1. Sharp test functions and a crucial estimate

The tricky part of the Brezis-Nirenberg original arguments is related to the inequality (2.2), which is in connection with an estimate of the critical level of the energy functional associated with the problem. The same difficulty appears in the nonlocal setting. The present subsection is devoted to this.

As first shown in [4], in order to get the estimate of the critical level of the functional $\mathcal{J}_{s,\lambda}$, it is enough to prove that there exists $u_{0,s} \in X_0$ such that

$$S_{s,\lambda}(u_{0,s}) < S_s := \inf_{v \in H^s(\mathbb{R}^n)\setminus\{0\}} S_s(v), \tag{3.4}$$

where

$$X_0 \ni v \mapsto S_{s,\lambda}(v) := \frac{\displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_{\mathbb{R}^n} |v(x)|^2\,dx}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2_s^*} dx\Big)^{2/2_s^*}},$$

and

$$S_s(v) := S_{s,0}(v).$$

For this purpose, as it happens in the classical local case, the idea consists of considering the function

$$\tilde{u}_s(x) = \kappa\Big(\mu^2 + |x|^2\Big)^{-(n-2s)/2}, \quad x \in \mathbb{R}^n,$$

where $\kappa \in \mathbb{R} \setminus \{0\}$ and $\mu > 0$. As proved in [46, Theorem 1.1], the function $\tilde{u}_s$ attains the fractional best critical Sobolev constant S_s.

Then, we construct an explicit solution of the limiting problem

$$(-\Delta)^s u = |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^n$$

as follows:

$$u_s^*(x) = \bar{u}_s\left(\frac{x}{S_s^{1/(2s)}}\right), \qquad x \in \mathbb{R}^n,$$

where

$$\bar{u}_s(x) = \frac{\tilde{u}_s(x)}{\|\tilde{u}_s\|_{L^{2_s^*}(\mathbb{R}^n)}}, \quad x \in \mathbb{R}^n.$$

Later, starting from u_s^*, we define the family of functions $U_{\varepsilon,s}$ as follows:

$$U_{\varepsilon,s}(x) = \varepsilon^{-(n-2s)/2}\, u_s^*(x/\varepsilon), \qquad x \in \mathbb{R}^n$$

for any $\varepsilon > 0$.

Now, for our aims, we need to appropriately put $U_{\varepsilon,s}$ to zero outside Ω. For this, let us fix $\delta > 0$ such that $B_{4\delta} \subset \Omega$, and let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $0 \le \eta \le 1$ in $\mathbb{R}^n$, $\eta \equiv 1$ in B_δ, and $\eta \equiv 0$ in $CB_{2\delta}$, where $B_\delta = B(0,\delta)$ and $CB_\delta = \mathbb{R}^n \setminus B_\delta$.

Finally, for every $\varepsilon > 0$, we put

$$u_{\varepsilon,s}(x) = \eta(x)U_{\varepsilon,s}(x), \quad x \in \mathbb{R}^n.$$

It is easily seen that

$$u_{\varepsilon,s} \in H^s(\mathbb{R}^n),\ u_{\varepsilon,s} \ge 0 \text{ a.e. in } \mathbb{R}^n \text{ and } u_{\varepsilon,s} = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega,$$

so that

$$u_{\varepsilon,s} \in X_0.$$

For our purpose, it is important that the function $u_{\varepsilon,s}$ satisfies the following crucial estimates:

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \le S_s^{n/(2s)} + O(\varepsilon^{n-2s}), \tag{3.5}$$

$$\int_{\mathbb{R}^n} |u_{\varepsilon,s}(x)|^2\,dx \ge \begin{cases} C_s\varepsilon^{2s} + O(\varepsilon^{n-2s}) & \text{if} \quad n > 4s \\ C_s\varepsilon^{2s}\big|\log\varepsilon\big| + O(\varepsilon^{2s}) & \text{if} \quad n = 4s \\ C_s\varepsilon^{n-2s} + O(\varepsilon^{2s}) & \text{if} \quad n < 4s, \end{cases} \tag{3.6}$$

and

$$\int_{\mathbb{R}^n} |u_{\varepsilon,s}(x)|^{2_s^*}\,dx = S_s^{n/(2s)} + O(\varepsilon^n), \tag{3.7}$$

as $\varepsilon \to 0$, for some positive constant C_s depending on s.

Observe that (3.5)–(3.7) are the nonlocal counterparts of (2.3)–(2.5), respectively. In particular, we stress that in the nonlocal setting, the estimate (3.5) of the Gagliardo seminorm of $u_{\varepsilon,s}$ plays the role of the (2.3). With respect to the classical case of the Laplacian, in the nonlocal setting, the proof of (3.5) is more delicate than the one of (2.3), due to the nonlocal nature of the operator $(-\Delta)^s$. Indeed, in order to control the Gagliardo seminorm of $u_{\varepsilon,s}$, pointwise estimates on $u_{\varepsilon,s}$ are not enough. Moreover, we also need some integral estimates that allow us to control the interactions between B_δ and $\mathbb{R}^n \setminus B_\delta$.

The proof of (3.5) is, in several points, different than the one of the classical case of the Laplacian: these additional complications are not only technical, but, sometimes, they somehow reflect the different distribution of the energy density due to the nonlocal interactions. For instance, the far-away part of the energy in our case, though it is small, gives a contribution to the functional that cannot be neglected, since it is of order comparable with the perturbation. For further comments and the detailed calculations, we refer to [4].

Using the estimates (3.5)–(3.7), it can be proved that (3.4) is satisfied by taking $u_{0,s} = u_{\varepsilon,s}$ when ε is sufficiently small. The arguments used here are very delicate and require a subtle analysis and particular attention both to the dimension n of the space and to the resonant/non-resonant case.

Having all the ingredients (compactness condition, geometric structure, and estimate of the critical level), the minimax argument can be applied to the energy functional $\mathcal{J}_{s,\lambda}$ and, as a consequence, the Mountain Pass or the Linking Theorems give the existence of a nontrivial solution for the nonlocal Brezis-Nirenberg problem (1.5). In conclusion, the following existence result (see [2–6]) holds true:

Theorem 2. *The critical problem* (1.5) *has a nontrivial solution provided*

- $n > 4s$ *and* $\lambda > 0$*;*
- $n = 4s$ *and* $\lambda > 0$ *is different from the eigenvalues of* $(-\Delta)^s$*;*
- $2s < n < 4s$ *and* $\lambda > 0$ *is sufficiently large.*

Note that Theorem 2 can be seen as the nonlocal counterpart of Theorem 1. As it happens in the classical context, it can be proven that, when $\lambda > 0$ is less than the first eigenvalue of $(-\Delta)^s$ with homogeneous Dirichlet datum, the solution of (1.5) obtained in Theorem 2 is positive. For further details on these results and their generalization to integro-differential operators and related topics, we refer also to the monographs [47].

As mentioned in the Introduction, the celebrated work [1] of Brezis and Nirenberg has given rise to a wide literature in various contexts. Furthermore, it has also generated numerous works that have raised questions regarding the multiplicity of solutions, as well as their regularity. It would be interesting to see whether, in this case too, the arguments and strategies used in the various contexts are the same across all contexts or whether there are significant differences.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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