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Research article

Modeling and analysis of a prey-predator system with prey habitat selection in an environment subject to stochastic disturbances

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Abstract: In natural ecosystems, the external environment is constantly changing, and is affected by various factors, thus presenting a certain degree of randomness and uncertainty. Therefore, having a suitable habitat is essential for the reproductive success of many species. Understanding the impact of habitat selection provides valuable insights into how species locate and adapt to suitable living environments based on their specific needs. For this purpose, a prey-predator system model with prey habitat selection in an environment subject to stochastic disturbances is formulated. The properties of the proposed model without and with stochastic disturbances are investigated, including the existence of a unique ergodic stationary distribution, the stochastically ultimate bounded-ness of the solutions, and the extinction and persistence of the populations. The study demonstrates that prey can persist at a low intensity noise, whereas stronger stochastic disturbances may lead to the extinction of both the prey and predator species. To illustrate the theoretical results, numerical simulations are presented step by step. This work provides a theoretical reference for further studies on populations with habitat selection in an environment subject to stochastic disturbances.

Keywords: stochastic prey-predator model; habitat selection; stationary distribution; stochastically ultimate bounded-ness; persistence

1. Introduction

The study of population dynamics is an important branch of ecosystem models and has attracted the special interest of ecologists and applied mathematics at historical and contemporary stages [1, 2]. Mathematical models play a key role in understanding changes in biological systems [3, 4] and the effect of control measures [5]. In ecosystems, predation is an important way for species to interact and depend on each other. The pioneer work in mathematics that describes the interplay or dynamics between predator and prey belongs to the Lotka [6] and Volterra [7], which is well known as the Lotka-Voterra model. Subsequently, scholars have refined and enhanced the model in the application of different scenarios and proposed different kinds of uptake functions to characterize the predator's hunting ability [8–11]. The uptake function with a Holling type is representative and applicable to predation phenomena of different species, ranging from lower organisms such as algae and unicellular organisms to invertebrates and vertebrates, such as Holling-I [12], Holling-II [13], Holling-III [14], Holling-(n+1) [15] and so on. The Holling-III functional response showed that the predator's predation reached saturation when the number of prey reached a certain level [14], which is consistent with most of the actual situations. Therefore, it is of a great significance to study the dynamics of a predator-prey model with a Holling-III functional response function.

Over the long period of evolution, prey have evolved with a series of anti-predator behaviors. The anti-predator behaviors of prey can be classified into two categories: (i) defensive counterattacks [16] and (ii) morphological or behavioral changes, including camouflage [17], seeking refuge [18, 19], and so on. The benefit of anti-predator behaviors is that they reduce the risk of predation. Red colobus monkeys show siege when they are threatened by chimpanzees [16], cuttlefish choose to camouflage by matching background features [17], northern pigtailed macaques select trees with abundant branches and elevated sleeping locations in order to minimize the threat of predation [18], and white-headed langurs opt for cliff edges or caves as their nocturnal habitats, thus seeking refuge from predators such as leopard cats [19]. Anti-predator behaviors are widespread in nature and are essential to consider in the modeling process. Tang and Xiao [20] proposed a model of the first type of anti-predation behaviors by adding anti-predation terms to reduce the rate of predator growth. It was demonstrated that anti-predator behaviors inhibit predator-prey oscillations and decreased the likelihood of the coexistence between prey and predator [21]. However, many animals lack strong defenses against attacks and they often exhibit the second type anti-predation behavior. An anti-predator model for habitat selection or vigilance was proposed by Ives and Dobson [22], which takes the form:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left[r\left(1 - \dfrac{x}{K}\right) - v - e^{-\varepsilon v}\dfrac{qy}{1 + ax}\right], \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{ce^{-\varepsilon v}qxy}{1 + ax} - my. \end{cases} \tag{1.1}$$

The existence of the anti-predator effect modulates the reproductive capacity of prey through the factor vx. For example, yellow-eyed juncos exhibit a heightened vigilance by dedicating more time to scanning for predators upon the release of a trained Harris hawk in their vicinity, thus resulting in a reduced time spent foraging for food [23]. It is hypothesized that the diminished feeding rates will ultimately either heighten the likelihood of starvation or hinder the ability to successfully rear offspring. Werner et al. discovered that when bass are present, bluegills will select not rich in resources but more hidden habitats, moreover, bluegills reared in the presence of bass attain only approximately 80% of the mass compared to those raised without the bass present [24]. Additionally, predators pay a price for the anti-predator effect, which is defined by the term $e^{-\varepsilon v}$ on the predator's hunting ability. Based on this consideration, a prey-predator model with a prey habitat selection is considered in this study.

In reality, population systems are typically subjected to a wide range of random disturbances. May has pointed out that the rate of growth, environmental carrying capacity, coefficient of competition, and other pertinent system parameters are all impacted by environmental noise to differing degrees [25]. The deterministic model has certain limitations, in some situations, frequently yields disappointing results. To more accurately characterize and predict the population ecosystems, it is necessary to investigate population models with stochastic disturbances. In natural system, the stochastic disturbances can be described by white noise [26], colored noise [27], Levy jumps [28], Markov switching [29], and others. Population systems are often subject to environmental fluctuations. Generally speaking, such fluctuations could be modeled by a colored noise. Moreover, if the colored noise is not strongly correlated, then it can be approximately modeled by a white noise, and the approximation works quite well [30]. Mao et al. observed that even the smallest white noise can suppress the population explosion and provided a classical method for the uniqueness of the solution [31]. Ji and Jiang [32] explored a stochastic predatory system that incorporated Beddington-DeAngelis uptake function and discussed the asymptotic property of the system model. Liu et al. [33] put forward a prey-predator system model which followed the Holling type-II uptake function in an environment that involved stochastic disturbances. Zhang et al. [34] analyzed a prey-predator system model by considering the Holling type-II uptake function and hyperbolic mortality in an environment subject to stochastic disturbances. Liu [35] analyzed the dynamics of a stochastic regime-switching predator-prey model with modified Leslie-Gower Holling-type II schemes and prey harvesting. Li et al. [36] proposed a Melnikov-type method for chaos in a class of hybrid piecewise-smooth systems with impact and noise excitation under unilateral rigid constraint. In the present investigation, it is assumed that the environment is influenced by white noise in the modeling process.

The structure of this paper is organized as follows: in Section 2, a prey-predator system model with prey habitat selection and stochastic disturbances is formulated, which is followed by a presentation of some basic notations, definitions, and crucial lemmas used in this study; in Sections 3 and 4, the basic properties and dynamic behavior of the system model with or without stochastic disturbances are investigated; in Section 5, numerical simulations with discussions are presented to illustrate the main results; and finally, the work is summarized and further research directions are put forward.

2. Mathematical model and basic knowledge

2.1. Mathematical model

When prey habitat selection is considered, the prey-predator model can be described as follows:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)\left[r\left(1-\dfrac{x(t)}{K}\right) - v - e^{-(vm+m_0)}\dfrac{x(t)y(t)}{1+ax(t)^2}\right] := xF_1(x,y), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t)\left[ce^{-(vm+m_0)}\dfrac{x(t)^2}{1+ax(t)^2} - by(t) - d\right] := yF_2(x,y), \end{cases} \tag{2.1}$$

where

- x — prey's densities;
- y — predator's densities;
- r — prey's intrinsic growth rate;
- K — prey's environmental carrying capacity;

- v — intensity of anti-predator effect;
- c — conversion efficiency;
- b — interspecific competition coefficient;
- d — natural mortality rate;
- m — efficiency of anti-predator behavior;
- e^{-m_0} — maximum predation rate.

Considering the changes in the natural environment and the universality of random disturbances, we introduce white noise into the Model (2.1) in order to analyze the effect of random disturbances on the Model (2.1). Let

$$r \to r + \sigma_1 \dot{B}_1(t), \quad -d \to -d + \sigma_2 \dot{B}_2(t),$$

where σ_i represents the white noise intensity, which describes the strength of random disturbance in the environment. $B_i(t)$ stands for a Brownian motion, $i = 1, 2$. Then, the model subject to stochastic disturbances is represented as follows:

$$\begin{cases} \mathrm{d}x(t) = x(t)\left[r\left(1 - \dfrac{x(t)}{K}\right) - v - e^{-(vm+m_0)}\dfrac{x(t)y(t)}{1+ax(t)^2}\right]\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = y(t)\left[ce^{-(vm+m_0)}\dfrac{x(t)^2}{1+ax(t)^2} - by(t) - d\right]\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t). \end{cases} \tag{2.2}$$

All the parameters in Model (2.1) and Model (2.2) are positive. In addition, it is assumed that $v \le r$ and $c \ge \underline{c} \triangleq ade^{vm+m_0}$ in Model (2.1) for biological restriction.

2.2. Basic knowledge

Let us denote the following:

$$\langle u(t)\rangle \triangleq \lim_{t\to+\infty}\int_0^t t^{-1}u(s)\mathrm{d}s, \ \ \langle u(t)\rangle^* \triangleq \lim_{t\to+\infty}\sup\int_0^t t^{-1}u(s)\mathrm{d}s, \ \ \langle u(t)\rangle_* \triangleq \lim_{t\to+\infty}\inf\int_0^t t^{-1}u(s)\mathrm{d}s.$$

Consider the following:

$$\mathrm{d}\mathbf{U}(t) = \mathbf{F}(\mathbf{U}(t), t)\mathrm{d}t + \mathbf{G}(\mathbf{U}(t), t)\mathrm{d}\mathbf{B}(t), \tag{2.3}$$

where $\mathbf{B}(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ equipped with $\{\mathcal{F}_t\}_{t\ge 0}$, where $\mathbf{F}(\mathbf{U}(t), t) \in \mathbb{R}^l \times [0, +\infty)$, $\mathbf{G}(\mathbf{U}(t), t)_{l\times q}$ is a matrix. Define $\mathcal{L}$ as follows:

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{l} F_i(\mathbf{U}(t), t)\frac{\partial}{\partial U_i} + \frac{1}{2}\sum_{i,j=1}^{l}[\mathbf{G}^T(\mathbf{U}(t), t)\mathbf{G}(\mathbf{U}(t), t)]_{i,j}\frac{\partial^2}{\partial U_i \partial U_j}.$$

For $V(\mathbf{U}(t), t) \in C^{2,1}(\mathbb{R}^l \times [0, +\infty), \mathbb{R}_+)$, there is

$$\mathcal{L}V(\mathbf{U}(t), t) = \frac{1}{2}\mathrm{TRACE}[\mathbf{G}^T(\mathbf{U}(t), t)V_{\mathbf{UU}}(\mathbf{U}(t), t)\mathbf{G}(\mathbf{U}(t), t)] + V_t(\mathbf{U}(t), t) + V_{\mathbf{U}}(\mathbf{U}(t), t)\mathbf{F}(\mathbf{U}(t), t),$$

where $V_t = \dfrac{\partial V}{\partial t}$, $V_{\mathbf{U}} = \left(\dfrac{\partial V}{\partial U_1}, \dfrac{\partial V}{\partial U_2}, \cdots, \dfrac{\partial V}{\partial U_l}\right)$, $V_{\mathbf{UU}} = \left(\dfrac{\partial^2 V}{\partial U_i \partial U_j}\right)_{l\times l}$. Then,

$$\mathrm{d}V(\mathbf{U}(t), t) = V_{\mathbf{U}}(\mathbf{U}(t), t)\mathbf{G}(\mathbf{U}(t), t)\mathrm{d}\mathbf{B}(t) + \mathcal{L}V(\mathbf{U}(t), t)\mathrm{d}t.$$

Definition 1 (Stochastically ultimate boundedness [31]). *The solution* $(u(t), v(t))$ *is defined as having the property of stochastically ultimate boundedness if for any* $\varepsilon \in (0, 1)$, *there exists a constant* $\chi > 0$ *such that*

$$\limsup_{t\to+\infty} P\{\|(u(t), v(t))\|_2 > \chi\} < \varepsilon, \quad \forall (u_0, v_0) \in \mathbb{R}^2_+.$$

Definition 2 (Extinction and persistence [26]). *The species u is defined as extinction when* $\lim\limits_{t\to+\infty} u(t) = 0$, *weakly persistence (in the mean) when* $\langle u(t)\rangle^* > 0$, *strong persistence (in the mean) when* $\langle u(t)\rangle_* > 0$, *persistence (in the mean) when* $\langle u(t)\rangle > 0$, *and weak persistence when* $\lim\limits_{t\to+\infty} \sup u(t) > 0$.

Assume that $\mathbf{U}(t) \in E_l$ is a regular time-homogeneous Markov process characterized by the following stochastic differential equation:

$$\mathrm{d}\mathbf{U}(t) = b(\mathbf{U})\mathrm{d}t + \sum_{k=1}^{q} g_r(\mathbf{U})\mathrm{d}\mathbf{B}_r(t).$$

The diffusion matrix of the process $\mathbf{U}(t)$ is denoted as follows:

$$\Lambda(\mathbf{U}) = \left(\lambda_{ij}(\mathbf{U})\right)_{l\times l}, \quad \lambda_{ij}(\mathbf{U}) = \sum_{k=1}^{q} g_r^i(\mathbf{U}) g_r^j(\mathbf{U}).$$

Lemma 1 (Ergodic stationary distribution [37]). *For a given Markov process* $\mathbf{U}(t) \in E_l$, *it is called possessing a unique ergodic stationary distribution* $\mu(\cdot)$ *if* $\exists I \subset E_l$ *with boundary* Γ *such that i)* $\min(eig(\Lambda(\mathbf{U})))$ *is bounded in I and some neighborhood of I; ii) For* $u \in E_l \backslash I$, *if a path originating from u can reach I in a finite average time* τ *and* $\sup_{u\in S} E_u\tau < \infty$, $\forall S \subset E_l$, *where S is a compact subset.*

Remark 1. *To establish the condition i), it is sufficient to demonstrate that* $\exists\theta > 0$ *such that*

$$\sum_{i,j=1}^{l} a_{ij}(u)\xi_i\xi_j \geq \theta\|\xi\|^2, u \in I, \xi \in \mathbb{R}^l;$$

To establish the condition ii), it is crucial to demonstrate the existence of a neighborhood I along with $V \in C^2$ *such that* $\mathcal{L}V(u) < 0$, $\forall u \in E_l \backslash I$.

Lemma 2 ([26]). *For a given* $u(t) \in C[\Omega \times [0, +\infty), R^+]$:

(1) If $\exists \rho_0, T > 0$ *with*

$$u(t) \leq \exp\left(\rho t - \rho_0 \int_0^t u(s)\mathrm{d}s + \sum_{i=1}^{n} \alpha_i B_i(t)\right)$$

for $t \geq T$, *then*

$$\begin{cases} \langle u\rangle^* \leq \dfrac{\rho}{\rho_0}, & \rho \geq 0 \\ \lim\limits_{t\to+\infty} u(t) = 0, & \rho < 0 \end{cases}$$

(2) If $\exists \rho_0$, *T and ρ with*

$$\ln u(t) \geq \rho t - \rho_0 \int_0^t u(s) \mathrm{d}s + \sum_{i=1}^{n} \alpha_i B_i(t)$$

for $t \geq T$, then $\langle u \rangle_* \geq \dfrac{\rho}{\rho_0}$.

Lemma 3 ([37]). *For $T_0 > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$, it has*

$$P\left\{ \sup_{0 \leq t \leq T_0} \left[\int_0^t g(s) dB(s) - \frac{\alpha_1}{2} \int_0^t |g(s)|^2 ds \right] > \alpha_2 \right\} \leq e^{-\alpha_1 \alpha_2}.$$

3. Dynamic properties of Model (2.1)

As the population densities are non-negative, the discussion on Model (2.1) is restricted in the region $\mathbb{R}^2/\mathbb{R}^2_-$. The equilibrium of Model (2.1) satisfies the following:

$$\begin{cases} x\left[r\left(1 - \dfrac{x}{K}\right) - v - e^{-(vm+m_0)} \dfrac{xy}{1+ax^2} \right] = 0, \\ y\left[ce^{-(vm+m_0)} \dfrac{x^2}{1+ax^2} - by - d \right] = 0. \end{cases}$$

Obviously, Model (2.1) always has two equilibria: $O(0,0)$ and $E_B(\overline{K}, 0)$, where $\overline{K} \triangleq (1 - v/r)K$. $O(0,0)$ is a saddle and constantly unstable.

Define the following:

$$\psi(x) \triangleq (r-v)e^{vm+m_0}(1+ax^2)\left(\frac{1}{x} - \frac{1}{\overline{K}}\right), \quad \phi(x) \triangleq \frac{1}{b}\left(ce^{-vm-m_0}\frac{x^2}{1+ax^2} - d\right).$$

Then, $x = 0$ and $y = \psi(x)$ are two x-isolines, $y = 0$ and $y = \phi(x)$ are two y-isolines.

Since

$$\psi'(x) = \frac{(r-v)e^{vm+m_0}}{x^2}\left(-\frac{2a}{\overline{K}}x^3 + ax^2 - 1\right),$$

then it has $\psi'(x) \to -\infty$ as $x \to 0$. Denote $\overline{v} \triangleq r(1 - 3\sqrt{3/a\overline{K}^{-2}})$. For $v \geq \overline{v}$, there is $\psi'(x) \leq 0$ for $x \in (0, \overline{K}]$; for $0 \leq v < \overline{v}$, it has $\psi'(\overline{K}/3) > 0$. Then, there exists $x_1 \in (0, \overline{K}/3)$ and $x_2 \in (\overline{K}/3, \overline{K})$ such that $\psi(x_1) = \psi(x_2) = 0$.

Similarly, it has $\phi(0) = -d/b$, $\phi'(x) > 0$ for $x \in [0, \overline{K}]$ and $\phi(x) = 0$ if and only if $x = \overline{x} \triangleq \sqrt{\frac{de^{vm+m_0}}{c-ade^{vm+m_0}}}$. Obviously, $\overline{x} < \overline{K}$ if and only if $c > \overline{c} \triangleq de^{vm+m_0}(a + \overline{K}^{-2})$.

Theorem 1. *The boundary equilibrium $E_B(\overline{K}, 0)$ is globally asymptotically stable for $\underline{c} \leq c \leq \overline{c}$.*

Proof. At $E_B(\overline{K}, 0)$, the Jacobian matrix is as follows:

$$J_{E_B(\overline{K},0)} = \begin{pmatrix} v - r & -e^{-(vm+m_0)}\dfrac{\overline{K}^2}{1+a\overline{K}^2} \\ 0 & ce^{-(vm+m_0)}\dfrac{\overline{K}^2}{1+a\overline{K}^2} - d \end{pmatrix},$$

and its characteristic roots are as follows:

$$\lambda_1 = v - r < 0, \quad \lambda_2 = d\left(\frac{c}{\bar{c}} - 1\right).$$

If $c < \bar{c}$, then there is $\lambda_2 < 0$, then $E_B(\overline{K}, 0)$ is a node and locally asymptotically stable. If $c = \bar{c}$, there is $\lambda_2 = 0$, then $E_B(\overline{K}, 0)$ is a saddle-node. Since for $c \leq \bar{c}$, there is $\phi(x) < 0$, i.e., $\mathrm{d}y/\mathrm{d}t < 0$ for $x \in [0, \overline{K})$, then $y(t) \to 0$ as $t \to \infty$. While $\mathrm{d}x/\mathrm{d}t > 0$ for $x \in [0, K)$ and $y = 0$, then $x(t) \to K$, i.e., $E_B(\overline{K}, 0)$ is globally asymptotically stable. □

For $c > \bar{c}$, there is $\bar{x} < \overline{K}$. In such case, $E_B(\overline{K}, 0)$ is unstable.

Theorem 2. *For $c > \bar{c}$, Model* (2.1) *has at least one and no more than three interior equilibria. Moreover, if $\bar{v} \leq v < r$, then the interior equilibrium is unique. In addition, the interior equilibrium $\widehat{E}(\hat{x}, \hat{y})$ is locally asymptotically stable if and only if $\phi(\hat{x}) < \psi'(\hat{x}) < bc\hat{y}/(b\hat{y} + d)$.*

Proof. Since there are $\psi(\bar{x}) > 0$ and $\psi(\overline{K}) = 0$ for $c > \bar{c}$, $\phi(\bar{x}) = 0$, $\phi(\overline{K}) > 0$, according to the mediocrity theorem, there exists at least one $x^* \in (\bar{x}, \overline{K})$ such that $\psi(x^*) = \phi(x^*)$, i.e., Model (2.1) has at least one interior equilibrium $E^*(x^*, y^*)$, where $y^* = \psi(x^*)$, as illustrated in Figure 1.

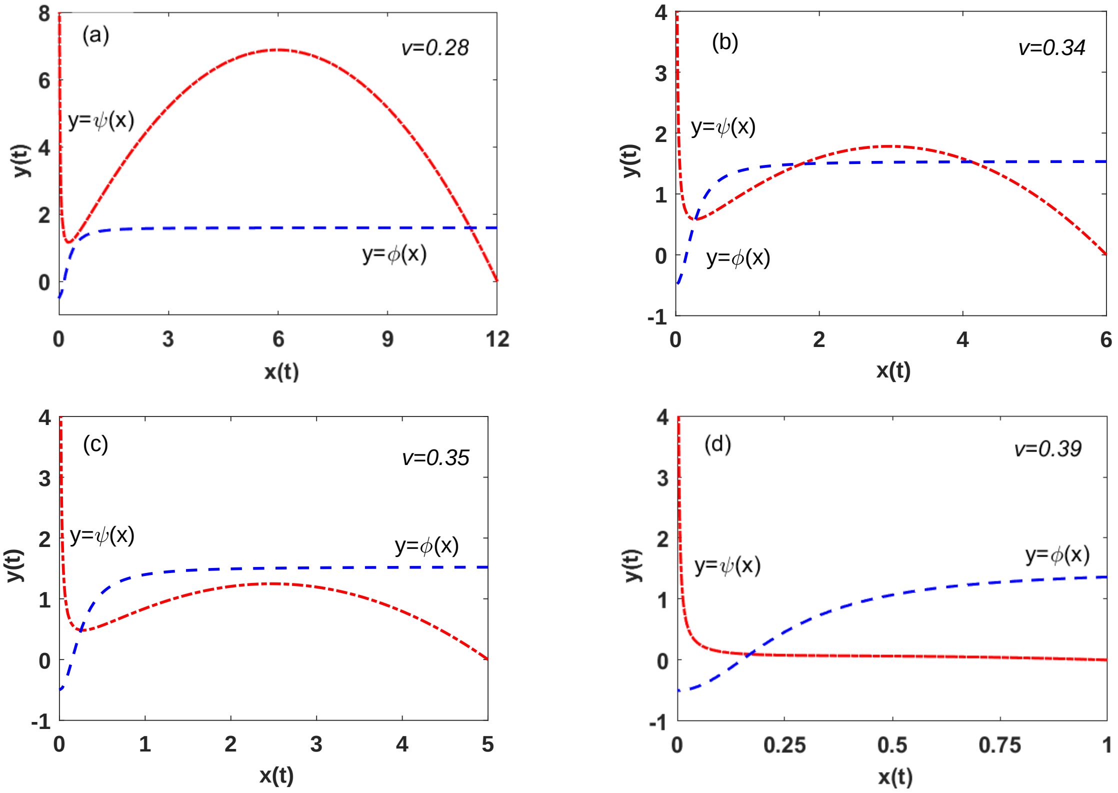

Figure 1. Illustration for $y = \psi(x)$ and $y = \phi(x)$ for the following given model parameters: $r = 0.4$, $K = 40$, $m = 0.5$, $m_0 = 0.1$, $a = 15$, $c = 0.8$, $b = 0.02$, $d = 0.01$.

For $\bar{v} \leq v < r$, $y = \psi(x)$ is monotonically decreasing on $(\bar{x}, \overline{K})$ and $y = \phi(x)$ is monotonically increasing on $(\bar{x}, \overline{K})$; therefore, $y = \psi(x)$ and $y = \phi(x)$ have a unique intersection point, i.e., the interior equilibrium $E(x^*, \psi(x^*))$ is unique (Figure 1(d)).

For $0 < v < \overline{v}$, $y = \phi(x)$ is monotonically increasing and tends to saturation $y = y_{\text{sup}} \triangleq \frac{ce^{-vm-m_0}-ad}{ab}$. Alternatively, $y = \psi(x)$ is monotonically decreasing on $(0, x_1)$, monotonically increasing on (x_1, x_2), and monotonically decreasing on $(x_2, \overline{K})$, which forms a S-type on $(0, \overline{K})$. Thus, $y = \psi(x)$ and $y = \phi(x)$ have no more than three intersections, i.e., Model (2.1) has no more than three interior equilibria (Figure 1(b)).

For the given interior equilibrium $\widehat{E}(\hat{x}, \hat{y})$, the Jacobian matrix is as follows:

$$J_{\widehat{E}(\hat{x},\hat{y})} = \begin{pmatrix} \hat{x}\dfrac{\partial F_1}{\partial x}(\hat{x}, \hat{y}) & \hat{x}\dfrac{\partial F_1}{\partial y}(\hat{x}, \hat{y}) \\ \hat{y}\dfrac{\partial F_2}{\partial x}(\hat{x}, \hat{y}) & \hat{y}\dfrac{\partial F_2}{\partial y}(\hat{x}, \hat{y}) \end{pmatrix} = \begin{pmatrix} \dfrac{e^{-vm-m_0}\hat{x}}{1 + a\hat{x}^2}\psi'(\hat{x}) & -\dfrac{e^{-vm-m_0}\hat{x}}{1 + a\hat{x}^2} \\ b\hat{y}\phi'(\hat{x}) & -b\hat{y} \end{pmatrix}.$$

Since

$$\text{Trace}(J_{\widehat{E}(\hat{x},\hat{y})}) = \frac{\hat{x}}{e^{vm+m_0}(1 + a\hat{x}^2)}\psi'(\hat{x}) - b\hat{y} = \hat{y}\left[\frac{b\hat{y} + d}{c\hat{y}}\psi'(\hat{x}) - b\right],$$

$$\text{Det}(J_{\widehat{E}(\hat{x},\hat{y})}) = \frac{b\hat{x}\hat{y}}{e^{vm+m_0}(1 + a\hat{x}^2)}\left[\phi'(\hat{x}) - \psi'(\hat{x})\right],$$

if $\phi(\hat{x}) < \psi'(\hat{x}) < bc\hat{y}/(b\hat{y} + d)$, then there are $\text{Trace}(J_{\widehat{E}(\hat{x},\hat{y})}) < 0$ and $\text{Det}(J_{\widehat{E}(\hat{x},\hat{y})})$ (i.e., the interior equilibrium $\widehat{E}(\hat{x}, \hat{y})$ is locally asymptotically stable). □

4. Dynamical properties of Model (2.2)

In this segment, we will examine various characteristics of the stochastic prey-predator model.

4.1. Global positivity

Theorem 3. *For* $\forall(x_0, y_0) \in \mathbb{R}^2_+$, *Model* (2.2) *possesses a unique, global solution* $(x(t), y(t))$ *for* $t > 0$, *which keeps positive with a probability with one.*

Proof. Denote $x(t) = e^{p(t)}$, $y(t) = e^{q(t)}$. Consider the following:

$$\begin{cases} \mathrm{d}p(t) = \left[r\left(1 - \dfrac{e^{p(t)}}{K}\right) - v - e^{-(vm+m_0)}\dfrac{e^{p(t)+q(t)}}{1 + ae^{2p(t)}}\right]\mathrm{d}t + \sigma_1\mathrm{d}B_1(t), \\ \mathrm{d}q(t) = \left[ce^{-(vm+m_0)}\dfrac{e^{2p(t)}}{1 + ae^{2p(t)}} - be^{q(t)} - d\right]\mathrm{d}t + \sigma_2\mathrm{d}B_2(t). \end{cases} \tag{4.1}$$

It can be verified that Model (4.1) fulfills the Local Lipschitz Condition; then, for given $p_0 = \ln x_0$, $q_0 = \ln y_0$, it guarantees the existence of a unique, locally positive solution $(p(t), q(t))$ on the interval $[0, \tau_e)$, where τ_e is the time of explosion. The Itô formula implies that $(x(t), y(t)) = (e^{p(t)}, e^{q(t)})$ is exactly the solution of Model (2.2).

Next, it is sufficient to demonstrate that $\tau_e = \infty$. Let n_0 be an integer satisfy $(x_0, y_0) \subset \left[\frac{1}{n_0}, n_0\right]$. For $n \geq n_0$, define the following:

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{x(t), y(t)\} \leq \frac{1}{n} \ or \ \max\{x(t), y(t)\} \geq n\right\}.$$

For an empty set ϕ, define $\inf \phi = \infty$. It is evident that τ_n increases with $n \to \infty$. Define $\tau_\infty = \lim\limits_{n\to\infty} \tau_n$. Then, $\tau_\infty \leq \tau_e$. It is only necessary to prove that $\tau_\infty = \infty$.

By contradiction, it is assumed that $\tau_\infty < \infty$. Then, there exist $\varepsilon_\infty \in (0,1)$ and $T_\infty > 0$ satisfying $P\{\tau_\infty \leq T_\infty\} > \varepsilon_\infty$. Thus, $\exists n_\infty \geq n_0$ such that

$$P\{\tau_n \leq T_\infty\} > \varepsilon_\infty, \quad n \geq n_\infty. \tag{4.2}$$

Define the following:

$$V(x,y) \triangleq (x - 1 - \ln x) + \frac{1}{c}(y - 1 - \ln y).$$

Clearly, $V(x,y)$ is nonnegative because $s - 1 - \ln s \geq 0$ when $s > 0$. Then,

$$\mathrm{d}V(x,y) = \mathcal{L}V\mathrm{d}t + (x-1)\sigma_1 \mathrm{d}B_1(t) + \frac{1}{c}(y-1)\sigma_2 \mathrm{d}B_2(t),$$

where

$$\begin{aligned}
\mathcal{L}V =&(x-1)\left[r\left(1-\frac{x}{K}\right) - v - e^{-(vm+m_0)}\frac{xy}{1+ax^2}\right] + \frac{\sigma_1^2}{2}\\
&+ \frac{1}{c}(y-1)\left[ce^{-(vm+m_0)}\frac{x^2}{1+ax^2} - by - d\right] + \frac{\sigma_2^2}{2c}\\
=&rx - \frac{rx^2}{K} - vx + \frac{rx}{K} + e^{-(vm+m_0)}\frac{xy}{1+ax^2} - \frac{dy}{c} - \frac{by^2}{c}\\
&+ \frac{by}{c} - e^{-(vm+m_0)}\frac{x^2}{1+ax^2} + \left(v + \frac{d}{c} - r\right) + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2c}\\
\leq&rx - vx - \frac{rx^2}{K} + \frac{rx}{K} + e^{-(vm+m_0)}\frac{y}{ax} + y\left(\frac{b}{c} - \frac{d}{c} - \frac{by}{c}\right) + v + \frac{d}{c} - r + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2c}\\
\leq&\Theta.
\end{aligned}$$

Thus, it can be concluded that

$$\mathrm{d}V \leq \Theta\mathrm{d}t + (x-1)\sigma_1\mathrm{d}B_1(t) + \frac{1}{c}(y-1)\sigma_2\mathrm{d}B_2(t).$$

By taking expectations after integrating over the interval $(0, \tau_n \wedge T_\infty)$, we have the following:

$$EV(x(\tau_n \wedge T_\infty), y(\tau_n \wedge T_\infty)) \leq V(x_0,y_0) + \Theta E(\tau_n \wedge T_\infty) \leq V(x_0,y_0) + \Theta T_\infty. \tag{4.3}$$

Denote $\Omega_n = \{w \in \Omega | \tau_n = \tau_n(w) \leq T_\infty\}$. Then, $P(\Omega_n) > \varepsilon$ by Eq (4.2). It can be obtained that $x(\tau_n, w)$ and $y(\tau_n, w)$ equal to either $\frac{1}{n}$ or n for all $w \in \Omega_n$. From Eq (4.3), we obtain the following:

$$V(x_0,y_0) + \Theta T_\infty \geq E(1_{\Omega_n} V(x(\tau_n), y(\tau_n))) \geq \varepsilon_\infty \min\left\{\frac{1}{n} - 1 - \ln\frac{1}{n}, n - 1 - \ln n\right\},$$

which implies

$$\infty > V(x_0,y_0) + \Theta T_\infty = \infty \quad \text{as} \quad n \to \infty.$$

Therefore, we can obtain the following:

$$P\{\tau_\infty = \infty\} = 1.$$

□

4.2. Stationary distribution

Define the following:

$$\lambda \triangleq r - v - d - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} - \delta e^{-(vm+m_0)} - \frac{(\frac{r}{K} + r - v - 2e^{-(vm+m_0)}\sqrt{\delta c})^2}{4(\frac{r}{K} - \delta a e^{-(vm+m_0)})} - \frac{(b-d)^2}{4b},$$

where $\delta \in (0, \frac{r}{Ka})$.

Theorem 4. *For* $\forall (x_0, y_0) \in \mathbb{R}_+^2$, *Model* (2.2) *possesses a unique, ergodic stationary distribution for* $t \geq 0$ *if* $\lambda > 0$.

Proof. Since

$$\mathrm{d}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\left[r\left(1 - \dfrac{x}{K}\right) - v - e^{-(vm+m_0)}\dfrac{xy}{1+ax^2}\right] \\ y\left[ce^{-(vm+m_0)}\dfrac{x^2}{1+ax^2} - by - d\right] \end{pmatrix}\mathrm{d}t + \begin{pmatrix} \sigma_1 x \\ 0 \end{pmatrix}\mathrm{d}B_1(t) + \begin{pmatrix} 0 \\ \sigma_2 y \end{pmatrix}\mathrm{d}B_2(t),$$

its diffusion matrix is given by the following

$$\Lambda(x,y) = \begin{pmatrix} \sigma_1^2 x^2 & 0 \\ 0 & \sigma_2^2 y^2 \end{pmatrix}.$$

Let $\theta \triangleq \min\{\sigma_1^2 x^2, \sigma_2^2 y^2\}$. Then, it has

$$\sum_{i,j=1}^{2} \lambda_{ij}(xy)\xi_i\xi_j = \sigma_1^2 x^2 \xi_1^2 + \sigma_2^2 y^2 \xi_2^2 \geq \theta|\xi|^2,$$

so Lemma 1 condition (i) holds.

To show Lemma 1 condition (ii) holds, set the following:

$$H(x,y) = -M_1(\ln x + \ln y - x - y) + \frac{1}{2}(cx+y)^2 = H_1(x,y) + H_2(x,y),$$

where $H_1(x) = -M_1(\ln x + \ln y - x - y)$, $H_2(y) = \frac{1}{2}(cx+y)^2$, $M_1 = \frac{2}{\lambda}\max\{2, \sup\limits_{(x,y)\in R_+^2}\{-\frac{b_1}{2}x^3 + b_2 x^2 - \frac{b_3}{2}y^3 + b_4 y^2\}\} > 0$, and $b_i > 0$ (for $i = 1,2,3,4$), will be specified lately. $H(x,y)$ tends to infinity when (x,y) tends to the boundary of $\mathbb{R}_+^2$, so $H(x,y)$ has a lower bound at $(x', y') \in \mathbb{R}_+^2$. Define

$$W(x,y) = H(x,y) - H(x', y').$$

For $\forall \delta \in (0, \frac{r}{ka})$, there are $\frac{xy}{1+ax^2} \leq xy$, $\frac{-cx^2}{1+ax^2} \leq -2\sqrt{\delta c}x + \delta(1+ax^2)$ and $\frac{cx^2 y}{1+ax^2} \leq \frac{cxy}{2\sqrt{a}}$.

By the Itô formula, it has the following:

$$
\begin{aligned}
\mathcal{L}H_1 =& -M_1\Bigg[(1-x)\left(r-\frac{rx}{K}-v-e^{-(vm+m_0)}\frac{xy}{1+ax^2}\right)-\frac{\sigma_1^2}{2}\\
&+(1-y)(ce^{-(vm+m_0)}\frac{x^2}{1+ax^2}-by-d)-\frac{\sigma_2^2}{2}\Bigg]\\
=& -M_1\Bigg[r-\frac{rx}{K}-v-e^{-(vm+m_0)}\frac{xy}{1+ax^2}-rx+\frac{rx^2}{K}+vx+e^{-(vm+m_0)}\frac{x^2y}{1+ax^2}\\
&+ce^{-(vm+m_0)}\frac{x^2}{1+ax^2}-by-d-ce^{-(vm+m_0)}\frac{x^2y}{1+ax^2}+by^2+dy-\frac{\sigma_1^2}{2}-\frac{\sigma_2^2}{2}\Bigg]\\
\leq& M_1\left(e^{-(vm+m_0)}+\frac{ce^{-(vm+m_0)}}{2\sqrt{a}}\right)xy-M_1\left(r-v-d-\frac{\sigma_1^2}{2}-\frac{\sigma_2^2}{2}-\delta e^{-(vm+m_0)}\right)\\
&-M_1\left(-\frac{rx}{K}-rx+vx+\frac{rx^2}{K}+2e^{-(vm+m_0)}\sqrt{\delta c}x-\delta ae^{-(vm+m_0)}x^2-by+by^2+dy\right)\\
\leq& M_1\left(e^{-(vm+m_0)}+\frac{ce^{-(vm+m_0)}}{2\sqrt{a}}\right)xy-M_1\Bigg(r-v-d-\frac{\sigma_1^2}{2}-\frac{\sigma_2^2}{2}-\delta e^{-(vm+m_0)}\\
&-\frac{(\frac{r}{K}+r-v-2e^{-(vm+m_0)}\sqrt{\delta c})^2}{4(\frac{r}{K}-\delta ae^{-(vm+m_0)})}-\frac{(b-d)^2}{4b}\Bigg)\\
=& M_1\left(e^{-(vm+m_0)}+\frac{ce^{-(vm+m_0)}}{2\sqrt{a}}\right)xy-M_1\lambda,
\end{aligned}
\tag{4.4}
$$

where $\lambda = r - v - d - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} - \delta e^{-(vm+m_0)} - \frac{\left[\frac{r}{K}+r-v-2e^{-(vm+m_0)}\sqrt{\delta c}\right]^2}{4\left[\frac{r}{K}-\delta ae^{-(vm+m_0)}\right]} - \frac{(b-d)^2}{4b}$.

Similarly,

$$
\begin{aligned}
\mathcal{L}H_2 =& (cx+y)\Bigg[cx\left(r-\frac{rx}{K}-v-e^{-(vm+m_0)}\frac{xy}{1+ax^2}\right)\\
&+y\left(ce^{-(vm+m_0)}\frac{x^2}{1+ax^2}-by-d\right)\Bigg]+\frac{\sigma_1^2}{2}c^2x^2+\frac{\sigma_2^2}{2}y^2\\
=& (cx+y)\left(rcx-\frac{rcx^2}{K}-vcx-by^2-dy\right)+\frac{\sigma_1^2}{2}c^2x^2+\frac{\sigma_2^2}{2}y^2\\
\leq& rc^2x^2-\frac{rc^2}{K}x^3+rcxy-by^3+\frac{\sigma_1^2}{2}c^2x^2+\frac{\sigma_2^2}{2}y^2\\
=& -\frac{rc^2}{K}x^3+c^2\left(r+\frac{\sigma_1^2}{2}\right)x^2-by^3+\frac{\sigma_2^2}{2}y^2+crxy.
\end{aligned}
\tag{4.5}
$$

Combining (4.4) and (4.5), we have the following:

$$
\begin{aligned}
\mathcal{L}W \leq& -M_1\lambda-\frac{rc^2}{K}x^3+c^2\left(r+\frac{\sigma_1^2}{2}\right)x^2-by^3+\frac{\sigma_2^2}{2}y^2+\left[cr+M_1\left(e^{-(vm+m_0)}+\frac{ce^{-(vm+m_0)}}{2\sqrt{a}}\right)\right]xy\\
=& -M_1\lambda-b_1x^3+b_2x^2-b_3y^3+b_4y^2+b_5xy,
\end{aligned}
\tag{4.6}
$$

where $b_1=\frac{rc^2}{K}$, $b_2=c^2\left(r+\frac{\sigma_1^2}{2}\right)$, $b_3=b$, $b_4=\frac{\sigma_2^2}{2}$, and $b_5=cr+M_1\left[e^{-(vm+m_0)}+\frac{ce^{-(vm+m_0)}}{2\sqrt{a}}\right]$.

Denote $U = [\varepsilon, 1/\varepsilon] \times [\varepsilon, 1/\varepsilon]$ for the given

$$0 < \varepsilon < \min\left\{\frac{M_1\lambda}{4b_5}, \frac{b_3}{2b_5}, \frac{b_1}{2b_5}\right\}, \tag{4.7}$$

$$1 - M_1\lambda + M_2 \leq \min\left\{\frac{b_1}{2\varepsilon^3}, \frac{b_3}{2\varepsilon^3}\right\}, \tag{4.8}$$

where $M_2 = \sup\limits_{(x,y)\in R_+^2} \left\{-\frac{b_1}{2}x^3 + b_2x^2 - \frac{b_3}{2}y^3 + b_4y^2 + \frac{b_5}{2}\left(x^2 + y^2\right)\right\}$. Then, $\mathbb{R}_+^2 \setminus U$ can be divided into four regions, $\mathbb{R}_+^2 \setminus U = \Omega_1 \bigcup \Omega_2 \bigcup \Omega_3 \bigcup \Omega_4$, in which

$$\Omega_1 = \{(x, y) \in \mathbb{R}_+^2 | 0 < x < \varepsilon\}, \Omega_2 = \{(x, y) \in \mathbb{R}_+^2 | 0 < y < \varepsilon\},$$

$$\Omega_3 = \left\{(x, y) \in \mathbb{R}_+^2 | x > \frac{1}{\varepsilon}\right\}, \Omega_4 = \left\{(x, y) \in \mathbb{R}_+^2 | y > \frac{1}{\varepsilon}\right\}.$$

Next, we will prove that $\mathcal{L}W \leq -1$ on each Ω_i $(i = 1, 2, 3, 4)$, respectively.

1) For $\forall(x, y) \in \Omega_1$, there is $xy \leq \varepsilon y \leq \varepsilon(1 + y^3)$. By (4.6) and (4.7), we obtain the following:

$$\begin{aligned}\mathcal{L}W &\leq -M_1\lambda - b_1x^3 + b_2x^2 - b_3y^3 + b_4y^2 + \varepsilon b_5 + \varepsilon b_5 y^3\\ &\leq -\frac{M_1\lambda}{4} - \left(\frac{M_1\lambda}{4} - \varepsilon b_5\right) - \left(\frac{b_3}{2} - \varepsilon b_5\right)y^3 - \frac{M_1\lambda}{2} - \frac{b_1}{2}x^3 + b_2x^2 - \frac{b_3}{2}y^3 + b_4y^2\\ &\leq -\frac{M_1\lambda}{4}\\ &\leq -1.\end{aligned}$$

2) For $\forall(x, y) \in \Omega_2$, there is $xy \leq \varepsilon x \leq \varepsilon(1 + x^3)$. By (4.6) and (4.7), we obtain the following:

$$\begin{aligned}\mathcal{L}W &\leq -M_1\lambda - b_1x^3 + b_2x^2 - b_3y^3 + b_4y^2 + \varepsilon b_5 + \varepsilon b_5 x^3\\ &\leq -\frac{M_1\lambda}{4} - \left(\frac{M_1\lambda}{4} - \varepsilon b_5\right) - \left(\frac{b_1}{2} - \varepsilon b_5\right)x^3 - \frac{M_1\lambda}{2} - \frac{b_1}{2}x^3 + b_2x^2 - \frac{b_3}{2}y^3 + b_4y^2\\ &\leq -\frac{M_1\lambda}{4}\\ &\leq -1.\end{aligned}$$

3) For $\forall(x, y) \in \Omega_3$, by (4.6)–(4.8), we obtain the following:

$$\begin{aligned}\mathcal{L}W &\leq -M_1\lambda - b_1x^3 + b_2x^2 - b_3y^3 + b_4y^2 + \frac{b_5}{2}\left(x^2 + y^2\right)\\ &\leq -M_1\lambda - \frac{b_1}{2}x^3 - \frac{b_1}{2}x^3 + b_2x^2 - \frac{b_3}{2}y^3 + b_4y^2 + \frac{b_5}{2}\left(x^2 + y^2\right)\\ &\leq -M_1\lambda - \frac{b_1}{2\varepsilon^3} + M_2\\ &\leq -1.\end{aligned}$$

4) For $\forall(x,y)\in\Omega_4$, by (4.6)–(4.8), we obtain the following:

$$\begin{aligned}\mathcal{L}W &\le -M_1\lambda - b_1x^3 + b_2x^2 - b_3y^3 + b_4y^2 + \frac{b_5}{2}\left(x^2+y^2\right)\\ &\le -M_1\lambda - \frac{b_3}{2}y^3 - \frac{b_3}{2}y^3 + b_2x^2 - \frac{b_1}{2}x^3 + b_4y^2 + \frac{b_5}{2}\left(x^2+y^2\right)\\ &\le -M_1\lambda - \frac{b_3}{2\varepsilon^3} + M_2\\ &\le -1.\end{aligned}$$

To sum up, condition (ii) in Lemma 1 holds. Thus the Model (2.2) possesses a unique, ergodic stationary distribution. □

4.3. *Stochastically ultimate boundedness*

In this section, we will go into more detail on whether the solution is always bounded.

Theorem 5. *Model* (2.2)*'s solutions are stochastically ultimate bounded.*

Proof. Let $V_1(x,y) = x^{\frac{1}{2}} + y^{\frac{1}{2}}$. Then,

$$\mathrm{d}V_1(x,y) = LV_1\mathrm{d}t + \frac{\sigma_1}{2}x^{\frac{1}{2}}\mathrm{d}B_1(t) + \frac{\sigma_2}{2}y^{\frac{1}{2}}\mathrm{d}B_2(t),$$

where

$$\begin{aligned}\mathcal{L}V_1 =&\frac{1}{2}x^{\frac{1}{2}}\left[r(1-\frac{x}{K}) - v - e^{-(vm+m_0)}\frac{xy}{1+ax^2}\right] - \frac{\sigma_1^2}{8}x^{\frac{1}{2}}\\ &+\frac{1}{2}y^{\frac{1}{2}}\left[ce^{-(vm+m_0)}\frac{x^2}{1+ax^2} - by - d\right] - \frac{\sigma_2^2}{8}y^{\frac{1}{2}}\\ \le&\frac{1}{2}\left(r - \frac{rx}{K} - \frac{\sigma_1^2}{4}\right)x^{\frac{1}{2}} + \frac{1}{2}\left(\frac{c}{a}e^{-(vm+m_0)} - by - d - \frac{\sigma_2^2}{4}\right)y^{\frac{1}{2}}\\ =&\frac{1}{2}\left(2 + r - \frac{rx}{K} - \frac{\sigma_1^2}{4}\right)x^{\frac{1}{2}} + \frac{1}{2}\left(2 + \frac{c}{a}e^{-(vm+m_0)} - by - d - \frac{\sigma_2^2}{4}\right)y^{\frac{1}{2}} - V_1(x,y)\\ \le&P_0 - V_1(x,y),\end{aligned}$$

where P_0 represents a positive constant, expressed as

$$P_0 = \sup_{(x,y)\in R_+^2}\left\{\frac{1}{2}\left(2 + r - \frac{rx}{K} - \frac{\sigma_1^2}{4}\right)x^{\frac{1}{2}} + \frac{1}{2}\left(2 + \frac{c}{a}e^{-(vm+m_0)} - by - d - \frac{\sigma_2^2}{4}\right)y^{\frac{1}{2}}\right\}.$$

Then,

$$\mathrm{d}V_1(x,y) \le (P_0 - V_1(x,y))\mathrm{d}t + \frac{\sigma_1}{2}x^{\frac{1}{2}}\mathrm{d}B_1(t) + \frac{\sigma_2}{2}y^{\frac{1}{2}}\mathrm{d}B_2(t),$$

so

$$\mathrm{d}(e^tV_1(x,y)) = e^t[V_1(x,y)\mathrm{d}t + \mathrm{d}V_1(x,y)] \le e^tP_0\mathrm{d}t + \frac{\sigma_1}{2}x^{\frac{1}{2}}e^t\mathrm{d}B_1(t) + \frac{\sigma_2}{2}y^{\frac{1}{2}}e^t\mathrm{d}B_2(t).$$

Thus,

$$E(e^tV_1(x,y)) = e^tE(V_1(x,y)) \le V_1(x_0,y_0) + P_0(e^t-1)$$

and

$$\limsup_{t\to+\infty} E(V_1(x,y)) \leq P_0 \Rightarrow \limsup_{t\to+\infty} E|(x,y)|^{\frac{1}{2}} \leq P_0.$$

Therefore, for any normal number ε, set $\chi = \dfrac{P_0^2}{\varepsilon^2}$. Utilizing Chebyshev's inequality [31,37], there is the following:

$$P\{|(x,y)| > \chi\} \leq \frac{E|(x,y)|^{\frac{1}{2}}}{\sqrt{\chi}} \Rightarrow \sup P\{|(x(t),y(t))| > \chi\} \leq \varepsilon \text{ as } t \to +\infty.$$

□

4.4. Extinction and persistence

The objective of this section is to explore the extinction and persistence for the populations.

Lemma 4. *For Model* (2.2)*, there are* $\lim_{t\to+\infty} \sup\{t^{-1}\ln x(t)\} \leq 0$ *and* $\lim_{t\to+\infty} \sup\{t^{-1}\ln y(t)\} \leq 0$.

Proof. Since

$$d(e^t \ln x(t)) = e^t\left(\ln x + r - \frac{rx}{K} - v - e^{-(vm+m_0)}\frac{xy}{1+ax^2} - \frac{\sigma_1^2}{2}\right)dt + e^t\sigma_1 dB_1(t), \tag{4.9}$$

then

$$e^t \ln x(t) = \int_0^t e^s\left(\ln x(s) + r - \frac{rx(s)}{k} - v - e^{-(vm+m_0)}\frac{x(s)y(s)}{1+ax^2(s)} - \frac{\sigma_1^2}{2}\right)ds + \int_0^t e^s\sigma_1 dB_1(s) + \ln x_0.$$

Let $M(t) = \sigma_1\int_0^t e^s dB_1(s)$. $M(t)$ can be regarded as a localized harness that exhibits the following quadratic variance function:

$$\langle M(t), M(t)\rangle = \sigma_1^2\int_0^t e^{2s}ds.$$

Take $\mu > 1$ and $\gamma > 1$. For $k \in \mathbb{N}$, denote $\alpha = e^{-\gamma k}$, $\beta = \mu e^{\gamma k}\ln k$, $T = \gamma k$. Then, according to the Exponential Martingale Inequality, it has the following:

$$P\left(\sup_{0\leq t\leq T}\{[M - \frac{\alpha}{2}\langle M(t), M(t)\rangle] > \beta\}\right) \leq \frac{1}{k^\mu}.$$

Since $\sum \frac{1}{k^\mu} < \infty$, then by the Borel-Cantelli lemma [31,37], there exists $\Omega \in \mathcal{F}$ with $P(\Omega) = 1$ and the integer-valued random variable $k_0(w)$ such that for any $\omega \in \Omega$ and $k \geq k_0(\omega)$, there is the following:

$$M(t) \leq \mu e^{\gamma k}\ln k + \frac{1}{2}e^{-\gamma k}\langle M(t), M(t)\rangle, \quad 0 \leq t \leq \gamma k.$$

Hence,

$$\begin{aligned} e^t \ln x(t) \leq{}& \int_0^t e^s\left(\ln x(s) + r - \frac{rx(s)}{K} - v - e^{-(vm+m_0)}\frac{x(s)y(s)}{1+ax^2(s)} - \frac{\sigma_1^2}{2}\right)ds \\ &+ \frac{1}{2}e^{-\gamma k}\langle M(t), M(t)\rangle + \mu e^{\gamma k}\ln k + \ln x_0 \\ \leq{}& \int_0^t e^s Q(x(s))ds + \mu e^{\gamma k}\ln k + \ln x_0, \end{aligned}$$

where $Q(x(s)) = r - \frac{rx(s)}{K} - v - \frac{\sigma_1^2}{2} + \frac{\sigma_1^2}{2}e^{s-\gamma k} + \ln x(s)$. For any s that satisfies $0 \le s \le \gamma k$, $x(s) > 0$, a k-independent positive constant Φ_0 can be found such that $Q(x(s)) \le \Phi_0$, then, $e^t \ln x(t) - \ln x_0 \le \Phi_0(e^t - 1) + \mu e^{\gamma k} \ln k$ and $\ln x(t) \le \Phi_0(1 - e^{-t}) + \mu e^{\gamma k - t} \ln k + e^{-t} \ln x_0$. Then,

$$\frac{\ln x(t)}{t} \le \frac{\Phi_0(1 - e^{-t})}{t} + \frac{\mu e^{\gamma k - \gamma(k-1)}}{t} \ln k + \frac{e^{-t} \ln x_0}{t}.$$

Let $t \to \infty$; we obtain $\lim\limits_{t\to\infty} sup \frac{\ln x(t)}{t} \le 0$. In a similar way, one can establish $\lim\limits_{t\to\infty} sup \frac{\ln y(t)}{t} \le 0$. □

Denote $\eta \triangleq \lim\limits_{t\to+\infty} \sup x(t)$ and define the following:

$$\Delta_1 \triangleq \frac{K}{r}\left(r - v - \frac{\sigma_1^2}{2}\right), \Delta_2 \triangleq \frac{ce^{-(vm+m_0)}}{a} - d - \frac{\sigma_2^2}{2}, \Delta_3 \triangleq \frac{ce^{-(vm+m_0)}}{a}\frac{\eta^2}{1 + a\eta^2} - d - \frac{\sigma_2^2}{2}.$$

Theorem 6. *For Model* (2.2)*'s solution with a given* $(x_0, y_0) \in \mathbb{R}_+^2$, *when* $\Delta_1 < 0$, *both species x and y will eventually go extinct; when* $\Delta_1 > 0$, *species x can keep weakly persistent; when* $\Delta_1 > 0$ *and* $\Delta_2 < 0$, *species x can keep persistent* $\langle x(t)\rangle = \Delta_1$ *and species y will eventually go extinct; and when* $\Delta_1 > 0$ *and* $\Delta_3 > 0$, *both species x and y can keep weakly persistent.*

Proof. 1) Since

$$\mathrm{d}\ln x = \left[r\left(1 - \frac{x}{K}\right) - v - e^{-(vm+m_0)}\frac{xy}{1 + ax^2} - \frac{\sigma_1^2}{2}\right]\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t),$$

then

$$\ln x(t) = \int_0^t \left(r - \frac{rx}{K} - v - e^{-(vm+m_0)}\frac{xy}{1 + ax^2} - \frac{\sigma_1^2}{2}\right)\mathrm{d}s + \sigma_1 B_1(t) + \ln x_0.$$

Subsequently, it has

$$\frac{\ln x(t)}{t} \le r - v - \frac{\sigma_1^2}{2} + \frac{\ln x_0}{t} + \frac{\sigma_1 B_1(t)}{t},$$

then

$$\limsup_{t\to+\infty} \frac{\ln x(t)}{t} \le r - v - \frac{\sigma_1^2}{2} = \frac{r}{K}\Delta_1 < 0.$$

Therefore, $\lim\limits_{t\to+\infty} x(t) = 0$.

When $\Delta_2 < 0$, it has the following:

$$\mathrm{d}\ln y = \left(ce^{-(vm+m_0)}\frac{x^2}{1 + ax^2} - by - d - \frac{\sigma_2^2}{2}\right)\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t).$$

Then,

$$\ln y(t) - \ln y_0 = \int_0^t \left(ce^{-(vm+m_0)}\frac{x^2}{1 + ax^2} - by - d - \frac{\sigma_2^2}{2}\right)\mathrm{d}s + \sigma_2 B_2(t).$$

Thus,

$$\frac{\ln y(t)}{t} \le \frac{\ln y_0}{t} + \frac{c}{a}e^{-(vm+m_0)} - d - \frac{\sigma_2^2}{2} + \frac{\sigma_2 B_2(t)}{t},$$

i.e.,

$$\limsup_{t\to+\infty}\frac{\ln y(t)}{t}\leq\frac{c}{a}e^{-(vm+m_0)}-d-\frac{\sigma_2^2}{2}<0.$$

Therefore, $\lim\limits_{t\to\infty}y(t)=0$.

When $\Delta_2>0$, then for $\forall\varepsilon_1$ and $\varepsilon_2>0$, $\exists T_1$ such that

$$ce^{-(vm+m_0)}\frac{x^2}{1+ax^2}\leq\varepsilon_1,\ \frac{\ln y_0}{t}\leq\varepsilon_2$$

hold for $t\geq T_1$. Thus,

$$\ln y(t)\leq\int_0^t\left(\varepsilon_1-by-d-\frac{\sigma_2^2}{2}\right)\mathrm{d}s+\varepsilon_2t+\sigma_2B_2(t)=\left(\varepsilon_1+\varepsilon_2-d-\frac{\sigma_2^2}{2}\right)t-b\int_0^t y(s)\mathrm{d}s+\sigma_2B_2(t).$$

By the arbitrariness of ε_1 and ε_2 and Lemma 2, there is $\lim\limits_{t\to\infty}y(t)=0$.

2) By contradiction. Assume $\Sigma=\{\limsup\limits_{t\to\infty}x(t)=0\}$ with $P(\Sigma)>0$. Thus,

$$\frac{\ln x(t)}{t}=\frac{\ln x_0}{t}+\frac{1}{t}\int_0^t\left(r-\frac{rx}{K}-v-e^{-(vm+m_0)}\frac{xy}{1+ax^2}-\frac{\sigma_1^2}{2}\right)\mathrm{d}s+\frac{\sigma_1B_1(t)}{t}.$$

Therefore, $\limsup\limits_{t\to\infty}t^{-1}\ln x(t)=r-v-\frac{\sigma_1^2}{2}=\frac{r}{K}\Delta_1>0$. That means that $\limsup\limits_{t\to\infty}t^{-1}\ln x(t)>0$ holds for any $w\in\Sigma$. There is an inclusion relation $\Sigma\subseteq\{w:\limsup\limits_{t\to\infty}t^{-1}\ln x(t)>0\}$, then $P\{w:\limsup\limits_{t\to\infty}t^{-1}\ln x(t)>0\}\geq P(\Sigma)>0$, which contradicts to $\limsup\limits_{t\to\infty}t^{-1}\ln x(t)\leq 0$. Therefore, species x is weakly persistent.

3) It is known that $y(t)$ is extinct as $\lim\limits_{t\to\infty}y(t)=0$ when $\Delta_2<0$. From $\lim\limits_{t\to+\infty}\frac{\ln x_0}{t}=0$, it is known that for $\forall\varepsilon_3,\varepsilon_4>0$, $\exists T_2$ such that $-\varepsilon_3\leq -e^{-(vm+m_0)}\frac{xy}{1+ax^2}\leq\varepsilon_3$ and $-\varepsilon_4\leq\frac{\ln x_0}{t}\leq\varepsilon_4$ hold for $t\geq T_2$. Obviously,

$$\begin{aligned}\ln x(t)&\leq t\varepsilon_4+\int_0^t\left(r-\frac{rx}{K}-v+\varepsilon_3-\frac{\sigma_1^2}{2}\right)\mathrm{d}s+\sigma_1B_1(t)\\&=\left(r-v+\varepsilon_3+\varepsilon_4-\frac{\sigma_1^2}{2}\right)t-\frac{r}{K}\int_0^t x(s)\mathrm{d}s+\sigma_1B_1(t),\\\ln x(t)&\geq -t\varepsilon_4+\int_0^t\left(r-\frac{rx}{K}-v-\varepsilon_3-\frac{\sigma_1^2}{2}\right)\mathrm{d}s+\sigma_1B_1(t)\\&=\left(r-v-\varepsilon_3-\varepsilon_4-\frac{\sigma_1^2}{2}\right)t-\frac{r}{K}\int_0^t x(s)\mathrm{d}s+\sigma_1B_1(t).\end{aligned}$$

Based on Lemma 2, it can be deduced that $\langle x(t)\rangle^*\leq\frac{K}{r}\left(r-v+\varepsilon_3+\varepsilon_4-\frac{\sigma_1^2}{2}\right)$ and $\langle x(t)\rangle_*\geq\frac{K}{r}\left(r-v-\varepsilon_3-\varepsilon_4-\frac{\sigma_1^2}{2}\right)$. Therefore, $\langle x(t)\rangle=\frac{K}{r}\left(r-v-\frac{\sigma_1^2}{2}\right)=\Delta_1>0$ (i.e., species x keeps in persistence (in the mean)).

4) Similarly, by using the converse method, one can show that y is also weakly persistent. □

5. Numerical simulations

This section presents numerical simulations to illustrate the theoretical consequences presented in the study.

5.1. Verification of the main results

For Model (2.1) with the model parameters $r = 0.4$, $K = 100$, $v = 0.36$, $m = 0.5$, $m_0 = 0.1$, $a = 15$, $c = 0.8$, $b = 0.02$ and $d = 0.01$, there exist three interior equilibria: $E_1^*(0.26, 0.46)$, $E_2^*(2.5, 1.5)$ and $E_3^*(7.5, 1.5)$, where E_1^* is a locally asymptotically stable focus, E_2^* is a saddle and unstable, and E_3^* is a locally asymptotically stable nodel, as illustrated in Figure 2.

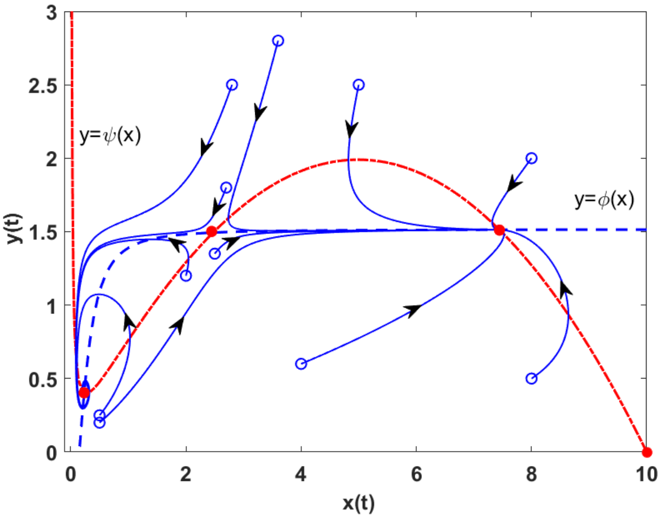

Figure 2. Verification of Theorem 2. The tendency of the solution of Model (2.1) with different initial value for $v = 0.36$ and $a = 15$.

For Model (2.2), let consider the following discrete form [38]:

$$\begin{cases} x_{j+1} = x_j + x_j\left(r - \dfrac{rx_j}{K} - e^{-(vm+m_0)}\dfrac{x_j y_j}{1+a(x_j)^2}\right)\Delta t + \sigma_1 x_j\sqrt{\Delta t}\varepsilon_{1j} + \frac{1}{2}\sigma_1^2 x_j(\varepsilon_{1j}^2 - 1)\Delta t, \\ y_{j+1} = y_j + y_j\left(ce^{-(vm+m_0)}\dfrac{(x_j)^2}{1+a(x_j)^2} - by_j - d\right)\Delta t + \sigma_2 y_j\sqrt{\Delta t}\varepsilon_{2j} + \frac{1}{2}\sigma_2^2 y_j(\varepsilon_{2j}^2 - 1)\Delta t, \end{cases}$$

where ε_{1j}, ε_{2j} follow $\mathcal{N}(0, 1)$, $j = 1, 2, \cdots, n$. Take $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, $\delta = 0.0001$. For the above model parameters, there is $\lambda = -0.00286 < 0$, and the condition in Theorem 4 does not hold. When v increases (e.g. $v = 0.38$), Model (2.1) has a unique interior equilibrium $E^*(0.193, 0.213)$, which is a globally asymptotically stable focus, as presented in Figure 3(a). In such case, the condition in Theorem 4 does not hold due to $\lambda = -0.00136$. For $a = 8$, Model (2.1) has a globally asymptotically stable focus $E^*(0.155, 0.2)$, as presented in Figure 3(b). In such case, there is $\lambda = 0.000144 > 0$, and the condition in Theorem 4 holds. As depicted in Figure 4, for Model (2.2), there exists an ergodic stationary distribution. It is observed that $(x(t), y(t))$ is stable at $(0.155, 0.2)$ without white noise. $x(t)$ varies around 0.155 and $y(t)$ fluctuates around 0.2 when the white noise is presented.

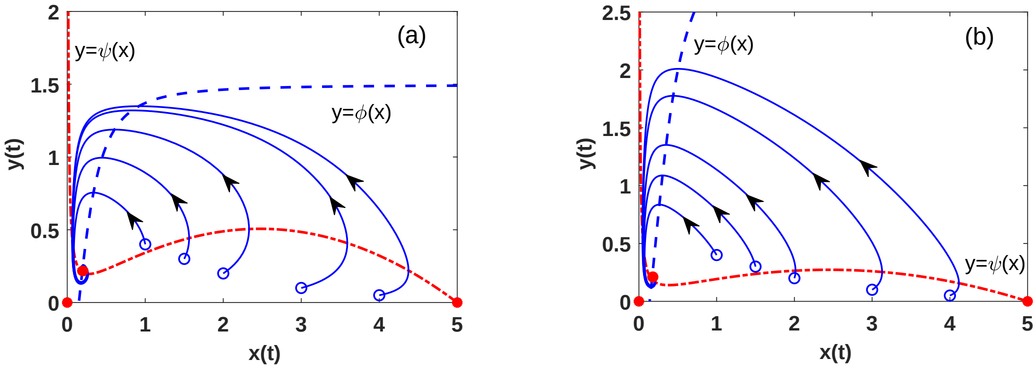

Figure 3. Verification of Theorem 2. The tendency of the solution of Model (2.1) with different initial value for given $v = 0.38$ and (a) $a = 15$, (b) $a = 8$.

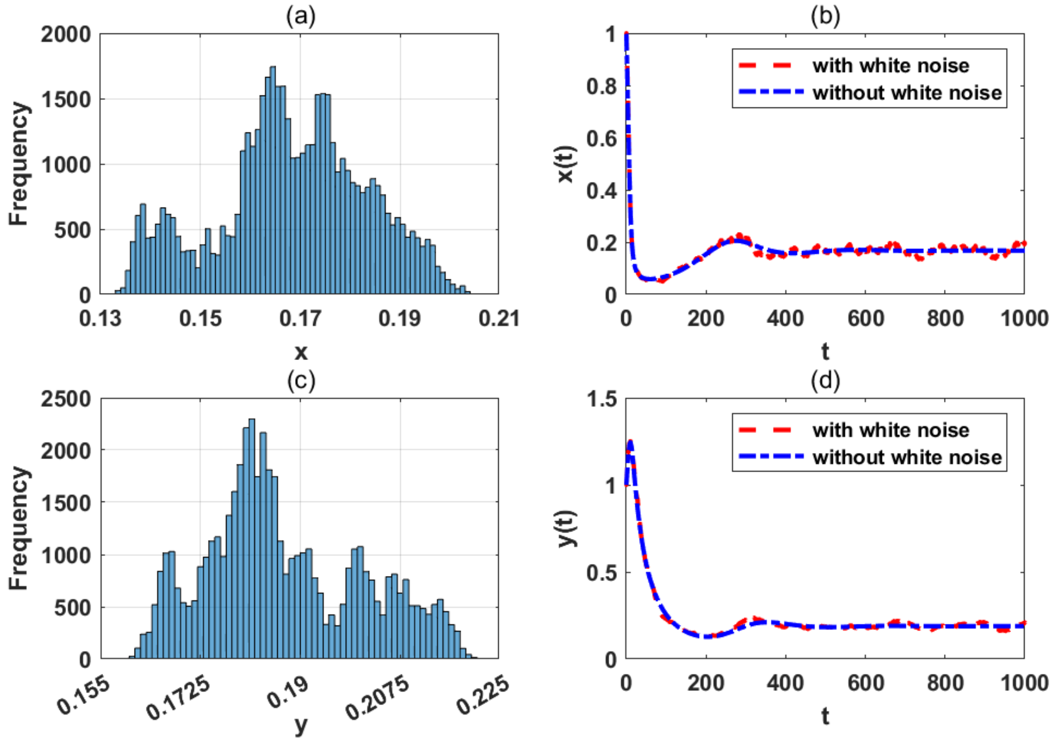

Figure 4. The distribution of $x(t)$ and $y(t)$ in Model (2.2) ((a) and (c)) and the densities of $x(t)$ and $y(t)$ in stochastic Model (2.2) and deterministic Model (2.1) with $(x_0, y_0) = (1, 1)$ ((b) and (d)).

5.2. *Impact of the noise level and parameters*

For Model (2.1) with the model parameters $r = 0.6$, $K = 100$, $a = 2$, $v = 0.3$, $m = 0.5$, $m_0 = 0.01$, $c = 0.8$, $b = 0.01$ and $d = 0.05$, there is a unique interior equilibrium $E^*(0.1914, 0.208)$, which is locally asymptotically stable.

Different levels of white noise are set to illustrate how white noise affects Model (2.2)'s dynamics. Taking $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, it has $\Delta_1 > 0$, $\Delta_3 > 0$. Then, species x and species y can keep in a weak persistence, as shown in Figure 5(a),(b). This means that the white noise will have little impact on the populations. Let $\sigma_2 \to 0.8$; it has $\Delta_1 > 0$, $\Delta_2 < 0$. Therefore, species x can keep in persistence (in the mean), while species y will eventually go extinct, as shown in Figure 5(c),(d). It is evident that a larger white noise induces the predator to be eventually extinct. Let $\sigma_1 \to 0.8$; it has $\Delta_1 < 0$. Then, species x and species y will eventually go to extinction, as shown in Figure 5(e),(f). Make it clear that the higher the noise level, the more significant the impact on the species of prey.

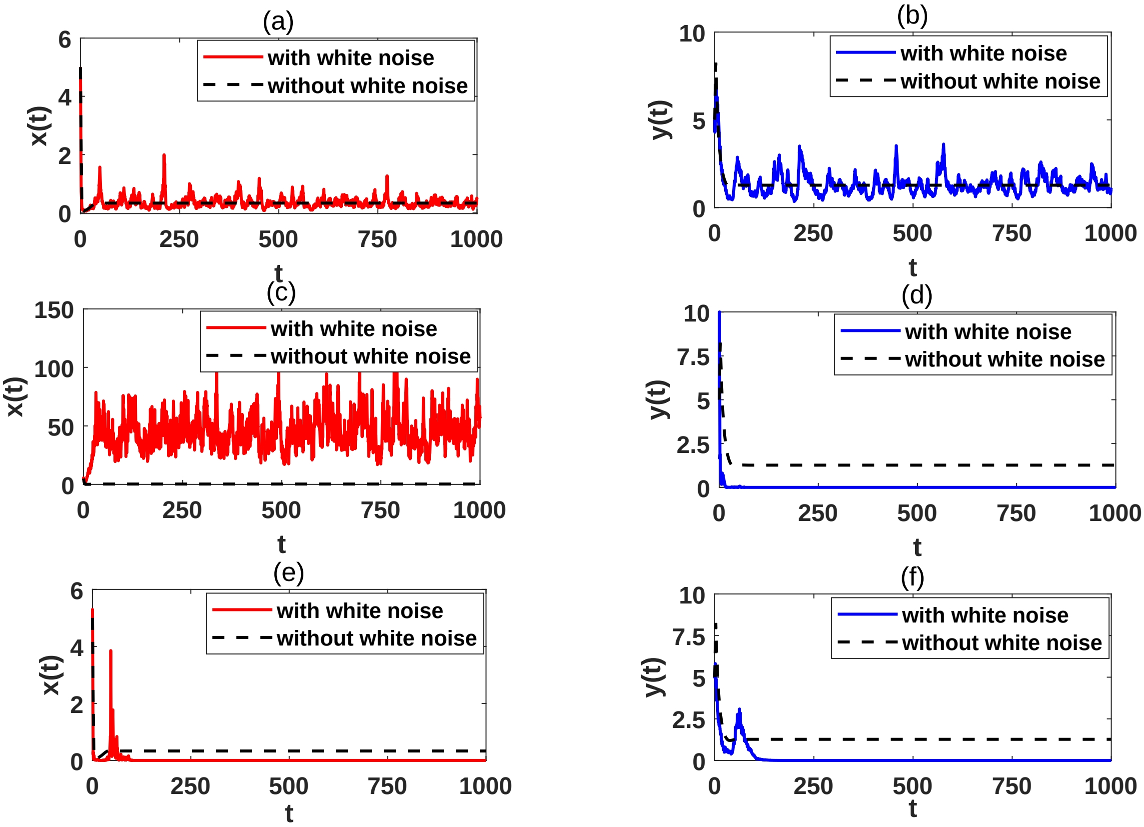

Figure 5. The densities of species x and y for $\sigma_1 = 0.2$, $\sigma_2 = 0.1$ (a,b), $\sigma_1 = 0.2$, $\sigma_2 = 0.8$ (c,d) and $\sigma_1 = 0.8$, $\sigma_2 = 0.1$ (e,f).

The predator's hunting ability is affected by a term e^{-vm}, which decreases when m or v increases. For the parameter m, we select a range of diverse values and take $\sigma_1 = 0.2$, $\sigma_2 = 0.1$. The solution is presented in Figure 6. It is evident that $x(t)$ greatly increases while $y(t)$ increases to be a lesser extent when m rises from 0.5 to 5. There is little change in $x(t)$ when m increases from 5 to 10, while $y(t)$ will change from weakly persistent to eventually extinct. As a result, a modest increase in m benefits y, a large increase in m leads to the extinction of y, and an increase in m promotes the growth of x.

By varying the parameter v, the solutions are presented in Figure 7. Obviously, for a smaller v, species x and species y are both persistent; species y becomes extinct when v increases, and when the anti-predator level becomes excessively strong, species x and species y will eventually go extinct.

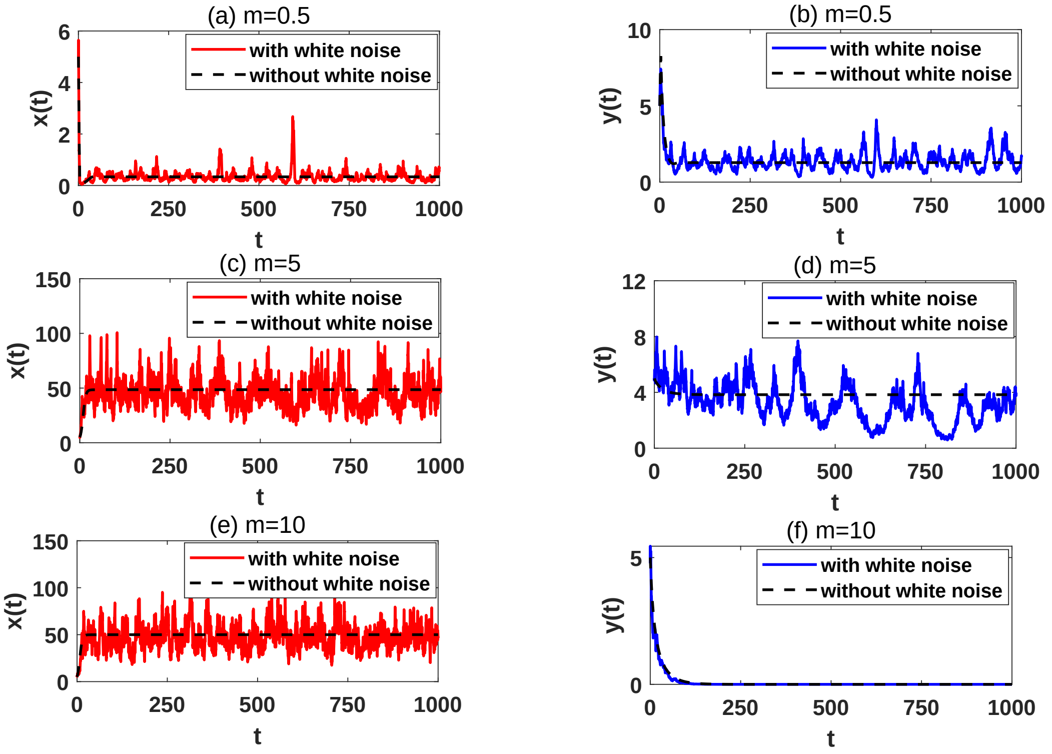

Figure 6. Time series of the stochastic Model (2.2) with different m.

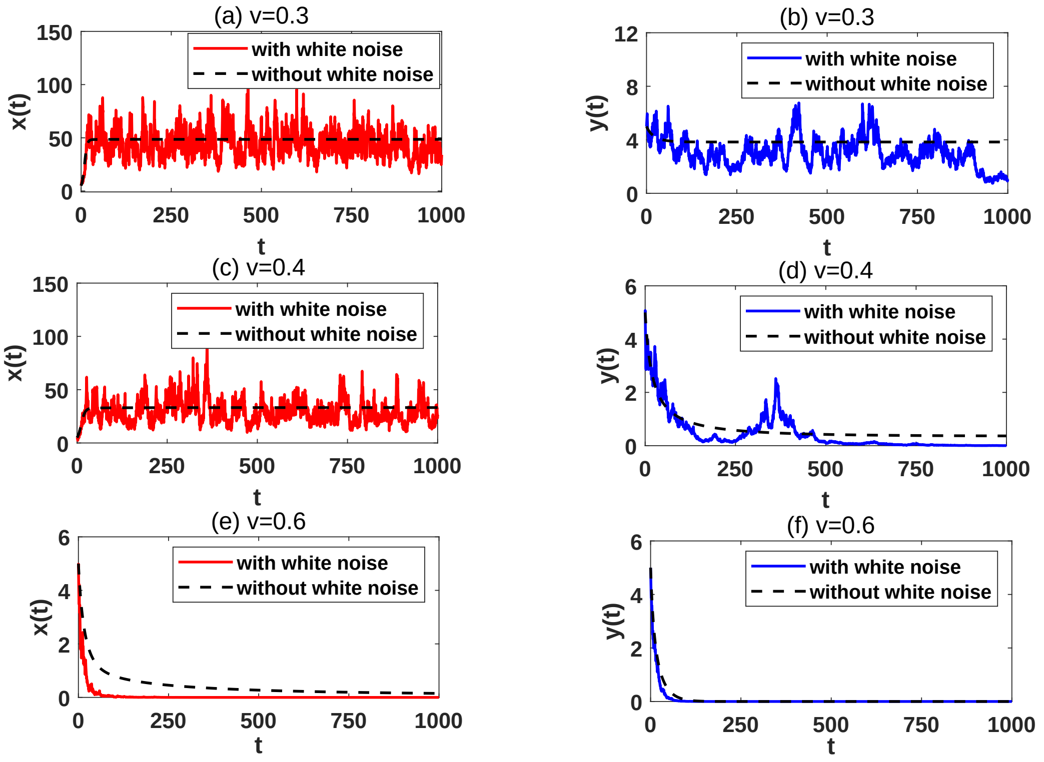

Figure 7. Time series of the stochastic Model (2.2) with different level v.

6. Conclusions

The current work explored a Holling-III prey-predator model by incorporating prey habitat selection in an environment with stochastic disturbances. For the model without stochastic disturbances, it was shown that $O(0,0)$ is constantly unstable, while $E(\overline{K},0)$ is stable when $c \leq \overline{c} \triangleq de^{vm+m_0}(a+\overline{K}^{-2})$. Additionally, it was shown that Model (2.1) possesses not more than three interior equilibria, and the interior equilibrium $\widehat{E}(\hat{x},\hat{y})$ is locally asymptotically stable if and only if $\phi(\hat{x}) < \psi'(\hat{x}) < bc\hat{y}/(b\hat{y}+d)$.

Additionally, we investigated the dynamics of Model (2.2). By devising an appropriate Lyapunov function, it obtained the conditions for an ergodic stationary distribution. Furthermore, it was demonstrated that Model (2.2)'s solutions are stochastically, eventually bounded. The results suggest that white noise will eventually cause the extinction of prey and predators when it has a large impact on the prey $x(t)$ ($\Delta_1 < 0$). When the white noise has more impact on the predators and less disturbance to the prey, the prey can persist (in the mean), and the predators become extinct ($\Delta_1 > 0$, $\Delta_2 < 0$). The prey and the predator are weakly persistent when white noise has a small effect on them.

Compared to deterministic models, it is well known that sometimes less intense white noise contributes to the survival of prey; however, white noise with a high intensity is unfavorable to the survival of the population. When the hunting ability of the predator is affected by the e^{-vm} term, an increase in m favors the growth of x, a small increase in m would benefit y, while a large increase in m would lead to the demise of y. The population survival is significantly impacted by anti-predatory behavior. It indicated that $x(t)$ and $y(t)$ can persistently survive when the anti-predator level is small. The population $y(t)$ becomes extinct with an increase of the anti-predator level. The prey and predator eventually go extinct when the anti-predator level becomes excessively strong. Incorporating habitat selection and white noise interference into predator-prey models allows for more accurate estimates of changes in natural populations and a deeper understanding of the mechanisms of species interactions and ecological balance. The results suggest that populations may tend to go extinct when the white noise is high or the habitat selection behavior is strong. Therefore, we can carry out an early warning and take effective intervention and protection measures, such as establishing protected areas, improving the habitat environment, controlling the intensity of the white noise interference, and so on.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. Y. Kuang, E. Beretta, Global qualitative analysis of a ratio-dependent predator-prey system, *J. Math. Biol.*, **36** (1998), 389–406. https://doi.org/10.1007/s002850050105
2. D. Xiao, S. Ruan, Global analysis in a predator-prey system with nonmonotonic functional response, *SIAM J. Appl. Math.*, **61** (2001), 1445–1472. https://doi.org/10.1137/S0036139999361896
3. T. Zheng, L. Nie, Modelling the transmission dynamics of two-strain Dengue in the presence awareness and vector control, *J. Theor. Biol.*, **443** (2018), 82–91. https://doi.org/10.1016/j.jtbi.2018.01.017
4. L. Nie, J. Shen, C. Yang. Dynamic behavior analysis of SIVS epidemic models with state-dependent pulse vaccination, *Nonlinear Anal: Hybrid Syst.*, **27** (2018), 258–270. https://doi.org/10.1016/j.nahs.2017.08.004
5. X. Yan, Y. Tian, K. Sun, Dynamic analysis of additional food provided non-smooth pest-natural enemy models based on nonlinear threshold control, *J. Appl. Math. Comput.*, (2025). https://doi.org/10.1007/s12190-024-02318-7
6. A. J. Lotka, *Elements of Physical Biology*, Williams and Wilkins Company, Baltimore, 1925.
7. V. Volterra, Fluctuations in the abundance of a species considered mathematically, *Nature*, **119** (1927), 12–13. https://doi.org/10.1038/119012b0
8. C. S. Holling, The functional response of predators to prey density and its role in mimicry and population regulation, *Mem. Entomol. Soc. Can.*, **97** (1965), 5–60. https://doi.org/10.4039/entm9745fv
9. M. P. Hassell, G. C. Varley, New inductive population model for insect parasites and its bearing on biological control, *Nature*, **223** (1969), 1133–1177. https://doi.org/10.1038/2231133a0
10. J. R. Beddington, Mutual interference between parasites or predator and its effect on searching efficiency, *J. Anim Ecol.*, **44** (1975), 331–340. https://doi.org/10.2307/3866
11. R. Arditi, L. R. Ginzburg, Coupling in predator-prey dynamics: Ratio-dependence, *J. Theor. Biol.*, **139** (1989), 311–326. https://doi.org/10.1016/S0022-5193(89)80211-5
12. A. Zegeling, R. E. Kooij, Singular perturbations of the Holling I predator-prey system with a focus, *J. Differ. Equations*, **269** (2020), 5434–5462. https://doi.org/10.1016/j.jde.2020.04.011
13. M. Lu, J. Huang, Global analysis in Bazykin's model with Holling II functional response and predator competition, *J. Differ. Equations*, **280** (2021), 99–138. https://doi.org/10.1016/j.jde.2021.01.025
14. J. Huang, S. Ruan, R. Song, Bifurcations in a predator-prey system of Leslie type with generalized Holling type III functional response, *J. Differ. Equations*, **257** (2014), 1721–1752. https://doi.org/10.1016/j.jde.2014.04.024
15. Q. Yang, X. Zhang, D. Jiang, M. Shao, Analysis of a stochastic predator-prey model with weak Allee effect and Holling-(n+1) functional response, *Commun. Nonlinear Sci. Numer. Simul.*, **111** (2022), 106454. https://doi.org/10.1016/j.cnsns.2022.106454

16. C. B. Stanford, The influence of chimpanzee predation on group size and anti-predator behaviour in red colobus monkeys, *Anim. Behav.*, **49** (1995), 577–587. https://doi.org/10.1016/0003-3472(95)80191-X

17. M. J. How, M. Santon, Cuttlefish camouflage: Blending in by matching background features, *Curr. Biol.*, **32** (2022), 523–525. https://doi.org/10.1016/j.cub.2022.04.042

18. E. Gazagne, T. Savini, D. Ngoprasert, P. Poncin, M. Huynem, F. Brotcorne, When northern pigtailed macaques (Macaca leonina) cannot select for ideal sleeping sites in a degraded habitat, *Int. J. Primatol.*, **41** (2020), 614–633. https://doi.org/10.1007/s10764-020-00173-4

19. D. Li, Q. Zhou, X. Tang, H. Huang, C. Huang, Sleeping site use of the white-headed langur Trachypithecus leucocephalus: The role of predation risk, territorial defense, and proximity to feeding sites, *Curr. Zool.*, **57** (2011), 260–268. https://doi.org/10.1093/czoolo/57.3.260

20. B. Tang, Y. Xiao, Bifurcation analysis of a predator-prey model with anti-predator behaviour, *Chaos Solitons Fractals*, **70** (2015), 58–68. https://doi.org/10.1016/j.chaos.2014.11.008

21. Y. Tian, Y. Gao, K. Sun, A fishery predator-prey model with anti-predator behavior and complex dynamics induced by weighted fishing strategies, *Math. Biosci. Eng.*, **20** (2023), 1558–1579. https://doi.org/10.3934/mbe.2023071

22. A. R. Ives, A. P. Dobson, Antipredator behavior and the population dynamics of simple predator-prey systems, *Am. Nat.*, **130** (1987), 431–447. https://doi.org/10.1086/284719

23. T. Caraco, S. Martindale, H. R. Pulliam, Avian flocking in the presence of a predator, *Nature*, **285** (1980), 400–401. https://doi.org/10.1038/285400a0

24. E. E. Werner, J. F. Gilliam, D. J. Hall, G. G. Mittelbach, An experimental test of the effects of predation risk on habitat use in fish, *Ecology*, **64** (1983), 1540–1548. https://doi.org/10.2307/1937508

25. R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton university press, Princeton, 2001. https://doi.org/10.1515/9780691206912

26. M. Liu, K. Wang, Q. Wu, Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle, *Bull. Math. Biol.*, **73** (2011), 1969–2012. https://doi.org/10.1007/s11538-010-9569-5

27. W. Wang, W. Li, Z. Li, H. Zhang, The effect of colored noise on spatiotemporal dynamics of biological invasion in a diffusive predator-prey system, *Biosystems*, **104** (2011), 48–56. https://doi.org/10.1016/j.biosystems.2010.12.011

28. G. Li, Q. Yang, Dynamics of a stochastic Holling II predator-prey model with Lévy jumps and habitat complexity, *Int. J. Biomath.*, **14** (2021), 2150077. https://doi.org/10.1142/S1793524521500777

29. M. Li, G. Lan, C. Wei, Threshold dynamics of stochastic H7N9 model with Markov switching and hybrid strategy, *J. Franklin Inst.*, **361** (2024), 916–932. https://doi.org/10.1016/j.jfranklin.2023.12.034

30. L. Arnold, *Stochastic Differential Equations: Theory and Applications*, Wiley, New York, 1974.

31. X. Mao, G. Marion, E. Renshaw, Environmental Brownian noise suppresses explosions in population dynamics, *Stochastic Processes Appl.*, **97** (2002), 95–110. https://doi.org/10.1016/S0304-4149(01)00126-0
32. C. Ji, D. Jiang, Dynamics of a stochastic density dependent predator-prey system with Beddington-DeAngelis functional response, *J. Math. Anal. Appl.*, **381** (2011), 441–453. https://doi.org/10.1016/j.jmaa.2011.02.037
33. Q. Liu, L. Zu, D. Jiang, Dynamics of stochastic predator-prey models with Holling II functional response, *Commun. Nonlinear Sci. Numer. Simul.*, **37** (2016), 62–76. https://doi.org/10.1016/j.cnsns.2016.01.005
34. X. Zhang, Y. Li, D. Jiang, Dynamics of a stochastic Holling type II predator-prey model with hyperbolic mortality, *Nonlinear Dyn.*, **87** (2017), 2011–2020. https://doi.org/10.1007/s11071-016-3172-8
35. M. Liu, Dynamics of a stochastic regime-switching predator-prey model with modified Leslie-Gower Holling-type II schemes and prey harvesting, *Nonlinear Dyn.*, **96** (2019), 417–442. https://doi.org/10.1007/s11071-019-04797-x
36. Y. Li, Z. Wei, W. Zhang, T. Kapitaniak, Melnikov-type method for chaos in a class of hybrid piecewise-smooth systems with impact and noise excitation under unilateral rigid constraint, *Appl. Math. Modell.*, **122** (2023), 506–523. https://doi.org/10.1016/j.apm.2023.06.015
37. X. Mao, *Stochastic Differential Equations and Applications*, 2^{nd} edition, Woodhead Publishing, Cambridge, 2007.
38. D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2021), 525–546. https://doi.org/10.1137/S0036144500378302

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